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A NOTE ON THE GENERALIZED BERNOULLI POLYNOMIALS WITH (p, q)-POLYLOGARITHM FUNCTION†

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Abstract. In this article, we define a generating function of the generalized (p, q)-poly-Bernoulli polynomials with variable a by using the polylogarithm function. From the definition, we derive some properties that is concerned with other numbers and polynomials. Furthermore, we construct a special functions and give some symmetric identities involving the generalized (p, q)-poly-Bernoulli polynomials and power sums of the first integers.

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1. Introduction

Throughout this paper, we use the following standard notations. $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of natural numbers, $\mathbb{Z}_+$ denotes the set of nonnegative integers, $\mathbb{Z}$ denotes the set of integers, and $\mathbb{C}$ denotes the set of complex numbers, respectively.

The ordinary Bernoulli polynomials $B_n(x)$ are defined by the generating functions

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad \text{(see [1,2,4,5])}.$$

When $x = 0, B_{n,q}^{(k)} = B_{n,q}^{(k)}(0)$ are called poly-Bernoulli numbers.

In [4], we introduced a generalization of the ordinary Bernoulli polynomials $B_n(x; a)$ with variable a that are defined by

$$\frac{Li_k(1 - e^{-t})}{e^{at} - 1} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x; a) \frac{t^n}{n!}$$

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where

$$Li_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}, \quad (k \in \mathbb{Z}) \quad \text{(see [1,2,4,5,6,7])}$$

is polylogarithm function. When $a = 1$, it is equal to the poly-Bernoulli polynomials. If $k = 1$, then we have $Li_1(x) = -log(1-x)$ and $Li_1(1-e^{-t}) = t$. Using the result of polylogarithm function, we observe that the poly-Bernoulli polynomials is identical to the ordinary Bernoulli polynomials $B_n(x)$.

For $n \in \mathbb{C}$, the (p,q)-integer $[n]_{p,q}$ is defined as follows

$$[n]_{p,q} = \frac{p^n - q^n}{p-q}.$$

Note that $\lim_{p\to 1}[n]_{p,q} = [n]_q$ and $\lim_{q\to 1}[n]_q = n$.

From the definition of the (p,q)-integer, the (p,q)-analogue of the polylogarithm function $Li_{k,p,q}$ is given by

$$Li_{k,p,q}(x) = \sum_{n=1}^{\infty} \frac{x^n}{[n]_{p,q}^k}, (k \in \mathbb{Z}) \quad \text{(see [6])}. \tag{1.1}$$

The Stirling numbers of the second kind $S_2(n,m)$ are defined as below

$$x^n = \sum_{m=0}^{n} S_2(n,m)(x)_m,$$

where $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$ is falling factorial. The generating function of the Stirling numbers of the second kind is given by

$$\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} = \frac{(e^t-1)^m}{m!} \quad \text{(see [3,4,5,8])}. \tag{1.2}$$

In this paper, by using the (p,q)-polylogarithm function, we define a generalized (p,q)-poly-Bernoulli polynomials with variable a. We show that is related with the other numbers and polynomials. We also find some identities that are concerned with the Stirling numbers and the weighted Stirling numbers of the second kind. Moreover, by using special functions and power sums of first integers, we derive some symmetric properties of the generalized (p,q)-poly-Bernoulli numbers and polynomials.

2. Generalized of (p,q)-poly-Bernoulli polynomials with variable a

In this section, by using the Equation (1.2), we construct a generalized (p,q)-poly-Bernoulli polynomials $B_{n,p,q}^{(k)}(x;a)$ with variable a by the following generating functions. We investigate some properties of the polynomials and find several relations that are connected with other polynomials.

Definition 2.1. For $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $0 < q < p \leq 1$, we define a generalized (p, q)-poly-Bernoulli polynomials with variable a by

$$\frac{Li_{k,p,q}(1-e^{-t})}{e^{at}-1}e^{xt} = \sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(x;a)\frac{t^n}{n!} \tag{2.1}$$

where

$$Li_{k,p,q}(t) = \sum_{n=1}^{\infty} \frac{t^n}{[n]_{p,q}^k}$$

is the k-th (p, q)-polylogarithm function. When $x = 0$, $B_{n,p,q}^{(k)}(0; a) = B_{n,p,q}^{(k)}(a)$ are called the generalized (p, q)-poly-Bernoulli numbers with variable a.

From the Equation (2.1), we have a relation between the generalized poly-Bernoulli numbers and polynomials.

Theorem 2.2. *Let $n \geq 0$, $m \geq 1$ and $k \in \mathbb{Z}$. We have*

$$B_{n,p,q}^{(k)}(mx;a) = \sum_{l=0}^{n} \binom{n}{l} m^{n-1} B_{l,p,q}^{(k)}(a) x^{n-l}.$$

Proof. For $n \geq 0$, $m \geq 1$ and $k \in \mathbb{Z}$, we get

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(mx;a)\frac{t^n}{n!} &= \frac{Li_{k,p,q}(1-e^{-t})}{e^{at}-1}e^{mxt} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} m^{n-1} B_{l,p,q}^{(k)}(a)x^{n-l}\right)\frac{t^n}{n!}.\end{aligned}$$

Thus, we obtain the above result. $\square$

When $m = 1$, it satisfies

$$B_{n,p,q}^{(k)}(x;a) = \sum_{l=0}^{n} \binom{n}{l} B_{l,p,q}^{(k)}(a) x^{n-l}.$$

Theorem 2.3. *Let $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$. Then we get*

$$B_{n,p,q}^{(k)}(x+y;a) = \sum_{l=0}^{n} \binom{n}{l} B_{l,p,q}^{(k)}(x;a) y^{n-l}. \tag{2.2}$$

Proof. For $n \in \mathbb{Z}_+$, $k \in \mathbb{Z}$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(x+y;a)\frac{t^n}{n!} &= \frac{Li_{k,p,q}(1-e^{-t})}{e^{at}-1}e^{(x+y)t} \\ &= \sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(x;a)\frac{t^n}{n!}\sum_{n=0}^{\infty} y^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}B_{l,p,q}^{(k)}(x;a)y^{n-l}\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore, we have the obvious result. □

If mx is replaced $x+y$ in Theorem 2.3, the next theorem is obtained.

Corollary 2.4. *For $n>0$, $m \geq 1$ and $k \in \mathbb{Z}$, we get*

$$B_{n,p,q}^{(k)}(mx;a) = \sum_{l=0}^{n}\binom{n}{l}(m-1)^{n-1}B_{l,p,q}^{(k)}(x;a)x^{n-l}.$$

Proof. Let $n>0$, $m \geq 1$ and $k \in \mathbb{Z}$. We derive

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(mx;a)\frac{t^n}{n!} &= \frac{Li_{k,p,q}(1-e^{-t})}{e^{at}-1}e^{mxt} \\ &= \sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(x;a)\frac{t^n}{n!}\sum_{n=0}^{\infty}((m-1)x)^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(m-1)^{n-1}B_{l,p,q}^{(k)}(x;a)x^{n-l}\right)\frac{t^n}{n!}.\end{aligned}$$

□

From the Equation (2.2), we find a recurrence relation as below.

Theorem 2.5. *Let $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$. We obtain*

$$B_{n,p,q}^{(k)}(x+1;a) - B_{n,p,q}^{(k)}(x;a) = \sum_{l=0}^{n-1}\binom{n}{l}B_{l,p,q}^{(k)}(x;a).$$

Proof. For $n \in \mathbb{Z}_+, k \in \mathbb{Z}$, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(x+1;a)\frac{t^n}{n!} - \sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(x;a)\frac{t^n}{n!} \\
&= \frac{Li_{k,p,q}(1-e^{-t})}{e^{at}-1}e^{xt}(e^t-1) \\
&= \sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(x;a)\frac{t^n}{n!}\sum_{n=1}^{\infty}\frac{t^n}{n!} \\
&= \sum_{n=1}^{\infty}\sum_{l=0}^{n-1}\binom{n}{l}B_{l,p,q}^{(k)}(x;a)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficient on both sides, we have the above theorem. □

By using the binomials series and the definition of polylogarithm function, we derive next result.

Theorem 2.6. *For $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, we have*

$$B_{n,p,q}^{(k)}(x;a) = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{r=0}^{m+1}\binom{m+1}{r}\frac{(-1)^{r+1}(x-r+al-am))^n}{[m+1]_{p,q}^n}.$$

Proof. Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$. From the Equation (1.4), we obtain

$$\begin{aligned}
\frac{Li_{k,p,q}(1-e^{-t})}{e^{at}-1}e^{xt} &= \left(-\sum_{l=0}^{\infty}e^{lat}\right)\left(\sum_{n=0}^{\infty}\frac{(1-e^{-t})^{n+1}}{[n+1]_{p,q}^k}\right)e^{xt} \\
&= -\sum_{l=0}^{\infty}\sum_{m=0}^{l}e^{(l-m)at}\frac{(1-e^{-t})^{m+1}}{[m+1]_{p,q}^k}e^{xt} \\
&= \left(-\sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{e^{(l-m)at}}{[m+1]_{p,q}^k}\right)\left(\sum_{r=0}^{m+1}\binom{m+1}{r}(-1)^r e^{(x-r)t}\right) \\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{r=0}^{m+1}\binom{m+1}{r}\frac{(-1)^{r+1}(x-r+al-am)^n}{[m+1]_{p,q}^k}\frac{t^n}{n!}
\end{aligned}$$

So, we get the desired result.

$$B_{n,p,q}^{(k)}(x;a) = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{r=0}^{m+1}\binom{m+1}{r}\frac{(-1)^{r+1}(x-r+al-am)^n}{[m+1]_{p,q}^k}.$$

□

In similar method, we get result that is related with the generalized classical Bernoulli polynomials with variable a.

Theorem 2.7. *Let $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$. Then we have*

$$B^{(k)}_{n,p,q}(x;a) = \sum_{l=0}^{\infty} \frac{1}{[l+1]^k_{p,q}} \sum_{r=0}^{l+1} (-1)^r \binom{l+1}{r} \frac{B_{n+1}(x-r;a)}{n+1}.$$

Proof. For $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$,

$$\begin{aligned}
\frac{Li_{k,p,q}(1-e^{-t})}{e^{at}-1} e^{xt} &= \sum_{l=1}^{\infty} \frac{(1-e^{-t})^l}{[l]^k_{p,q}} \frac{e^{xt}}{e^{at}-1} \\
&= \sum_{n=0}^{\infty} \frac{1}{[l+1]^k_{p,q}} \sum_{r=0}^{l+1} \binom{l+1}{r} (-1)^r \frac{e^{(x-r)t}}{e^{at}-1} \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{\infty} \frac{1}{[l+1]^k_{p,q}} \sum_{r=0}^{l+1} \binom{l+1}{r} (-1)^r \frac{B_{n+1}(x-r;a)}{n+1} \right) \frac{t^n}{n!}.
\end{aligned}$$

Hence, we obtain the result.

$$B^{(k)}_{n,p,q}(x;a) = \sum_{l=0}^{\infty} \frac{1}{[l+1]^k_{p,q}} \sum_{r=0}^{l+1} \binom{l+1}{r} (-1)^r \frac{B_{n+1}(x-r;a)}{n+1}.$$

□

3. Relations with the Stirling numbers of the second kind

In this section, we investigate some identities that is concerned with the Stirling numbers of the second kind by using the generating function. Furthermore, the definition of the weighted Stirling numbers of the second kind gives some interesting results that is associated with the generalized poly Bernoulli polynomials with variable a.

From the Equation (1.2), the polylogarithm function $Li_{k,p,q}(x)$ is represented by the following formula.

$$\begin{aligned}
\frac{1}{t} Li_{k,p,q}(1-e^{-t}) &= \frac{1}{t} \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{(-1)^{n+m}}{[m]^k_{p,q}} m! S_2(n,m) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{m=1}^{n+1} \frac{(-1)^{n+1+m}}{[m]^k_{p,q}} m! \frac{S_2(n+1,m)}{n+1} \frac{t^n}{n!}
\end{aligned} \tag{3.1}$$

The above equation gives the following identity that is related with the Stirling numbers of the second kind.

Theorem 3.1. *Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$. Then we have*

$$B^{(k)}_{n,p,q}(x;a) = \sum_{r=0}^{n} \sum_{l=1}^{r+1} \binom{n}{r} \frac{(-1)^{r+1+l} l! S_2(r+1,l)}{[l]^k_{p,q}(r+1)} B_{n-r}(x;a).$$

Proof. Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$. From the Equation (3.1), the generalized (p, q)-poly-Bernoulli polynomials $B^{(k)}_{n,p,q}(x; a)$ is represented with the Stirling numbers and the generalized ordinary Bernoulli polynomials with variable a

$$\begin{aligned}
\sum_{n=0}^{\infty} B^{(k)}_{n,p,q}(x; a)\frac{t^n}{n!} &= \frac{Li_{k,p,q}(1 - e^{-t})}{t}\frac{te^{xt}}{e^{at} - 1} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=1}^{n+1}\frac{(-1)^{n+1+m}}{[m]^k_{p,q}}m!\frac{S_2(n + 1, m)}{n + 1}\right)\frac{t^n}{n!}\sum_{n=0}^{\infty} B_n(x; a)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{m=1}^{l+1}\binom{n}{l}\frac{(-1)^{l+1+m}m!S_2(l + 1, m)}{[m]^k_{p,q}(l + 1)}B_{n-l}(x; a)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides, the proof is complete. □

Theorem 3.2. *If $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, then we get*

$$B^{(k)}_{n,p,q}(x; a) = \sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}(x)_l S_2(m, l)B^{(k)}_{n-m,p,q}(a). \tag{3.2}$$

where $(x)_l = x(x - 1)(x - 2)\cdots(x - l + 1)$ is falling factorial.

Proof. Let $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$. The generalized (p, q)-poly-Bernoulli numbers and polynomials can be indicated by the following formula that is related with the Stirling numbers.

$$\begin{aligned}
\sum_{n=0}^{\infty} B^{(k)}_{n,p,q}(x; a)\frac{t^n}{n!} &= \frac{Li_{k,p,q}(1 - e^{-t})}{e^{at} - 1}\sum_{l=0}^{\infty}(x)_l\frac{(e^t - 1)^l}{l!} \\
&= \sum_{n=0}^{\infty} B^{(k)}_{n,p,q}(a)\frac{t^n}{n!}\sum_{l=0}^{\infty}(x)_l\sum_{r=l}^{\infty} S_2(r, l)\frac{t^r}{r!} \\
&= \sum_{n=0}^{\infty} B^{(k)}_{n,p,q}(a)\frac{t^n}{n!}\sum_{n=0}^{\infty}\sum_{l=0}^{n}(x)_l S_2(n, l)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}(x)_l S_2(m, l)B^{(k)}_{n-m,p,q}(a)\right)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficient on both sides, we have the explicit result.

$$B^{(k)}_{n,p,q}(x; a) = \sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}(x)_l S_2(m, l)B^{(k)}_{n-m,p,q}(a).$$

□

From Definition 2.1 and the Equation (3.2), we have the another recurrence formula that is different from the result of Theorem 2.5.

Theorem 3.3. *For $n \geq 1, k \in \mathbb{Z}$, we get*

$$\begin{aligned} &B_{n,p,q}^{(k)}(x+a;a) - B_{n,p,q}^{(k)}(x;a) \\ &\quad = \sum_{r=1}^{n-r}\sum_{l=0}^{r}\binom{n}{r}\frac{(-1)^{r+l}l!S_2(r,l)}{[l]_{p,q}^{k}}x^{n-r}. \end{aligned}$$

Proof. Let $n \geq 1, k \in \mathbb{Z}$. From the definition of the generalized (p,q)-ploy-Bernoulli polynomials, we obtain

$$\begin{aligned} &\sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(x+a;a)\frac{t^n}{n!} - \sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(x;a)\frac{t^n}{n!} \\ &\quad = \frac{Li_{k,p,q}(1-e^{-t})}{e^{at}-1}e^{(x+a)t} - \frac{Li_{k,p,q}(1-e^{-t})}{e^{at}-1}e^{xt} \\ &\quad = \sum_{n=1}^{\infty}\sum_{l=0}^{n-1}\frac{(-1)^{n+l+1}}{[l+1]_{p,q}^{k}}(l+1)!S_2(n,l+1)\frac{t^n}{n!}\sum_{m=0}^{\infty}x^m\frac{t^m}{m!} \\ &\quad = \sum_{n=1}^{\infty}\sum_{l=1}^{n}\frac{(-1)^{n+l}}{[l]_{p,q}^{k}}l!S_2(n,l)\frac{t^n}{n!}\sum_{m=0}^{\infty}x^m\frac{t^m}{m!} \\ &\quad = \sum_{n=1}^{\infty}\sum_{r=1}^{n-r}\sum_{l=1}^{r}\binom{n}{r}\frac{(-1)^{r+l}}{[l]_{p,q}^{k}}l!S_2(r,l)x^{n-r}\frac{t^n}{n!}. \end{aligned}$$

Therefore, we get the following recurrence formula.

$$B_{n,p,q}^{(k)}(x+a;a) - B_{n,p,q}^{(k)}(x;a) = \sum_{r=1}^{n-r}\sum_{l=1}^{r}\binom{n}{r}\frac{(-1)^{r+l}l!S_2(r,l)}{[l]_{p,q}^{k}}x^{n-r}.$$

□

Theorem 3.4. *For $n \geq 1, k \in \mathbb{Z}$, we get*

$$E_{n,p,q}^{(k)}(x+1) + E_{n,p,q}^{(k)}(x) = 2B_{n,p,q}^{(k)}(x+a;a) - 2B_{n,p,q}^{(k)}(x;a).$$

Proof. Let $n \geq 1, k \in \mathbb{Z}$. From the following equation

$$\frac{2Li_{k,p,q}(1-e^{-t})}{e^t+1}(1+e^t)e^{xt} = \frac{2Li_{k,p,q}(1-e^{-t})}{e^{at}-1}(e^{at}-1)e^{xt},$$

we have

$$\begin{aligned} &\sum_{n=0}^{\infty}E_{n,p,q}^{(k)}(x+1)\frac{t^n}{n!} + \sum_{n=0}^{\infty}E_{n,p,q}^{(k)}(x)\frac{t^n}{n!} \\ &\quad = 2\sum_{n=0}^{\infty}B_{n,p,q}^{(k)}(x+a;a)\frac{t^n}{n!} - 2\sum_{n=0}^{\infty}B_{n,p,q}^{(k)}(x;a)\frac{t^n}{n!}. \end{aligned}$$

□

In [3], Carlitz introduced the weighted Stirling numbers of the second kind $S_2(n, m, x)$ as follows

$$\frac{(e^t-1)^m}{m!}e^{xt} = \sum_{n=m}^{\infty} S_2(n, m, x)\frac{t^n}{n!} \tag{3.3}$$

Theorem 3.5. *If $n \in \mathbb{N}, k \in \mathbb{Z}$. then we have*

$$B_{n,p,q}^{(k)}(x;a) = \sum_{m=0}^{n} \frac{(m+1)!}{[m+1]_{p,q}^k} \frac{S_2(n+1, m+1, x-(m+1))}{n+1}.$$

Proof. Let $n \in \mathbb{N}, k \in \mathbb{Z}$. From the definition 2.1, we observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(x;a)\frac{t^n}{n!} &= \frac{Li_{k,p,q}(1-e^{-t})}{e^{at}-1}e^{xt} \\
&= \sum_{m=0}^{\infty} \frac{(1-e^{-t})^{m+1}}{[m+1]_{p,q}^k} \frac{e^{xt}}{e^{at}-1} \\
&= \sum_{m=0}^{\infty} \sum_{n=m+1}^{\infty} \frac{(m+1)!}{[m+1]_{p,q}^k} \frac{1}{e^{at}-1} S_2(n, m+1, x-(m+1))\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(m+1)!}{[m+1]_{p,q}^k} B_{n,p,q}(a) \frac{S_2(n+1, m+1, x-(m+1))}{n+1} \frac{t^n}{n!}
\end{aligned}$$

Comparing the coefficient on both sides, we get the above result. □

Corollary 3.6. *Let $n \in \mathbb{N}, k \in \mathbb{Z}$. We obtain*

$$B_{n,p,q}^{(k)}(x;a) = \sum_{l=1}^{n} \sum_{m=0}^{l} \binom{n}{l} \frac{(m+1)!}{[m+1]_{p,q}^k} B_{l,p,q}(a) \frac{S_2(l+1, m+1)}{l+1}(x-(m+1))^{n-l}.$$

Proof. For $n \in \mathbb{N}, k \in \mathbb{Z}$, the generalized (p, q)-poly-Bernoulli polynomials are satisfied as follows

$$\begin{aligned}
\sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(x;a)\frac{t^n}{n!} &= \sum_{m=0}^{\infty} \frac{(1-e^{-t})^{m+1}}{[m+1]_{p,q}^k} \frac{e^{xt}}{e^{at}-1} \\
&= \sum_{m=0}^{\infty} \frac{(m+1)!}{[m+1]_{p,q}^k} B_{n,p,q}(a) \frac{1}{t} \sum_{n=m+1}^{\infty} S_2(n, m+1)\frac{t^n}{n!} \sum_{n=0}^{\infty} (x-(m+1))^n \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \frac{(m+1)!}{[m+1]_{p,q}^k} B_{l,p,q}(a) \frac{S_2(l+1, m+1)}{l+1}(x-(m+1))^{n-l}\frac{t^n}{n!}.
\end{aligned}$$

Thus, we obtain the desired result. □

4. Symmetric identities of the generalized (p,q)-poly-Bernoulli polynomials with variable a

In this section, we construct special functions and find symmetric properties of the generalized (p,q)-poly-Bernoulli polynomials with variable a from the special functions.

Theorem 4.1. *For $m_1, m_2 > 0$ $(m_1 \neq m_2)$, $n \in \mathbb{Z}_+, k \in \mathbb{Z}$, we have*

$$\sum_{r=0}^{n}\binom{n}{r} m_1^{n-r} m_2^{r} B_{r,p,q}^{(k)}(m_1 x; a) B_{n-r,p,q}^{(k)}(m_2 x; a)$$
$$= \sum_{r=0}^{n}\binom{n}{r} m_1^{r} m_2^{n-r} B_{n-r,p,q}^{(k)}(m_1 x; a) B_{r,p,q}^{(k)}(m_2 x; a).$$

Proof. For $n \in \mathbb{Z}_+, k \in \mathbb{Z}$, $m_1, m_2 > 0$ and $m_1 \neq m_2$, we construct a special function as below

$$F(t) = \frac{Li_{k,p,q}(1-e^{-m_1 t}) Li_{kp,q}(1-e^{-m_2 t})}{(e^{a m_1 t}-1)(e^{a m_2 t}-1)} e^{2 m_1 m_2 x t}.$$

From the special function $F(t)$, we derive the following equation.

$$\begin{aligned} F(t) &= \frac{Li_{k,p,q}(1-e^{-m_1 t})}{(e^{a m_1 t}-1)} e^{m_1 m_2 x t} \frac{Li_{k,p,q}(1-e^{-m_2 t})}{(e^{a m_2 t}-1)} e^{m_1 m_2 x t} \\ &= \sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(m_2 x; a) \frac{(m_1 t)^n}{n!} \sum_{r=0}^{\infty} B_{r,p,q}^{(k)}(m_1 x; a) \frac{(m_2 t)^r}{r!} \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \binom{n}{r} m_1^{n-r} m_2^{r} B_{r,p,q}^{(k)}(m_1 x; a) B_{n-r,p,q}^{(k)}(m_2 x; a) \frac{t^n}{n!}. \end{aligned} \tag{4.1}$$

In similar method, we obtain

$$F(t) = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \binom{n}{r} m_1^{r} m_2^{n-r} B_{n-r,p,q}^{(k)}(m_1 x; a) B_{r,p,q}^{(k)}(m_2 x; a) \frac{t^n}{n!}. \tag{4.2}$$

By the Equation (4.1) and (4.2), it is easy to get the above theorem. □

Note that $\widetilde{S}_n(x,m) = x^n + (x+1)^n + \cdots + (x-(m-1))^m = \sum_{k=0}^{m-1}(x+k)^n$, $x \in \mathbb{C}$ is the power sums on an arithmetic progression. The power sums of the first integers are expressed by $\widetilde{S}_n(m) = \widetilde{S}_n(0,m) = \sum_{k=0}^{m-1} k^n$ (see [8]). The exponential generating function of the power sums are expressed by

$$\sum_{n=0}^{\infty} \widetilde{S}_n(x,m) \frac{t^n}{n!} = \frac{e^{mt}-1}{e^t-1} e^{xt}$$

From the generating function of the power sums of first integers, we have the symmetric identity of the generalized (p,q)-poly-Bernoulli polynomials with variable a.

Theorem 4.2. *For $n \in \mathbb{Z}_+, k \in \mathbb{Z}$, $m_1, m_2 \in \mathbb{N}$ and $m_1 \neq m_2$, we get*

$$Li_{k,p,q}(1-e^{-m_2 t})\sum_{r=0}^{n}\binom{n}{r}a^{n-r}m_1^r m_2^{n-r}B_{r,p,q}^{(k)}(m_2 x;a)\widetilde{S}_{n-r}(m_1)$$
$$= Li_{k,p,q}(1-e^{-m_1 t})\sum_{r=0}^{n}\binom{n}{r}a^{n-r}m_1^{n-r}m_2^{r}B_{r,p,q}^{(k)}(m_1 x;a)\widetilde{S}_{n-r}(m_2)$$

Proof. Let $n \in \mathbb{Z}_+$, $m_1, m_2 > 0$ and $m_1 \neq m_2$. If we consider a special function that is given below, then we get

$$F(t) = \frac{Li_{k,p,q}(1-e^{-m_1 t})Li_{k,p,q}(1-e^{-m_2 t})(e^{m_1 m_2 t}-1)(e^{a m_1 m_2 x t})}{(e^{a m_1 t}-1)(e^{a m_2 t}-1)}$$
$$= \sum_{n=0}^{\infty} Li_{k,p,q}(1-e^{-m_2 t})\sum_{r=0}^{n}\binom{n}{r}a^{n-r}m_1^r m_2^{n-r}B_{r,p,q}^{(k)}(m_2 x;a)\widetilde{S}_{n-r}(m_1)\frac{t^n}{n!}.$$

In analogous method, we have

$$F(t) = Li_{k,p,q}(1-e^{-m_1 t})\sum_{n=0}^{\infty}B_{n,p,q}^{(k)}(m_1 x)\frac{(m_2 t)^n}{n!}\sum_{r=0}^{\infty}\widetilde{S}_r(m_2)\frac{(a m_1 t)^r}{r!}$$
$$= \sum_{n=0}^{\infty} Li_{k,p,q}(1-e^{-m_1 t})\sum_{r=0}^{n}\binom{n}{r}a^{n-r}m_1^{n-r}m_2^{r}B_{r,p,q}^{(k)}(m_1 x;a)\widetilde{S}_{n-r}(m_2)\frac{t^n}{n!}.$$

Comparing the coefficient of $\frac{t^n}{n!}$, then we get the above symmetric identity. $\square$

Theorem 4.3. *If $n \in \mathbb{Z}_+, k \in \mathbb{Z}$, m_1, m_2 $(m_1 \neq m_2)$ $\in \mathbb{N}$, then we get*

$$Li_{k,p,q}(1-e^{-m_2 t})\sum_{r=0}^{n}\binom{n}{r}a^{n-r}m_1^r m_2^{n-r}B_{r,p,q}^{(k)}(a)\widetilde{S}_{n-r}(\frac{m_1}{a}x, m_1)$$
$$= Li_{k,p,q}(1-e^{-m_1 t})\sum_{r=0}^{n}\binom{n}{r}a^{n-r}m_1^{n-r}m_2^{r}B_{r,p,q}^{(k)}(a)\widetilde{S}_{n-r}(\frac{m_2}{a}x, m_2)$$

Proof. For $n \in \mathbb{Z}_+$, $m_1, m_2 \in \mathbb{N}$ and $m_1 \neq m_2$. From the special function that is given in theorem 4.2, then we obtain

$$F(t) = \frac{Li_{k,p,q}(1-e^{-m_1 t})Li_{k,p,q}(1-e^{-m_2 t})(e^{m_1 m_2 t}-1)(e^{a m_1 m_2 x t})}{(e^{a m_1 t}-1)(e^{a m_2 t}-1)}$$
$$= \sum_{n=0}^{\infty} Li_{k,p,q}(1-e^{-m_2 t})\sum_{r=0}^{n}\binom{n}{r}a^{n-r}m_1^r m_2^{n-r}B_{r,p,q}^{(k)}(a)\widetilde{S}_{n-r}(\frac{m_1}{a}x, m_1)\frac{t^n}{n!}.$$

In similar method, we get

$$F(t) = Li_{k,p,q}(1-e^{-m_1 t})\sum_{n=0}^{\infty}B_{n,p,q}^{(k)}(m_1 x)\frac{(m_2 t)^n}{n!}\sum_{r=0}^{\infty}\widetilde{S}_r(m_2)\frac{(a m_1 t)^r}{r!}$$
$$= \sum_{n=0}^{\infty} Li_{k,p,q}(1-e^{-m_1 t})\sum_{r=0}^{n}\binom{n}{r}a^{n-r}m_1^{n-r}m_2^{r}B_{r,p,q}^{(k)}(a)\widetilde{S}_{n-r}(\frac{m_2}{a}x, m_2)\frac{t^n}{n!}.$$

Therefore, we have the above result. □

Theorem 4.4. *Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $m_1, m_2 \in \mathbb{N}$ $(m_1 \neq m_2)$, we have*

$$\sum_{r=0}^{n} \binom{n}{r} a^{n-r} m_1^{n-r} m_2^{r-1} B_{r,p,q}(m_1 x) \widetilde{S}_{n-r}(m_2) = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} m_1^{r-1} m_2^{n-r} B_{r,p,q}(m_2 x) \widetilde{S}_{n-r}(m_1).$$

Proof. Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $m_1, m_2 \in \mathbb{N}$ $(m_1 \neq m_2)$. Consider a special function $F(t)$ as follows, then we get

$$\begin{aligned}
F(t) &= \frac{Li_{k,p,q}(1-e^{-m_1 t}) Li_{k,p,q}(1-e^{-m_2 t})(e^{am_1 m_2 t}-1)(e^{am_1 m_2 x t})t}{(e^{am_1 t}-1)^2 (e^{am_2 t}-1)^2} \\
&= \sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(a) \frac{(m_1 t)^n}{n!} \sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(a) \frac{(m_2 t)^n}{n!} \\
&\qquad \times \sum_{r=0}^{\infty} \widetilde{S}_r(m_2) \frac{(am_1 t)^r}{r!} a^{-1} m_2^{-1} \sum_{n=0}^{\infty} B_{n,p,q}(m_1 x) \frac{(am_2 t)^n}{n!} \\
&= \sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(a) \frac{(m_1 t)^n}{n!} \sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(a) \frac{(m_2 t)^n}{n!} \\
&\qquad \times \sum_{n=0}^{\infty} \sum_{r=0}^{n} \binom{n}{r} a^{n-1} m_1^{n-r} m_2^{r-1} B_{r,p,q}(m_1 x) \widetilde{S}_{n-r}(m_2) \frac{t^n}{n!}.
\end{aligned}$$

In similar method, $F(t)$ is expressed by

$$\begin{aligned}
F(t) &= \sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(a) \frac{(m_1 t)^n}{n!} \sum_{n=0}^{\infty} B_{n,p,q}^{(k)}(a) \frac{(m_2 t)^n}{n!} \\
&\qquad \times \sum_{n=0}^{\infty} \sum_{r=0}^{n} \binom{n}{r} a^{n-1} m_2^{n-r} m_1^{r-1} B_{r,p,q}(m_2 x) \widetilde{S}_{n-r}(m_1) \frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficient of both sides, we find the symmetric identity:

$$\sum_{r=0}^{n} \binom{n}{r} a^{n-l} m_1^{n-r} m_2^{r-1} B_{r,p,q}(m_1 x) \widetilde{S}_{n-r}(m_2) = \sum_{r=0}^{n} \binom{n}{r} a^{n-l} m_1^{r-1} m_2^{n-l} B_r(m_2 x) \widetilde{S}_{n-r}(m_1).$$

□

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