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SOME PROPERTIES OF GENERALIZED q-POLY-EULER NUMBERS AND POLYNOMIALS WITH VARIABLE a

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Abstract. In this paper, we discuss generalized q-poly-Euler numbers and polynomials. To do so, we define generalized q-poly-Euler polynomials with variable a and investigate its identities. We also represent generalized q-poly-Euler polynomials $E_{n,q}^{(k)}(x;a)$ using Stirling numbers of the second kind. So we explore the relation between generalized q-poly-Euler polynomials and Stirling numbers of the second kind through it. At the end, we provide symmetric properties related to generalized q-poly-Euler polynomials using alternating power sum.

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1. Introduction

A number of mathematicians have studied Euler numbers and polynomials, Bernoulli numbers and polynomials, tangent numbers and polynomials, poly-Euler numbers and polynomials, and poly-tanent numbers and polynomials(see [1-14]). Mathematicians also used polylogarithm function to redefine Euler numbers and polynomials. This paper is also one of the studies of poly-Euler numbers and polynomials using polylogarithm function.

In this paper, we use the following notations: $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers. The q-number is defined as follows:

$$[n]_q = \frac{1-q^n}{1-q},$$

where $n \in C$ and $0 < q < 1$. For any n, we note that $\lim_{q\to 1}[n]_q = n$.

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The classical Euler polynomials $E_n(x)$ are defined by the following generating function:

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \quad (|t| < \pi).$$

In [8], we know that generating function of generalized Euler polynomials $E_n(x;a)$ are defined by:

$$\frac{2}{e^{at}+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x;a)\frac{t^n}{n!}.$$

$E_n(a) = E_n(0;a)$ is generalized Euler numbers. If we put $a = 1$, then generalized Euler polynomials reduced to classical Euler polynomials.

For $k \in \mathbb{Z}$, polylogarithm function $Li_k(x)$ [1,4,5,6] is defined by

$$Li_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}.$$

For $k \leq 1$, the polylogarithm functions are given

$$Li_1(x) = -\log(1-x), \quad Li_0(x) = \frac{x}{1-x}, \quad Li_{-1}(x) = \frac{x}{(1-x)^2},$$

$$Li_{-2}(x) = \frac{x^2+x}{(1-x)^3}, \quad Li_{-3}(x) = \frac{x^3+4x^2+x}{(1-x)^4}, \cdots .$$

Poly-Euler polynomials are defined by Hamahata [4], as follows:

$$\frac{2Li_k(1-e^{-t})}{t(e^t+1)}e^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x)\frac{t^n}{n!}.$$

In [7], generalized poly-Euler polynomials are defined as the following generating function:

$$\frac{2Li_k(1-e^{-t})}{t(e^{at}+1)}e^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x;a)\frac{t^n}{n!}.$$

This polynomial is generalized through poly-Euler polynomials defined by Hamata [4]. For $k \in \mathbb{Z}$, k-th q-analogue of polylogarithm function $Li_{k,q}(x)$ [5,9] is defined as follows:

$$Li_{k,q}(t) = \sum_{n=1}^{\infty} \frac{t^n}{[n]_q^k}.$$

The k-th q-analogue of polylogarithm functions are given for a nonnegative integer k:

$$\begin{aligned} Li_{k,0} &= \frac{x}{1-x}, \\ Li_{k,-1} &= \frac{x}{(1-x)(1-xq)}, \\ Li_{k,-2} &= \frac{x(1+xq)}{(1-x)(1-xq)(1-xq^2)}, \\ Li_{k,-3} &= \frac{x(q+2xq+2xq^2+x^2q^3)}{(1-x)(1-xq)(1-xq^2)(1-xq^3)(1-xq^4)}, \cdots . \end{aligned}$$

For nonnegative integers k and n, the Stirling numbers of the second kind [2,3,8,10] are defined as the following relation

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k,$$

where $(x)_k = x(x-1)(x-2)\cdots(x-k+1)$ is falling factorial.

Generating function of the Stirling numbers $S_2(n,k)$ is also defined as follows:

$$\frac{(e^t-1)^k}{k!} = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}.$$

The equations

$$\sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!} = \sum_{n=0}^{\infty} S_2(n,k)\frac{t^n}{n!}$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} S_2(n,k)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} S_2(n,k)\frac{t^n}{n!}$$

are satisfied for the reason that $S_2(n,k) = 0$ when $n < k$. Recurrence relation of Stirling numbers of the second kind is

$$S_2(n,k) = kS_2(n-1,k) + S_2(n-1,k-1),$$

where $S_2(0,0) = 1$, $S_2(n,0) = 0\,(n \neq 0)$ and $S_2(n,k) = 0$ when $n < k$. By the above relation, we express some values of Stirling numbers of the second kind $S_2(n,k)$ in the table below(OEIS, sequence A008277, [13]):

$n \setminus k$	0	1	2	3	4	5
0	1					
1	0	1				
2	0	1	1			
3	0	1	3	1		
4	0	1	7	6	1	
5	0	1	15	25	10	1

In this paper, we define generalized q-poly-Euler polynomials with variable a through generalized poly-Euler polynomials defined by [7] and explore several properties. To be specific, we get some identites from generalized q-poly-Euler polynomials. Also, we utilize the generating function of Stirling numbers of the second kind to describe the relation between generalized q-poly-Euler polynomials and Stirling numbers of the second kind. At the end, we examine symmetric properies of generalized q-poly-Euler polynomials by using alternating power sum.

2. Generalized q-poly-Euler numbers and polynomials with variable a

In this section, we define generalized q-poly-Euler polynomials with variable a. In addition, we derive some properties from expressing generalized q-poly-Euler polynomials $E_{n,q}^{(k)}(x;a)$ in several ways.

Definition 2.1. For $k \in \mathbb{Z}$ and $0 < q < 1$, generalized q-poly-Euler polynomials with variable a are defined as the following generating function

$$\frac{2Li_{k,q}(1-e^{-t})}{t(e^{at}+1)}e^{xt} = \sum_{n=0}^{\infty} E_{n,q}^{(k)}(x;a)\frac{t^n}{n!},$$

where $Li_{k,q}(t) = \sum_{n=1}^{\infty} \frac{t^n}{[n]_q^k}$ is k-th q-analogue of polylogarithm function.

$E_{n,q}^{(k)}(a) = E_{n,q}^{(k)}(0;a)$ are called generalized q-poly-Euler numbers with variable a when $x = 0$. If we set $a = 1$, $k = 1$, and $q \to 1$ in Definition 2.1, then the generalized q-poly-Euler polynomials are reduced to classical Euler polynomials because of $\lim_{q\to 1} Li_{1,q}(1-e^{-t}) = t$. That is,

$$\lim_{q\to 1} E_{n,q}^{(1)}(x;1) = E_n(x).$$

Theorem 2.2. *For $k \in \mathbb{Z}$ and a nonnegative integer n and m, we get*

$$E_{n,q}^{(k)}(mx;a) = \sum_{l=0}^{n} \binom{n}{l} E_{l,q}^{(k)}(a)\, m^{n-l}x^{n-l}.$$

Proof. From Definition 2.1, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} E_{n,q}^{(k)}(mx;a)\frac{t^n}{n!} &= \frac{2Li_{k,q}(1-e^{-t})}{t(e^{at}+1)}e^{mxt} \\
&= \left(\sum_{n=0}^{\infty} E_{n,q}^{(k)}(a)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(mx)^n\frac{t^n}{n!}\right) \qquad (2.1) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}E_{l,q}^{(k)}(a)\, m^{n-l}x^{n-l}\right)\frac{t^n}{n!}.
\end{aligned}$$

Therefore, we finish the proof of Theorem 2.2 by comparing the coefficients of $\frac{t^n}{n!}$. □

If $m=1$ in Theorem 2.2, then we get the following corollary.

Corollary 2.3. *For $k\in\mathbb{Z}$ and a nonnegative integer n, we have*

$$E_{n,q}^{(k)}(x;a)=\sum_{l=0}^{n}\binom{n}{l}E_{l,q}^{(k)}(a)\,x^{n-l}.$$

Theorem 2.4. *For $k\in\mathbb{Z}$ and a nonnegative integer n and m, we obtain*

$$E_{n,q}^{(k)}(mx;a)=\sum_{l=0}^{n}\binom{n}{l}E_{l,q}^{(k)}(x;a)\,(m-1)^{n-l}x^{n-l}.$$

Proof. By utlizing Definition 2.1, we have

$$\begin{aligned}\sum_{n=0}^{\infty}E_{n,q}^{(k)}(mx;a)\frac{t^n}{n!}&=\frac{2Li_{k,q}(1-e^{-t})}{t(e^{at}+1)}e^{xt}e^{(m-1)xt}\\&=\left(\sum_{n=0}^{\infty}E_{n,q}^{(k)}(x;a)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(m-1)^nx^n\frac{t^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}E_{l,q}^{(k)}(x;a)\,(m-1)^{n-l}x^{n-l}\right)\frac{t^n}{n!}.\end{aligned}\tag{2.2}$$

Therefore, we end the proof by comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation (2.2). □

As a result of Theorem 2.2 and Theorem 2.4, $E_{n,q}^{(k)}(mx;a)$ can be presented as generalized q-poly-Euler polynomials and generalized q-poly-Euler numbers, respectively.

Theorem 2.5. *For $k\in\mathbb{Z}$ and a nonnegative integer n, we get*

$$E_{n,q}^{(k)}(x+y;a)=\sum_{l=0}^{n}\binom{n}{l}E_{l,q}^{(k)}(x;a)\,y^{n-l}.$$

Proof. Proof is omitted since it is a similar method of Theorem 2.2. □

Theorem 2.6. *For $k\in\mathbb{Z}$ and $n\in\mathbb{N}$, we have*

$$E_{n,q}^{(k)}(x+1;a)-E_{n,q}^{(k)}(x;a)=\sum_{l=0}^{n-1}\binom{n}{l}E_{l,q}^{(k)}(x;a).$$

Proof. By using Definition 2.1, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}E_{n,q}^{(k)}(x+1;a)\frac{t^n}{n!}-\sum_{n=0}^{\infty}E_{n,q}^{(k)}(x;a)\frac{t^n}{n!}\\&=\frac{2Li_{k,q}(1-e^{-t})}{t(e^{at}+1)}e^{xt}\left(e^{t}-1\right)\\&=\sum_{n=1}^{\infty}\left(\sum_{l=0}^{n-1}\binom{n}{l}E_{l,q}^{(k)}(x;a)\right)\frac{t^n}{n!}.\end{aligned}\tag{2.3}$$

Then we compare the coefficients of $\frac{t^n}{n!}$ for $n \geq 1$. The reason both sides of the above equation (2.3) can be compared the coefficients is that $E_{0,q}^{(k)}(x+1;a) - E_{0,q}^{(k)}(x;a) = 0$. Thus, the proof is done. $\square$

Theorem 2.7. *For $k \in \mathbb{Z}$ and a nonnegative integer n, we get*

$$nE_{n-1,q}^{(k)}(x;a) = \sum_{l=0}^{\infty}\sum_{m=0}^{l+1}\binom{l+1}{m}\frac{(-1)^m}{[l+1]_q^k}E_n(x-m;a),$$

where $E_n(x;a)$ is generalized Euler polynomials.

Proof. By using Definition 2.1, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}E_{n,q}^{(k)}(x;a)\frac{t^n}{n!}\\
&= \frac{1}{t}\sum_{l=0}^{\infty}\frac{(1-e^{-t})^{l+1}}{[l+1]_q^k}\frac{2}{e^{at}+1}e^{xt}\\
&= \frac{1}{t}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{m=0}^{l+1}\binom{l+1}{m}\frac{(-1)^m}{[l+1]^k}E_n(x-m;a)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.4}$$

Because of the identity $\sum_{n=0}^{\infty}E_{n,q}^{(k)}(x;a)\frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty}nE_{n-1,q}^{(k)}(x;a)\frac{t^n}{n!}$, we multiply both sides of the above equation (2.4) by t and compare the coefficients of $\frac{t^n}{n!}$. Hence, we end the proof. $\square$

Theorem 2.8. *For $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, we obtain*

$$nE_{n-1,q}^{(k)}(x;a) = 2\sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{r=0}^{m+1}\binom{m+1}{r}\frac{(-1)^{l-m+r}}{[m+1]_q^k}(al-am-r+x)^n.$$

Proof. From Definition 2.1, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}E_{n,q}^{(k)}(x;a)\frac{t^n}{n!}\\
&= \frac{2}{t}\left(\sum_{l=0}^{\infty}\frac{(1-e^{-t})^{l+1}}{[l+1]_q^k}\right)\left(\sum_{m=0}^{\infty}(-1)^m e^{(am+x)t}\right)\\
&= \frac{2}{t}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{r=0}^{m+1}\binom{m+1}{r}\frac{(-1)^r e^{-rt}}{[l+1]_q^k}(-1)^{l-m}e^{(al-am+x)t}\\
&= \frac{2}{t}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{r=0}^{m+1}\binom{m+1}{r}\frac{(-1)^{l-m+r}}{[l+1]_q^k}(al-am-r+x)^n\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.5}$$

If we multiply both sides of the above equation (2.5) by t, then we can compare the coefficients. The reason is that $\sum_{n=0}^{\infty}E_n^{(k)}(x;a)\frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty}nE_{n-1}^{(k)}(x;a)\frac{t^n}{n!}$. Therefore, the proof is done. $\square$

3. Relation between generalized q-poly-Euler polynomials and Stirling numbers of the second kind

In this section, we examine the relation of generalized q-poly-Euler polynomials and Stirling numbers of the second kind.

Theorem 3.1. *For $k \in \mathbb{Z}$ and a nonnegative integer n, we get*

$$E_{n,q}^{(k)}(x;a) = \sum_{l=0}^{n} \sum_{m=1}^{l+1} \binom{n}{l} \frac{(-1)^{l+m+1} m!}{[m]_q^k} \frac{S_2(l+1,m)}{l+1} E_{n-l}(x;a),$$

where $E_n(x;a)$ is generalized Euler polynomials.

Proof. By utilizing Definition 2.1, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} E_{n,q}^{(k)}(x;a) \frac{t^n}{n!} \\
&= \frac{1}{t} \sum_{m=1}^{\infty} \frac{(1-e^{-t})^m}{[m]_q^k} \frac{2}{e^{at}+1} e^{xt} \\
&= \frac{1}{t} \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{(-1)^{n+m}\, m!}{[m]_q^k} S_2(n,m) \frac{t^n}{n!} \frac{2}{e^{at}+1} e^{xt} \\
&= \left(\sum_{n=0}^{\infty} \sum_{m=1}^{n+1} \frac{(-1)^{n+m+1} m!}{[m]_q^k} \frac{S_2(n+1,m)}{n+1} \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} E_n(x;a) \frac{t^n}{n!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{m=1}^{l+1} \binom{n}{l} \frac{(-1)^{l+m+1} m!}{[m]_q^k} \frac{S_2(l+1,m)}{l+1} E_{n-l}(x;a) \right) \frac{t^n}{n!}.
\end{aligned} \tag{3.1}$$

In (3.1), the reason equation

$$\sum_{n=0}^{\infty} \sum_{l=1}^{\infty} S_2(n,l) = \sum_{n=1}^{\infty} \sum_{l=1}^{n} S_2(n,l)$$

can be satisfied is that $S_2(n,l) = 0$ when $n < l$. Thus, the proof is done by comparing the coefficients of $\frac{t^n}{n!}$. □

Theorem 3.2. *For $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, we obtain*

$$nE_{n-1,q}^{(k)}(x;a) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \frac{(-1)^{l+m+1}(m+1)!}{[m+1]_q^k} S_2(l,m+1) E_{n-l}(x;a),$$

where $E_n(x;a)$ is generalized Euler polynomials.

Proof. From Definition 2.1, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} E_{n,q}^{(k)}(x;a)\frac{t^n}{n!} \\
&= \frac{1}{t}\sum_{m=1}^{\infty}\frac{(-1)^m\, m!}{[m]_q^k}\frac{(e^{-t}-1)^m}{m!}\frac{2}{e^{at}+1}e^{xt} \\
&= \frac{1}{t}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n+m+1}(m+1)!}{[m+1]_q^k}S_2(n,m+1)\frac{t^n}{n!}\frac{2}{e^{at}+1}e^{xt} \\
&= \frac{1}{t}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{(-1)^{l+m+1}(m+1)!}{[m+1]_q^k}S_2(l,m+1)E_{n-l}(x;a)\right)\frac{t^n}{n!}.
\end{aligned} \tag{3.2}$$

If we multiply both sides of the equation (3.2) by t, then we can compare the coefficients because of the identity $\sum_{n=0}^{\infty} E_{n,q}^{(k)}(x;a)\frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} nE_{n-1,q}^{(k)}(x;a)\frac{t^n}{n!}$. Consequently, the proof is complete. □

Theorem 3.3. *For $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, we obtain*

$$E_{n,q}^{(k)}(x;a) = \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}(x)_m S_2(l,m)E_{n-l,q}^{(k)}(a),$$

where $(x)_m = x(x-1)\cdots(x-m+1)$ is falling factorial.

Proof. From Definition 2.1, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} E_{n,q}^{(k)}(x;a)\frac{t^n}{n!} &= \frac{2Li_{k,q}(1-e^{-t})}{t(e^{at}+1)}\Big\{(e^t-1)+1\Big\}^x \\
&= \frac{2Li_{k,q}(1-e^{-t})}{t(e^{at}+1)}\sum_{m=0}^{\infty}\binom{x}{m}(e^t-1)^m \\
&= \frac{2Li_{k,q}(1-e^{-t})}{t(e^{at}+1)}\sum_{m=0}^{\infty}(x)_m\frac{(e^t-1)^m}{m!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}(x)_m S_2(l,m)E_{n-l}^{(k)}(a)\right)\frac{t^n}{n!}.
\end{aligned} \tag{3.3}$$

Thus, we finish the proof by comparing the coefficients of $\frac{t^n}{n!}$. □

Theorem 3.4. *For $k \in \mathbb{Z}$ and a nonnegative integer n, we obtain*

$$\begin{aligned}
&nE_{n-1,q}^{(k)}(x+a;a) + nE_{n-1,q}^{(k)}(x;a) \\
&\quad = 2\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{(-1)^{l+m+1}(m+1)!}{[m+1]_q^k}S_2(l,m+1)x^{n-l}.
\end{aligned}$$

Proof. By using Definition 2.1, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} E_{n,q}^{(k)}(x+a;a)\frac{t^n}{n!}+\sum_{n=0}^{\infty} E_{n,q}^{(k)}(x;a)\frac{t^n}{n!}\\
&=\frac{2Li_{k,q}(1-e^{-t})}{t(e^{at}+1)}e^{xt}\left(e^{at}+1\right)\\
&=\frac{2}{t}\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n+m+1}(m+1)!}{[m+1]_q^k}S_2(n,m+1)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}x^n\frac{t^n}{n!}\right) \qquad (3.4)\\
&=\frac{2}{t}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{(-1)^{l+m+1}(m+1)!}{[m+1]_q^k}S_2(l,m+1)x^{n-l}\right)\frac{t^n}{n!}.
\end{aligned}$$

Let us multiply both sides of the above equation (3.4) by t. Then we can compare the coefficients of $\frac{t^n}{n!}$ because of the identity $\sum_{n=0}^{\infty} E_n^{(k)}(x+a;a)\frac{t^{n+1}}{n!}+\sum_{n=0}^{\infty} E_n^{(k)}(x;a)\frac{t^{n+1}}{n!}=\sum_{n=0}^{\infty} nE_{n-1}^{(k)}(x+a;a)\frac{t^n}{n!}+\sum_{n=0}^{\infty} nE_{n-1}^{(k)}(x;a)\frac{t^n}{n!}$. Hence, we end the proof. □

4. Symmtric properties of the generalized q-poly-Euler polynomials using alternating power sum.

In this section, we first offer a well-known alternating power sum and utilize it to provide symmetric identities of generalized q-poly-Euler polynomials. Furthermore, we investigate the symmetric identity of generalized q-poly-Euler polynomials.

Let w is an odd number. Then we can easily see

$$\sum_{n=0}^{\infty}\tilde{A}_n(w)\frac{t^n}{n!}=\frac{e^{wt}+1}{e^t+1}, \qquad (4.1)$$

where $\tilde{A}_n(w)=\sum_{l=0}^{w-1}(-1)^l\, l^n$ is called alternating power sum(see [14]).

Theorem 4.1. *Let w_1 and w_2 be an odd number and n be a nonnegative integer. Then we get*

$$\begin{aligned}
&Li_{k,q}(1-e^{-w_1t})\sum_{l=0}^{n}\binom{n}{l}a^{n-l}w_1^{n-l}w_2^{l+1}E_{l,q}^{(k)}(w_1x;a)\tilde{A}_{n-l}(w_2)\\
&=Li_{k,q}(1-e^{-w_2t})\sum_{l=0}^{n}\binom{n}{l}a^{n-l}w_2^{n-l}w_1^{l+1}E_{l,q}^{(k)}(w_2x;a)\tilde{A}_{n-l}(w_1).
\end{aligned}$$

Proof. Let us show that symmetric property of generalized q-poly-Euler polynomials by using alternating power sum. To do this we suppose that

$$F_1(t)=\frac{2Li_{k,q}(1-e^{-w_1t})Li_{k,q}(1-e^{-w_2t})(e^{aw_1w_2t}+1)}{t(e^{aw_1t}+1)(e^{aw_2t}+1)}e^{w_1w_2xt}. \qquad (4.2)$$

Then we obtain

$$
\begin{aligned}
&F_1(t)\\
&= Li_{k,q}(1-e^{-w_1t})\frac{2Li_{k,q}(1-e^{-w_2t})}{t(e^{aw_2t}+1)}e^{w_1w_2xt}\frac{e^{aw_1w_2t}+1}{e^{aw_1t}+1}\\
&= Li_{k,q}(1-e^{-w_1t})\left(w_2\sum_{n=0}^{\infty}E_{n,q}^{(k)}(w_1x;a)\frac{(w_2t)^n}{n!}\right)\left(\sum_{n=0}^{\infty}\tilde{A}_n(w_2)\frac{(aw_1t)^n}{n!}\right)\\
&= Li_{k,q}(1-e^{-w_1t})\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}a^{n-l}w_1^{n-l}w_2^{l+1}E_{l,q}^{(k)}(w_1x;a)\tilde{A}_{n-l}(w_2)\frac{t^n}{n!}.
\end{aligned}
\tag{4.3}
$$

From similar method of the equation (4.3), we get

$$
\begin{aligned}
&F_1(t)\\
&= Li_{k,q}(1-e^{-w_2t})\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}a^{n-l}w_2^{n-l}w_1^{l+1}E_{l,q}^{(k)}(w_2x;a)\tilde{A}_{n-l}(w_1)\frac{t^n}{n!}.
\end{aligned}
\tag{4.4}
$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the equations (4.3) and (4.4), we finish the proof. This theorem is symmetric property. $\square$

Theorem 4.2. *Let w_1 and w_2 be an odd number and n be a nonnegative integer. Then we have*

$$
\begin{aligned}
&\sum_{l=0}^{n}\binom{n}{l}a^{n+1}w_1^{l+1}w_2^{n-l}E_l(w_2x)\tilde{A}_{n-l}(w_1)\\
&=\sum_{l=0}^{n}\binom{n}{l}a^{n+1}w_2^{l+1}w_1^{n-l}E_l(w_1x)\tilde{A}_{n-l}(w_2),
\end{aligned}
$$

where $E_n(x)$ is classical Euler polynomials.

Proof. First, let us assume that

$$
F_2(t)=\frac{8Li_{k,q}(1-e^{-w_1t})Li_{k,q}(1-e^{-w_2t})(e^{aw_1w_2t}+1)}{t^2(e^{aw_1t}+1)^2(e^{aw_2t}+1)^2}e^{aw_1w_2xt}. \tag{4.5}
$$

Then we calculate

$$
\begin{aligned}
F_2(t) &= \frac{2Li_{k,q}(1-e^{-w_1t})}{t(e^{aw_1t}+1)}\frac{2Li_{k,q}(1-e^{-w_2t})}{t(e^{aw_2t}+1)}\\
&\quad\times\frac{2}{(e^{aw_1t}+1)}e^{aw_1w_2xt}\frac{e^{aw_1w_2t}+1}{e^{aw_2t}+1}\\
&=\left(\sum_{n=0}^{\infty}w_1^{n+1}E_{n,q}^{(k)}(a)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}w_2^{n+1}E_{n,q}^{(k)}(a)\frac{t^n}{n!}\right)\\
&\quad\times\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}a^{n+1}w_1^{l+1}w_2^{n-l}E_l(w_2x)\tilde{A}_{n-l}(w_1)\frac{t^n}{n!}.
\end{aligned}
\tag{4.6}
$$

In a similar way to the above equation (4.6), we get

$$\begin{aligned} F_2(t) &= \left(\sum_{n=0}^{\infty} w_1^{n+1} E_{n,q}^{(k)}(a)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} w_2^{n+1} E_{n,q}^{(k)}(a)\frac{t^n}{n!}\right) \\ &\quad \times \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l} a^{n+1} w_2^{l+1} w_1^{n-l} E_l(w_1 x)\tilde{A}_{n-l}(w_2)\frac{t^n}{n!}. \end{aligned} \tag{4.7}$$

Hence, the proof is complete by comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the equations (4.6) and (4.7). □

Theorem 4.3. *Let n be a nonnegative integer and $w_1, w_2 > 0\,(w_1 \neq w_2)$. Then we have*

$$\begin{aligned} &\sum_{l=0}^{n}\binom{n}{l} w_1^l w_2^{n-l} E_{l,q}^{(k)}(w_2 x; a)\, E_{n-l,q}^{(k)}(w_1 x; a) \\ &= \sum_{l=0}^{n}\binom{n}{l} w_2^l w_1^{n-l} E_{l,q}^{(k)}(w_1 x; a)\, E_{n-l,q}^{(k)}(w_2 x; a). \end{aligned}$$

Proof. Let us consider the function

$$F_3(t) = \frac{4Li_{k,q}(1-e^{-w_1 t})Li_{k,q}(1-e^{-w_2 t})}{t^2(e^{aw_1 t}+1)(e^{aw_2 t}+1)} e^{2w_1 w_2 x t}. \tag{4.8}$$

Then we obtain

$$\begin{aligned} F_3(t) &= \left(\frac{2Li_{k,q}(1-e^{-w_1 t})}{t(e^{aw_1 t}+1)} e^{w_1 w_2 x t}\right)\left(\frac{2Li_k(1-e^{-w_2 t})}{t(e^{aw_2 t}+1)} e^{w_1 w_2 x t}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} w_1^{l+1} w_2^{n-l+1} E_{l,q}^{(k)}(w_2 x; a) E_{n-l,q}^{(k)}(w_1 x; a)\right)\frac{t^n}{n!}. \end{aligned} \tag{4.9}$$

By calculating in the same way as the above equation (4.9), we can get

$$F_3(t) = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} w_2^{l+1} w_1^{n-l+1} E_{l,q}^{(k)}(w_1 x; a) E_{n-l,q}^{(k)}(w_2 x; a)\right)\frac{t^n}{n!}. \tag{4.10}$$

The proof is complete as a result of the equations (4.9) and (4.10). □

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