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FITTED MESH METHOD FOR SINGULARLY PERTURBED DELAY DIFFERENTIAL TURNING POINT PROBLEMS EXHIBITING TWIN BOUNDARY LAYERS†

WONDWOSEN GEBEYAW MELESSE*, AWOKE ANDARGIE TIRUNEH, AND GETACHEW ADAMU DERESE

Abstract. In this paper, a class of linear second order singularly perturbed delay differential turning point problems containing a small delay (or negative shift) on the reaction term and when the solution of the problem exhibits twin boundary layers are examined. A hybrid finite difference scheme on an appropriate piecewise-uniform Shishkin mesh is constructed to discretize the problem. We proved that the method is almost second order ε-uniformly convergent in the maximum norm. Numerical experiments are considered to illustrate the theoretical results.

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1. Introduction

Many real life phenomena in different fields of science are modeled mathematically by delay differential equations (DDEs). These type of equations arises widely in scientific fields such as physics, bio-sciences, ecology, control theory, economics, material science, medicine, and robotics; in which the time evolution depends not only on present states but also on the states at or near a given time in the past. DDEs are also prominent in describing several aspects of infectious disease dynamics such as primary infection, drug therapy, immune response, etc. In addition, statistical analysis of ecological data has shown that there is evidence of delay effects in the population dynamics of many species, for the detail

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*Corresponding author.
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theory of DDEs one can refer the books [3, 6].
If we restrict the class of DDEs in which the highest derivative is multiplied by a small parameter, then it is said to be a singularly perturbed delay differential equations (SPDDEs) [10]. In the past less attention had been given for the solutions of SPDDEs. However, in recent years, there has been a growing interest in the treatment of such problems. This is due to their importance in the modeling of processes in various fields: such as, in optical bi-stable devices [5], variational problems in control theory [7, 17], in the hydrodynamics of liquid helium [8], in the first exit-time problem [15], to describe the human pupil-light reflex [18], in micro-scale heat transfer [27] and in a variety of models for physiological processes or diseases [19, 28].
The study of different classes of SPDDEs was initiated by Lange and Miura [14, 15, 16], where they used extension of the method of matched asymptotic expansions for approximating the solution. But in all the cases they excluded the occurrence of turning points and left it for future study. On the other hand, Kadalbajoo and Sharma [9, 10, 11] initiated the numerical study of SPDDEs with small shifts by constructing a variety of numerical schemes. In recent years, different Scholars further developed numerical schemes for SPDDEs with negative shift, to mention few [1, 20, 21]. Most of the works developed so far focuses only on SPDDEs without turning points. In contrast, there are few works on singularly perturbed delay turning point problems. As far as we know the papers by Ria and Sharma [22, 23, 24, 25] are the first and also the only notable works in the treatment of such problems when the solutions exhibit both interior and boundary layers, where the authors used a fitted mesh and fitted operator methods and obtained an almost first order uniform convergence. Therefore, its natural to look for a robust numerical method for SPDDEs with turning points having a better accuracy and efficiency.
Motivated by the work of Ria and Sharma [25], we consider the following second order linear singularly perturbed delay differential equation with a turning point at $x = 0$:

$$-\varepsilon y''(x) - a(x)y'(x) + b(x)y(x-\delta) = f(x), \; \forall x \in \Omega = (-1, 1), \tag{1}$$

$$y(x) = \phi(x), \quad -1-\delta \le x \le -1, \qquad y(1) = \gamma, \tag{2}$$

where $a(x)$, $b(x)$, $f(x)$ and $\phi(x)$ are sufficiently smooth functions on $\Omega = (-1, 1)$, γ is any real constant, $0 < \varepsilon \ll 1$ is the singular perturbation parameter, and $\delta = o(\varepsilon)$ is the delay parameter (or negative shift). When the shift is zero (i.e., $\delta = 0$), the solution of the resulting turning point problem exhibits twin boundary layers or interior layer behavior depending on ε and $\lambda = \frac{b(0)}{a'(0)}$ i.e., if $\lambda < 0$, then $y(x)$ is smooth near the turning point $x = 0$, whereas if $\lambda > 0$, then $y(x)$ has a large gradient near $x = 0$ resulting in an interior layer [4].
In this article we consider the case in which the turning point results into a twin exponential boundary layers in the solution of the problem, together with the

following assumptions

$$a(0) = 0, \qquad a'(0) < 0, \tag{3}$$

$$b(x) \geq \beta > 0, \quad \forall x \in \bar{\Omega} = [-1, 1], \tag{4}$$

$$|a'(x)| \geq \frac{|a'(0)|}{2}, \quad \forall x \in \bar{\Omega} = [-1, 1]. \tag{5}$$

Under the assumptions (3)–(5), the problem (1)–(2) possesses a unique solution having twin boundary layers of exponential type at $x = \pm 1$ i.e., at both end points [25].

Here for the numerical treatment of problem (1)–(2), we propose hybrid finite difference scheme on an appropriate piecewise-uniform Shishkin mesh. Further, we analyze the stability and uniform convergence of the proposed scheme. In addition, we investigate the effect of the delay parameter on the behavior of the solution.

Throughout this paper, M (sometimes sub-scripted) denote a generic positive constants independent of the singular perturbation parameter ε and in the case of discrete problems also independent of the mesh parameter N. These constants may assume different values but remains to be constant. The maximum norm (i.e., $\|f\| = \max_{-1 \leq x \leq 1} |f(x)|$) is used for studying the convergence of the approximate solution to the exact solution of the problem.

2. The Continuous Problem

Using Taylor's series expansion to approximate the term containing the delay parameter, gives us

$$y(x - \delta) \approx y(x) - \delta y'(x) + \frac{\delta^2}{2} y''(x), \tag{6}$$

Substituting (6) into (1)–(2) and simplifying, gives the following asymptotically equivalent two-point boundary value problem

$$\begin{gathered} -(\varepsilon - \frac{\delta^2}{2} b(x)) y''(x) - (a(x) + \delta b(x)) y'(x) + b(x) y(x) \approx f(x), \\ y(-1) \approx \phi(-1), \quad y(1) = \gamma. \end{gathered} \tag{7}$$

Since (7) is an approximation version of (1)–(2), it is good to use different notation (say $u(x)$) for the solution of this approximate equation. Thus (7) can be rewritten as

$$\begin{aligned} &Lu := -C_\varepsilon(x) u''(x) - A(x) u'(x) + b(x) u(x) = f(x), \\ &u(-1) = \phi(-1) = \phi, \qquad u(1) = \gamma(1) = \gamma, \end{aligned} \tag{8}$$

where $C_\varepsilon(x) = (\varepsilon - \frac{\delta^2}{2} b(x)) > 0$ and $A(x) = a(x) + \delta b(x)$. Furthermore, the terms $a(x)$, $b(x)$ and δ are such that $|A(x)| \geq 2\alpha > 0$, for $\tau < |x| \leq 1$, for some $\tau > 0$ and later on we will use the term C_ε to denote the constant part of $C_\varepsilon(x)$. (Here it is to be noted that since $b(x)$ is bounded and δ is a small order of ε, we have $C_\varepsilon = O(\varepsilon)$).

The solution of problem (8) is an approximation to the solution of the original problem (1)–(2).
We establish some a priori results about the solutions and their derivatives for the modified problem (8). Hereinafter, we divide the interval $\bar{\Omega}$ in to three sub intervals as $\Omega_1 = [-1, -\tau]$, $\Omega_2 = [-\tau, \tau]$ and $\Omega_3 = [\tau, 1]$ such that $\bar{\Omega} = \Omega_1 \cup \Omega_2 \cup \Omega_3$, where $0 < \tau \leq 1/2$. First, we consider the following property of the operator L of (8).

Lemma 2.1. *(Maximum principle)*
Let $\pi(x)$ be any sufficiently smooth function satisfying $\pi(-1) \geq 0$ and $\pi(1) \geq 0$, such that $L\pi(x) \geq 0$ for all $x \in \Omega$. Then $\pi(x) \geq 0$ for all $x \in \bar{\Omega}$.

Proof. Let x^* be an arbitrary point in $\Omega = (-1, 1)$ such that $\pi(x^*) = \min_{x\in\bar{\Omega}}\{\pi(x)\}$ and assume that $\pi(x^*) < 0$. Clearly $x^* \notin \{-1, 1\}$, and from the definition of x^*, we have $\pi'(x^*) = 0$ and $\pi''(x^*) \geq 0$. But then,

$$L\pi(x^*) = -C_\varepsilon(x)\pi''(x^*) - A(x^*)\pi'(x^*) + b(x^*)\pi(x^*) \leq 0,$$

which is a contradiction. It follows that our assumption $\pi(x^*) < 0$ is wrong. So, $\pi(x^*) \geq 0$. Since x^* is an arbitrary point, $\pi(x) \geq 0$, $\forall x \in \bar{\Omega} = [-1, 1]$. □

Using the maximum principle its easy to prove that:

Lemma 2.2. *(Stability Result)*
Let $u(x)$ be the solution of the TPP (8). Then $\forall C_\varepsilon > 0$ we have

$$\|u\| \leq \beta^{-1} \|f\| + \max(|\phi|, |\gamma|), \quad \forall x \in \bar{\Omega} = [-1, 1].$$

Proof. First we consider the barrier functions $\varphi^{\pm}$ defined by

$$\varphi^{\pm}(x) = \beta^{-1} \|f\| + \max(|\phi|, |\gamma|) \pm u(x),$$

Then its easy to show that $\varphi^{\pm}(-1) \geq 0$ and $\varphi^{\pm}(1) \geq 0$, and

$$\begin{aligned} L\varphi^{\pm}(x) &= -C_\varepsilon(x)(\varphi^{\pm}(x))'' - A(x)(\varphi^{\pm}(x))' + b(x)\varphi^{\pm}(x) \\ &= b(x)(\beta^{-1}\|f\| + \max(|\phi|, |\gamma|)) \pm Lu(x) \\ &= b(x)(\beta^{-1}\|f\| + \max(|\phi|, |\gamma|)) \pm f(x) \\ &\geq (\|f\| \pm f(x)) + \beta \max(|\phi|, |\gamma|) \geq 0. \end{aligned}$$

Therefore, from Lemma 2.1, we obtain $\varphi^{\pm}(x) \geq 0$ for all $x \in [-1, 1]$, which gives the desired estimate. □

The following theorem gives estimates for u and its derivatives in the interval Ω_1 and Ω_3 which exclude the turning point $x = 0$.

Theorem 2.3. *Let $A(x)$, $b(x)$ and $f(x) \in C^m(\bar{\Omega})$, $m > 0$. Then there exists positive constants α and M, such that for $A(x) > 0$ on Ω_1, the solution $u(x)$ of problem (8) satisfies*

$$\left|u^{(i)}(x)\right| \leq M[1 + C_\varepsilon^{-i} \exp(-\alpha(1 + x)/C_\varepsilon)], \quad for \;\; i = 1, ..., m + 1,$$

and for $A(x) < 0$ on Ω_3 the solution satisfies

$$\left|u^{(i)}(x)\right| \leq M[1 + C_\varepsilon^{-i} \exp\left(-\alpha(1-x)/C_\varepsilon\right)], \quad for \;\; i = 1, ..., m+1.$$

Proof. For the proof of this theorem the reader can refer [25]. □

If $\lambda = b(0)/a'(0) < 0$, then the solution $u(x)$ is smooth near the turning point $x = 0$ [4]. Using this the following Theorem gives the bound for the derivatives of the solution in the interval Ω_2 which contains the turning point $x = 0$.

Theorem 2.4. *Let $\lambda < 0$ and assume that A, b, $f \in C^m(\bar{\Omega})$, for $m > 0$. If $u(x)$ is the solution of (8) and satisfies all conditions from (3) to (5), then there exists a positive constant M, such that*

$$\left|u^{(i)}(x)\right| \leq M, \quad fori = 1, ..., m \;\; and \;\; \forall x \in \Omega_2.$$

Proof. For the proof one can see [4, 25]. □

Finally, to prove uniform convergence, we consider the following theorem which provides bounds for the smooth and singular components of the exact solution $u(x)$ of problem (8).

Theorem 2.5. *Let A, b and $f \in C^4(\bar{\Omega})$ and assume that the solution $u(x)$ of the problem (8) is decomposed in to smooth and singular components as*

$$u(x) := v(x) + w(x), \quad \forall x \in \bar{\Omega}.$$

Then, for all i, $0 \leq i \leq 4$ the smooth component satisfies

$$\left|v^{(i)}(x)\right| \leq M[1 + C_\varepsilon^{-(i-3)} e(x, \alpha)], \qquad \forall x \in \bar{\Omega},$$

and the singular component satisfies

$$\left|w^{(i)}(x)\right| \leq M C_\varepsilon^{-i} e(x, \alpha), \qquad \forall x \in \bar{\Omega},$$

where $e(x, \alpha) = (\exp\left(-\alpha(1+x)/C_\varepsilon\right) + \exp\left(-\alpha(1-x)/C_\varepsilon\right))$.

Proof. For the proof of this theorem the reader can refer [12, 25]. □

3. Discrete Problem

In this section, we describe the piecewise-uniform Shishkin mesh for the discretization of the domain and propose the hybrid difference scheme used to discretize the TPP (8).

3.1. Piecewise-uniform Shishkin mesh. .

Consider the domain $\bar{\Omega} = [-1, 1]$ and let $N = 8k$ and $k > 0$ is a positive integer. Since the TPP (8) has two boundary layers at $x = \pm 1$, we construct a piecewise-uniform Shishkin mesh by subdividing the domain $\bar{\Omega}$ into three subintervals $\Omega_L = [-1, -1 + \tau]$, $\Omega_C = [-1 + \tau, 1 - \tau]$ and $\Omega_R = [1 - \tau, 1]$ such that $\bar{\Omega} = \Omega_L \cup \Omega_C \cup \Omega_R$, where the transition parameter τ satisfies $0 < \tau \leq 1/2$ and defined by

$$\tau = \min\{\frac{1}{2}, \frac{2C_\varepsilon}{\alpha} \ln N\}, \qquad \text{where} \qquad C_\varepsilon = \varepsilon - \frac{\delta^2}{2} \|b\|. \tag{9}$$

Then the discrete mesh $\bar{\Omega}^N$ is obtained by putting a uniform mesh with $N/4$ mesh elements in both Ω_L and Ω_R, and a uniform mesh with $N/2$ mesh elements in Ω_C. Further, let $h_i = x_i - x_{i-1}$, for $i = 1, ..., N$, denotes the variable step size. Since the mesh is piecewise-uniform, the mesh elements are given by

$$x_i = \begin{cases} (-1+\tau) + (i - N/4)h & \text{for } i = 1, ..., N/4, \\ (i - N/2)H & \text{for } i = N/4, ..., 3N/4, \\ (1-\tau) + (i - 3N/4)h & \text{for } i = 3N/4, ..., N, \end{cases} \tag{10}$$

where $h = 4\tau/N$ and $H = 4(1-\tau)/N$ are the mesh lengths. If $C_\varepsilon > MN^{-1}$, then the mesh becomes equally spaced and then $\tau = 1/2$ resulting

$$h_i = H = h = 2N^{-1} \qquad \text{and} \qquad C_\varepsilon^{-1} < 4 \ln N/\alpha. \tag{11}$$

On the other hand, for $C_\varepsilon \leq MN^{-1}$ the mesh is piecewise-uniform and $\tau = \frac{2C_\varepsilon}{\alpha} \ln N$. Here we have $2N^{-1} \leq H \leq 4N^{-1}$ and

$$\frac{h}{C_\varepsilon} = \frac{8}{\alpha} N^{-1} \ln N \qquad \text{and} \qquad e^{-\alpha(1+x_{N/4})/C_\varepsilon} = e^{-\alpha(1-x_{3N/4})/C_\varepsilon} = N^{-2}. \tag{12}$$

3.2. Hybrid difference scheme. .

Before describing the scheme, for a given mesh function $y(x_i) = y_i$, we define the forward, backward , central difference operators D^+, D^- and D^0 by

$$D^+ y_i = \frac{y_{i+1} - y_i}{h_{i+1}}, \qquad D^- y_i = \frac{y_i - y_{i-1}}{h_i} \qquad \text{and} \qquad D^0 y_i = \frac{y_{i+1} - y_{i-1}}{\bar{h}_i},$$

respectively, and the second-order central difference operator δ^2 by

$$\delta^2 y_i = \frac{2(D^+ y_i - D^- y_i)}{\bar{h}_i},$$

where $\bar{h}_i = h_i + h_{i+1}$, for $i = 1, ..., N - 1$.

Further, we define the midpoint upwind schemes $L^N_{M\pm}$ and the classical central difference scheme L^N_C used to approximate the continuous operator L as:

$$\begin{aligned} L^N_{M+} y_i &= -C_{\varepsilon,i+1/2} \ \delta^2 y_i - A_{i+1/2} D^+ y_i + b_{i+1/2} y_{i+1/2} = f_{i+1/2}, \\ L^N_{M-} y_i &= -C_{\varepsilon,i-1/2} \ \delta^2 y_i - A_{i-1/2} D^- y_i + b_{i-1/2} y_{i-1/2} = f_{i-1/2}, \\ L^N_C y_i &= -C_{\varepsilon,i} \ \delta^2 y_i - A_i D^0 y_i + b_i y_i = f_i, \end{aligned} \tag{13}$$

where $A_{i\pm1/2} = (A_i + A_{i\pm1})/2$ and similarly for $C_{\varepsilon,i\pm1/2}$, $b_{i\pm1/2}$ and $f_{i\pm1/2}$.
Now, we propose the hybrid difference scheme to solve (8), which consists of the classical central difference scheme when $C_\varepsilon > MN^{-1}$, and a proper combination of the midpoint upwind schemes in the outer region Ω_C and the central difference scheme in the layer regions Ω_L and Ω_R, when $C_\varepsilon \leq MN^{-1}$. Hence, the proposed hybrid scheme on $\bar{\Omega}^N$ takes the following form:

$$\begin{cases} L_H^N U_i = \bar{f}_i, & \text{for } i = 1, ..., N-1, \\ U_0 = u_0, & U_N = u_N, \end{cases} \tag{14}$$

where

$$L_H^N U_i = \begin{cases} L_C^N U_i, & i = 1, ..., N-1, \text{ and } C_\varepsilon > MN^{-1}, \\ L_{M+}^N U_i, & i = N/4, ..., N/2, \text{ and } C_\varepsilon \leq MN^{-1}, \\ L_{M-}^N U_i, & i = N/2+1, ..., 3N/4, \text{ and } C_\varepsilon \leq MN^{-1}, \\ L_C^N U_i, & i = 1, .., N/4-1, i = 3N/4+1, ..., N-1, \text{ and } C_\varepsilon \leq MN^{-1}, \end{cases}$$

and the right hand side vector $\bar{f}_i$ as

$$\bar{f}_i = \begin{cases} f_i, & i = 1, ..., N-1, \text{ and } C_\varepsilon > MN^{-1}, \\ f_{i+1/2}, & i = N/4, ..., N/2, \text{ and } C_\varepsilon \leq MN^{-1}, \\ f_{i-1/2}, & i = N/2+1, ..., 3N/4, \text{ and } C_\varepsilon \leq MN^{-1}, \\ f_i, & i = 1, ..., N/4-1, i = 3N/4+1, ..., N-1, \text{ and } C_\varepsilon \leq MN^{-1}. \end{cases}$$

After rearranging the terms in (14), we obtain the following system of equations:

$$L_H^N U_i = r_i^- U_{i-1} + r_i^c U_i + r_i^+ U_{i+1} = \bar{f}_i, \tag{15}$$

where the coefficients are given by

$$\begin{cases} r_i^- = -\dfrac{2C_{\varepsilon,i}}{h_i \bar{h}_i} + \dfrac{A_i}{\bar{h}_i}, & r_i^+ = -\dfrac{2C_{\varepsilon,i}}{h_{i+1}\bar{h}_i} - \dfrac{A_i}{\bar{h}_i}, \\ r_i^c = \dfrac{2C_{\varepsilon,i}}{h_i h_{i+1}} + b_i, & \text{if } L_H^N \equiv L_C^N. \end{cases} \tag{16}$$

$$\begin{cases} r_i^- = -\dfrac{2C_{\varepsilon,i+1/2}}{h_i \bar{h}_i}, & r_i^+ = -\dfrac{2C_{\varepsilon,i+1/2}}{h_{i+1}\bar{h}_i} - \dfrac{A_{i+1/2}}{h_{i+1}} + \dfrac{b_{i+1/2}}{2} \\ r_i^c = \dfrac{2C_{\varepsilon,i+1/2}}{h_i h_{i+1}} + \dfrac{A_{i+1/2}}{h_{i+1}} + \dfrac{b_{i+1/2}}{2}, & \text{if } L_H^N \equiv L_{M+}^N. \end{cases} \tag{17}$$

$$\begin{cases} r_i^- = -\dfrac{2C_{\varepsilon,i-1/2}}{h_i \bar{h}_i} + \dfrac{A_{i-1/2}}{h_i} + \dfrac{b_{i-1/2}}{2}, & r_i^+ = -\dfrac{2C_{\varepsilon,i-1/2}}{h_{i+1}\bar{h}_i}, \\ r_i^c = \dfrac{2C_{\varepsilon,i-1/2}}{h_i h_{i+1}} - \dfrac{A_{i-1/2}}{h_i} + \dfrac{b_{i-1/2}}{2}, & \text{if } L_H^N \equiv L_{M-}^N. \end{cases} \tag{18}$$

In general, central difference schemes can be unstable on coarser meshes, but we use this scheme only on the fine part of the Shishikn mesh and thus attain stability under the mild assumption on the minimum number of mesh points N, considered in the following lemma:

Lemma 3.1. *Assume that*

$$\frac{N}{\ln N} \geq 4\frac{\|A\|}{\alpha} \quad \textit{and} \quad \frac{\alpha N}{2} \geq \|b\|. \tag{19}$$

Then the discrete operator defined by (14) satisfies a discrete maximum principle, i.e., if U_i and B_i are mesh functions that satisfies $U_0 \leq B_0$, $U_N \leq B_N$ and $L_H^N U_i \leq L_H^N B_i$, for $i = 1, ..., N-1$, then $U_i \leq B_i$, for $i = 0, ..., N$. Hence (14) has a unique solution.

Proof. To proof the results it is enough to show that the operator given by (15) is an M-matrix. For this, we need to show that (15) satisfies the conditions

$$r_i^- < 0, \quad r_i^+ < 0, \quad \textit{and} \quad r_i^- + r_i^c + r_i^+ > 0, \tag{20}$$

for all the operators defined in (16)–(18). Here , we separately consider the following two cases based on the relation between C_ε and N.
Case 1: when $C_\varepsilon > MN^{-1}$, the mesh is uniform and we used central difference scheme on the entire domain. Thus, for $M \geq \|A\|$ and using (11) into (16) we get

$$r_i^- = -\frac{C_{\varepsilon,i}}{h^2} + \frac{A_i}{2h} = \frac{1}{h^2}(-C_{\varepsilon,i} + 1/2A_i h) \leq \frac{1}{h^2}(-C_\varepsilon + \|A\| N^{-1}) < 0,$$
$$r_i^+ = -\frac{C_{\varepsilon,i}}{h^2} - \frac{A_i}{2h} = \frac{1}{h^2}(-C_{\varepsilon,i} - 1/2A_i h) \leq \frac{1}{h^2}(-C_\varepsilon + \|A\| N^{-1}) < 0,$$

and some simple calculations gives $r_i^- + r_i^c + r_i^+ = b_i > 0$, for all $1 \leq i \leq N-1$.

Case 2: when $C_\varepsilon \leq MN^{-1}$, different operators are used in the layer regions and outer region.
In the layer regions, it is apparent that $r_i^+ < 0$ and $r_i^- < 0$, for $1 \leq i < N/4$ and $3N/4 < i \leq N-1$ respectively. Further, using the first assumption of (19) and (12) in to (16) we get

$$r_i^- = -\frac{C_{\varepsilon,i}}{h^2} + \frac{A_i}{2h} \leq \frac{1}{h}(-\frac{C_\varepsilon}{h} + \frac{A_i}{2}) \leq \frac{1}{h}(-\frac{\alpha N}{8\ln N} + \frac{\|A\|}{2}) < 0,$$
$$r_i^+ = -\frac{C_{\varepsilon,i}}{h^2} - \frac{A_i}{2h} \leq \frac{1}{h}(-\frac{C_\varepsilon}{h} - \frac{A_i}{2}) \leq \frac{1}{h}(-\frac{\alpha N}{8\ln N} + \frac{\|A\|}{2}) < 0,$$

for $1 \leq i \leq N/4 - 1$ and $3N/4 + 1 \leq i \leq N-1$, respectively .

In both the layer regions we simply obtain $r_i^- + r_i^c + r_i^+ = b_i > 0$.
Finally, in the outer regions it is straightforward that $r_i^- < 0$, for $N/4 \leq i \leq N/2$ and $r_i^+ < 0$, for $N/2 + 1 \leq i \leq 3N/4$. In addition, using $H \leq 4N^{-1}$ and the second assumption of (19) in (17) and (18) gives us

$$r_i^+ = -\frac{2C_{\varepsilon,i+1/2}}{h_{i+1}\bar{h}_i} - \frac{A_{i+1/2}}{h_{i+1}} + \frac{b_{i+1/2}}{2} \leq -\frac{2C_{\varepsilon,i+1/2}}{h_{i+1}\bar{h}_i} - \frac{\alpha N}{4} + \frac{\|b\|}{2} < 0,$$
$$r_i^- = -\frac{2C_{\varepsilon,i-1/2}}{h_i\bar{h}_i} + \frac{A_{i-1/2}}{h_i} + \frac{b_{i-1/2}}{2} \leq -\frac{2C_{\varepsilon,i-1/2}}{h_i\bar{h}_i} - \frac{\alpha N}{4} + \frac{\|b\|}{2} < 0,$$

for $N/4 \leq i \leq N/2$ and $N/2 + 1 \leq i \leq 3N/4$, respectively .

Moreover, for all $N/4 \leq i \leq 3N/4$, it is easy to verify that $r_i^- + r_i^c + r_i^+ > 0$. For all the cases it is verified that the operator (15) satisfies the conditions in (20). Hence, the matrix is an M-matrix. Therefore, the solution of (14) exists and the maximum principle easily follows. For more details the reader can refer [13, 26]. □

Whenever, the conditions of the maximum principle are satisfied, we can take $\{B_i\}$ as a barrier function for $\{U_i\}$.

4. Convergence Analysis of The Proposed Method

In this section, we establish the ε-uniform error estimate of the hybrid scheme (14). For this, we consider the two cases $C_\varepsilon > MN^{-1}$ and $C_\varepsilon \leq MN^{-1}$ separately. For both the cases, analogous to the continuous solution u, we decompose the discrete solution U into a smooth component V and a singular component W, such that $U := V + W$. Where V is the solution of the non-homogeneous problem given by

$$L_H^N V_i = \bar{f}, \quad \text{for } i = 1, ..., N-1, \qquad V_0 = v(-1), \;\; V_N = v(1), \tag{21}$$

and W the solution of the homogeneous problem

$$L_H^N W_i = 0, \quad \text{for } i = 1, ..., N-1, \quad W_0 = w(-1), \;\; W_N = w(1). \tag{22}$$

Then the error at each mesh point is

$$U_i - u(x_i) = (V_i - v(x_i)) + (W_i - w(x_i)),$$

which implies

$$|U_i - u(x_i)| \leq |V_i - v(x_i)| + |W_i - w(x_i)|, \tag{23}$$

and so the error in the smooth and singular components of the solution can be estimated separately.

First, to bound the errors we need to consider the truncation error of associated with the discrete operators in (14). For any smooth function $y(x)$, the truncation errors $L_{M\pm}^N$ applied to y at $x_{i\pm 1/2}$ and L_C^N applied to y at y_i , becomes $T_{1\pm} := L_{M\pm}^N(y_i) - (Ly)(x_{i\pm 1/2})$ and $T_2 := L_C^N(y_i) - (Ly)(x_i)$ respectively, where $y_i := y(x_i)$. Thus, the bounds are given in the following Lemma:

Lemma 4.1. *Let $y(x)$ be a smooth function defined on $[-1, 1]$. Then there exists a positive constant M such that*

$$|T_{1+}| \leq MC_\varepsilon \int_{x_{i-1}}^{x_{i+1}} |y'''(t)| dt + Mh_{i+1} \int_{x_i}^{x_{i+1}} |y'''(t)|\, dt,$$

$$|T_{1-}| \leq MC_\varepsilon \int_{x_{i-1}}^{x_{i+1}} |y'''(t)| dt + Mh_i \int_{x_{i-1}}^{x_i} |y'''(t)|\, dt$$

and

$$|T_2| \leq Mh \int_{x_{i-1}}^{x_{i+1}} [C_\varepsilon \left| y^{(4)}(t) \right| + |y'''(t)|] dt, \qquad \textit{for} \quad h_i = h_{i+1} = h.$$

Proof. By repeated use of the fundamental theorem of calculus, one can obtain the proof as in Lemma 3.3 of [13] . □

To bound the truncation error of the scheme the comparison principle of Lemma 3.1 alone is not enough, so we should also consider the following Lemma which enable us to bound the error.

Lemma 4.2. *Assume that the conditions of (19) holds and let $Z_i = 2 + x_i$ for $0 \le i \le N$ be the mesh function for (14). Then there exists a positive constant M such that*

$$L_H^N Z_i \ge M, \quad \text{for } 1 \le i \le N-1$$

Proof. The proof is an easy computation. □

Sometimes the truncation error contains a term of magnitude greater than the desired order of convergence, when this happens we shall combine Lemma 3.1 with the following results. Whenever $C_\varepsilon \le MN^{-1}$, we define the auxiliary discrete function on the mesh elements (10) as

$$S_i := \begin{cases} 2\left(1 + \dfrac{\eta h}{C_\varepsilon}\right)^{-N/4}, & \text{for } i = 0, ..., \frac{3}{4}N, \\ \left(1 + \dfrac{\eta h}{C_\varepsilon}\right)^{-N/4} + \left(1 + \dfrac{\eta h}{C_\varepsilon}\right)^{-(N-i)}, & \text{for } i = \frac{3}{4}N, ..., N. \end{cases} \tag{24}$$

where η is a positive constant.

Lemma 4.3. *For any $\eta > 0$ the discrete function $\{S_i\}$ from (24), there exists a positive constant M such that*

$$e^{-\eta(1-x_i)/C_\varepsilon} \le S_i \le \begin{cases} MN^{-2\eta/\alpha}, & \text{for } i \le \frac{3}{4}N, \\ M, & \text{for } i > \frac{3}{4}N. \end{cases} \tag{25}$$

and

$$L_H^N S_i = \begin{cases} b_i S_i, & i < N/4, \\ b_{i+1/2} S_i, & N/4 \le i \le N/2, \\ b_{i-1/2} S_i, & N/2 < i < 3N/4, \\ \left(b_{i-1/2} - \dfrac{\eta}{h+H}\right) S_i, & i = 3N/4, \\ \left(1 + \dfrac{\eta h}{C_\varepsilon}\right)^{-(N-(i-1))} \left[\dfrac{\eta}{C_\varepsilon}(-A_i - \eta) - A_i \dfrac{\eta^2 h}{2C_\varepsilon^2}\right] + b_i S_i, & i > 3N/4. \end{cases} \tag{26}$$

Proof. The lower bound for S_i follows from the inequality $e^{-t} \le (1+t)^{-1}$ which holds true for $t \ge 0$. The upper bound for S_i is obvious for $i > 3N/4$. For

$i \leq 3N/4$, it follows from the inequality $(1+t)^{-1} \leq e^{-t+t^2}$, which holds true for $t \geq 0$. Setting $t := \eta h/C_\varepsilon$ and using (12), we get

$$\begin{aligned} S_i &= 2\left(1+\frac{\eta h}{C_\varepsilon}\right)^{-N/4} \leq 2e^{\left(-\frac{\eta h}{C_\varepsilon}+\frac{\eta h}{C_\varepsilon}^2\right)N/4} = 2e^{\eta\frac{h}{C_\varepsilon}\left(-1+\eta\frac{h}{C_\varepsilon}\right)N/4} \\ &= 2e^{\eta\frac{8N^{-1}\ln N}{\alpha}\left(-1+\eta\frac{8N^{-1}\ln N}{\alpha}\right)\frac{N}{4}} = 2e^{\frac{2\eta}{\alpha}\ln N\left(-1+\frac{8\eta}{\alpha}N^{-1}\ln N\right)} \\ &= 2e^{\ln N\left(-\frac{2\eta}{\alpha}+\frac{16\eta^2}{\alpha^2}N^{-1}\ln N\right)} = 2N^{-\frac{2\eta}{\alpha}}.N^{16\frac{\eta^2}{\alpha^2}N^{-1}\ln N} \leq MN^{-\frac{2\eta}{\alpha}}, \end{aligned}$$

because, the sequence $N^{16\frac{\eta^2}{\alpha^2}N^{-1}\ln N}$ is bounded for $N \geq 8$. This proves the upper bound for S_i. The property (26) can be obtained by direct calculation. □

Lemma 4.4. *Let $\bar{S}_i := \frac{2\alpha}{\beta H}S_{3N/4}+S_i$ be a discrete function, where $\{S_i\}$ is from (24) with $\eta = \alpha$, then there exists a positive constant M such that*

$$0 < \bar{S}_i \leq \begin{cases} MN^{-1}, & \text{for } i \leq \frac{3}{4}N, \\ M, & \text{for } i > \frac{3}{4}N. \end{cases}$$

and

$$L_H^N\bar{S}_i \geq \begin{cases} MNe^{-\alpha(1-x_i)/C_\varepsilon}, & \text{for } i \leq \frac{3}{4}N, \\ MC_\varepsilon^{-1}e^{-\alpha(1-x_i)/C_\varepsilon}, & \text{for } i > \frac{3}{4}N. \end{cases}$$

Proof. The proof is similarly to Lemma 3.3 of [2]. □

Lemma 4.5. *Let $\check{S}_i := \frac{2\alpha}{\beta(h+H)}S_{3N/4}+S_i$ be a discrete function, where $\{S_i\}$ is from (24) with $\eta = 2\alpha$, then there exists a positive constant M such that*

$$0 < \check{S}_i \leq \begin{cases} MN^{-3}, & \text{for } i \leq \frac{3}{4}N, \\ M, & \text{for } i > \frac{3}{4}N, \end{cases}$$

$$\check{S}_i \geq e^{-\alpha(1-x_i)/C_\varepsilon} \quad \text{for} \quad i = 0, N, \quad \text{and} \quad L_H^N\check{S}_i \geq 0.$$

Proof. For the proof one can follow similarly like Lemma 3.4 of [2]. □

Remark 4.1. Because of the symmetry of the mesh and the adaptive nature of the hybrid scheme, it is easy to derive a similar result like Lemma 4.4 and 4.5 using the mesh function $\{S_{N-i}\}$ related to the layer function $e^{-\alpha(1+x)/C_\varepsilon}$.

Now we have assembled the tools for the proof of the ε-uniform convergence.

Theorem 4.6. *Assume that the conditions of (19) holds true. Then the hybrid scheme (14) satisfies the following error estimates:*
Case 1: *for $C_\varepsilon > MN^{-1}$, we have*

$$|U_i - u(x_i)| \leq MN^{-2}\ln^3 N, \qquad \text{for } i = 0, ..., N. \tag{27}$$

Case 2: *for $C_\varepsilon \leq MN^{-1}$, we have*

$$|U_i - u(x_i)| \leq \begin{cases} MN^{-2}, & \text{for } N/4 \leq i \leq 3N/4, \\ MN^{-2}\ln^2 N, & \text{for } 0 \leq i < N/4 \text{ and } 3N/4 < i \leq N. \end{cases} \tag{28}$$

where U_i the solution of the discrete problem (14) and $u(x_i)$ is the solution of the continuous problem (8) at the mesh points in $\bar{\Omega}^N$.

Proof. Here we estimate the error bounds separately in cases.
Case 1: when $C_\varepsilon > MN^{-1}$, we employed the central difference scheme on the entire domain. Further, we bound the error separately in V and W. First let us compute the nodal error for the smooth part V_i. For this using Lemma 4.1 and the bound of v from Theorem 2.5 , the truncation error is bounded by

$$\begin{aligned}\left|L_H^N(V_i - v(x_i))\right| &\leq Mh\int_{x_{i-1}}^{x_{i+1}} [C_\varepsilon \left|v^{(4)}(t)\right| + |v'''(t)|]dt \\ &\leq Mh^2(C_\varepsilon+1) + Mh\int_{x_{i-1}}^{x_{i+1}} (e^{-\alpha(1+t)/C_\varepsilon} + e^{-\alpha(1-t)/C_\varepsilon})dt \\ &= Mh^2(C_\varepsilon+1) + MhC_\varepsilon(e^{-\alpha(1+x_i)/C_\varepsilon} + e^{-\alpha(1-x_i)/C_\varepsilon})\sinh(\alpha h/C_\varepsilon) \\ &\leq Mh^2(C_\varepsilon+1) + Mh^2(e^{-\alpha(1+x_i)/C_\varepsilon} + e^{-\alpha(1-x_i)/C_\varepsilon}) \\ &\leq Mh^2(C_\varepsilon+1) + Mh^2 \leq Mh^2,\end{aligned}$$

since $\sinh t \leq Mt$ for $0 \leq t \leq 1$. Using $h = 2N^{-1}$ on the above inequality, we obtain the following estimate

$$\left|L_H^N(V_i - v(x_i))\right| \leq MN^{-2}, \qquad \text{for } i = 1, ..., N-1. \tag{29}$$

Now, let's take $B_i := MN^{-2}(2 + x_i)$ as a barrier function for $|V_i - v(x_i)|$, then from (21) it is easy to see that $|V_0 - v(x_0)| = 0 \leq B_0$, $|V_N - v(x_N)| = 0 \leq B_N$ and (29) together with Lemma 4.2 we observe that $\left|L_H^N(V_i - v(x_i))\right| \leq L_H^N B_i$, for $i = 1, ..., N-1$. Thus, invoking Lemma 3.1 we get

$$|V_i - v(x_i)| \leq MN^{-2}, \qquad \text{for } i = 0, ..., N. \tag{30}$$

Next, we analyze the error bounds for the singular component W_i. The local truncation error is bounded in standard way as we done above. More precisely,

$$\begin{aligned}\left|L_H^N(W_i - w(x_i))\right| &= \left|L_C^N(W_i - w(x_i))\right| = \left|(L - L_C^N)w(x_i)\right| \\ &\leq Mh\int_{x_{i-1}}^{x_{i+1}} [C_\varepsilon \left|w^{(4)}(t)\right| + |w'''(t)|]dt\end{aligned}$$

Application of Theorem 2.5 and using (11) on the above inequality gives

$$\left|L_H^N(W_i - w(x_i))\right| \leq Mh^2 C_\varepsilon^{-3} \leq MN^{-2}\ln^3 N, \qquad \text{for } i = 1, ..., N-1.$$

Now, arguing similarly like the smooth part we obtain

$$|W_i - w(x_i)| \leq MN^{-2}\ln^3 N, \qquad \text{for } i = 0, ..., N. \tag{31}$$

Using (30) and (31) in to (23) gives the required result of the first case (27).

Case 2: for $C_\varepsilon \leq MN^{-1}$, the mesh becomes piecewise-uniform and we employed a combinations of mid point upwind and central difference schemes. Like that of the previous case we bound the error separately in V and W. First, let us compute the error for the smooth part V_i. Similarly like the smooth part of Case 1, the truncation error becomes

$$\left|L_H^N((V_i - v(x_i))\right| = \begin{cases} \left|(Lv)x_{i+1/2} - L_{M+}^N v(x_i)\right|, & \text{for } i = N/4, ..., N/2, \\ \left|(Lv)x_{i-1/2} - L_{M-}^N v(x_i)\right|, & \text{for } i = N/2+1, ..., 3N/4, \\ \left|(L - L_C^N)v(x_i)\right|, & \text{Otherwise,} \end{cases}$$

then using Lemma 4.1 and the bound for $v(x)$ from Theorem 2.5, we get

$$\left|L_H^N((V_i - v(x_i))\right| \leq \begin{cases} M(C_\varepsilon + h_{i+1})(h_i + h_{i+1}), & \text{for } i = N/4, ..., N/2, \\ M(C_\varepsilon + h_i)(h_i + h_{i+1}), & \text{for } i = N/2+1, ..., 3N/4, \\ Mh^2, & \text{Otherwise.} \end{cases}$$

Since, $C_\varepsilon \leq MN^{-1}$ and $h_i \leq 4N^{-1}$, then using these in the above inequality gives us

$$\left|L_H^N(V_i - v(x_i))\right| \leq MN^{-2}, \qquad \text{for } i = 1, ..., N-1. \tag{32}$$

Now, arguing similarly like the smooth part of the previous case we obtain

$$|V_i - v(x_i)| \leq MN^{-2}, \qquad \text{for } i = 0, ..., N. \tag{33}$$

Next, we analyze the error bounds for the singular component W_i. A different argument is used to bound $|W - w|$ in the outer and layer regions. In the outer region $\bar{\Omega}_C$, both W and w are small, and by the triangle inequality we have

$$|(W - w)(x_i)| \leq |W(x_i)| + |w(x_i)|, \tag{34}$$

so, it suffices to bound $W(x_i)$ and $w(x_i)$ separately. Theorem 2.5 for $i = N/4, ..., 3N/4$ gives

$$|w(x_i)| \leq M(e^{-\alpha(1+x_i)/C_\varepsilon} + e^{-\alpha(1-x_i)/C_\varepsilon}) \leq M(e^{-\alpha(1+x_{N/4})/C_\varepsilon} + e^{-\alpha(1-x_{3N/4})/C_\varepsilon}).$$

Then using (12) in the above inequality we get

$$|w(x_i)| \leq MN^{-2}, \qquad \text{for} \quad i = N/4, ..., 3N/4. \tag{35}$$

To bound $W(x_i)$, we set $B_i := M_1(N^{-1}\bar{S}_i + N^{-1}\bar{S}_{N-i} + \check{S}_i + \check{S}_{N-i})$ for $i = 0, ..., N$, where $\{\bar{S}_i\}$ and $\{\check{S}_i\}$ are from Lemma 4.4 and 4.5, respectively. Now for sufficiently large M_1, using Theorem 2.5 in (22) and Lemma 4.5 we get

$$|W_0| = |w(-1)| \leq M(e^{-\alpha(1+x_0)/C_\varepsilon} + e^{-\alpha(1-x_0)/C_\varepsilon}) \leq M_1(\check{S}_0 + \check{S}_N) \leq B_0, \tag{36}$$

and

$$|W_N| = |w(1)| \leq M(e^{-\alpha(1+x_N)/C_\varepsilon} + e^{-\alpha(1-x_N)/C_\varepsilon}) \leq M_1(\check{S}_N + \check{S}_0) \leq B_N. \tag{37}$$

Further, for $i = 1, ..., N-1$ the property of the discrete operator from Lemma 4.4 and 4.5 implies

$$L_H^N B_i \geq \begin{cases} M_1(MC_\varepsilon^{-1}N^{-1}e^{-\alpha(1+x_i)/C_\varepsilon} + e^{-\alpha(1-x_i)/C_\varepsilon}), & \text{for } i = 1, ..., N/4-1, \\ M_1(e^{-\alpha(1+x_i)/C_\varepsilon} + e^{-\alpha(1-x_i)/C_\varepsilon}), & \text{for } i = N/4, ..., 3N/4, \\ M_1(e^{-\alpha(1+x_i)/C_\varepsilon} + MC_\varepsilon^{-1}N^{-1}e^{-\alpha(1-x_i)/C_\varepsilon}), & \text{for } i = 3N/4+1, ..., N-1, \end{cases}$$

since $C_\varepsilon \leq MN^{-1}$ implies $MC_\varepsilon^{-1}N^{-1} \geq 1$, using this in the above inequality we get

$$L_H^N B_i \geq M_1(e^{-\alpha(1+x_i)/C_\varepsilon} + e^{-\alpha(1-x_i)/C_\varepsilon}) \geq 0 = L_H^N |W_i|, \quad 1 \leq i \leq N-1. \tag{38}$$

From (36)–(38) we observe that B_i is a barrier function for W_i for M_1 sufficiently large. Therefore, by the discrete maximum principle of Lemma 3.1 we get

$$|W_i| \leq B_i, \qquad \text{for } i = 0, ..., N. \tag{39}$$

In particular in the coarser region, (39) and Lemma 4.4 and 4.5 together imply that

$$|W_i| \leq B_i \leq MN^{-2}, \qquad \text{for } i = N/4, ..., 3N/4. \tag{40}$$

Therefore, combining (34), (35) and (40) we get

$$|W_i - w(x_i)| \leq MN^{-2}, \qquad \text{for } i = N/4, ..., 3N/4. \tag{41}$$

It remains to prove the bound for the singular component in the layer regions $\bar{\Omega}_L$ and $\bar{\Omega}_R$. First we estimate the bound in $\bar{\Omega}_R$. Since we employ central difference scheme in $\bar{\Omega}_R$, so as we did for the smooth component we use the truncation error to bound the error. Thus, from Lemma 4.1 and Theorem 2.5 we get

$$\begin{aligned}
\left|L_H^N(W_i - w(x_i))\right| &\leq Mh\int_{x_{i-1}}^{x_{i+1}} [C_\varepsilon \left|w^4(t)\right| + |w'''(t)|]dt \\
&\leq MhC_\varepsilon^{-3}\int_{x_{i-1}}^{x_{i+1}} e^{-\alpha(1-t)/C_\varepsilon}dt \\
&= MhC_\varepsilon^{-2}(e^{-\alpha(1-x_{i+1})/C_\varepsilon} - e^{-\alpha(1-x_{i-1})/C_\varepsilon}) \\
&= MhC_\varepsilon^{-2}e^{-\alpha(1-x_i)/C_\varepsilon}\sinh(\alpha h/C_\varepsilon).
\end{aligned}$$

Clearly, the first assumption of (19) implies $\alpha h/C_\varepsilon \leq 1$ and since $\sinh t \leq Mt$ for $0 \leq t \leq 1$, so $\sinh(\alpha h/C_\varepsilon) \leq M\alpha h/C_\varepsilon$. Thus, for $i = 3N/4+1, ..., N-1$ the above inequality is reduced to

$$\left|L_H^N(W_i - w(x_i))\right| \leq M\left(\frac{h}{C_\varepsilon}\right)^2 C_\varepsilon^{-1}e^{-\alpha(1-x_i)/C_\varepsilon}. \tag{42}$$

Further, taking $i = 3N/4$ in (42), we get $\left|W_{3N/4} - w(x_{3N/4})\right| \leq MN^{-2}$, and for $i = N$ the boundary condition in (22) gives $|W_N - w(x_N)| = 0$. Now let

$$B_i := M_2\left(N^{-2} + \left(\frac{h}{C_\varepsilon}\right)^2 \bar{S}_i\right), \qquad \text{for } i = 3N/4, ..., N,$$

be the mesh function, where $\{\bar{S}_i\}$ is from Lemma 4.4. If M_2 is chosen large enough, our estimates shows that B_i is a barrier function for $|W_i - w(x_i)|$. So by using the discrete maximum principle of Lemma 3.1 and Lemma 4.4, together with (12) we get

$$|W_i - w(x_i)| \leq MN^{-2}\ln^2 N, \qquad \text{for } i = 3N/4, ..., N. \tag{43}$$

Similarly the proof follows for the left boundary layer region, $\bar{\Omega}_L$ i.e.,

$$|W_i - w(x_i)| \leq MN^{-2}\ln^2 N, \qquad \text{for } i = 0, ..., N/4. \tag{44}$$

Finally, properly using (33), (41), (43) and (44) in to (23) gives the required bound of the second case (28), which completes the proof. □

5. Test Problems and Numerical Results

To demonstrate the applicability of the proposed method we have implemented it on two problems of the form (1)–(2). Since the exact solution for the problems are not available, the point-wise errors ($\hat{e}_i^N$) and maximum absolute errors ($\hat{E}^N$) are calculated by using the double mesh principle given by

$$\hat{e}_i^N = |U_N(x_i) - U_{2N}(x_i)|\,, \ \text{ and } \ \hat{E}^N = \max_i\{\hat{e}_i^N\}.$$

where U_N and U_{2N} denotes the numerical solutions obtained using N and $2N$ meshes points respectively. Further, we determine the corresponding rate of convergence by

$$\hat{R}^N = \log_2{}^{(\hat{E}^N/\hat{E}^{2N})}.$$

Example 5.1. Consider the following homogeneous SPDDE with a turning point:

$$-\varepsilon y''(x) + xy'(x) + y(x-\delta) = 0, \ \ x \in (-1, 1)$$
$$y(x) = 1, \ \ -1-\delta \leq x \leq -1, \qquad y(1) = 1.$$

Example 5.2. Consider the following non-homogeneous SPDDE with a turning point:

$$-\varepsilon y''(x) - 2(1-2x)y'(x) + 4y(x-\delta) = 4(1-4x), \ \ x \in (0, 1)$$
$$y(x) = 1, \ \ -\delta \leq x \leq 0, \qquad y(1) = 1.$$

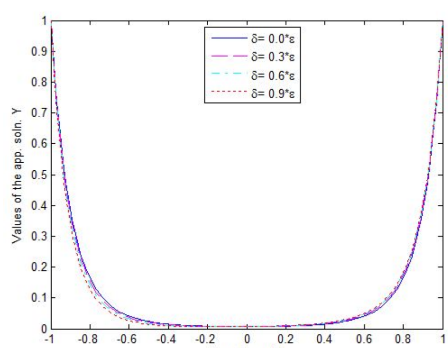

FIGURE 1. *Plot of the solutions of Example 5.1 for $\varepsilon = 0.1$ and $N = 128$.*

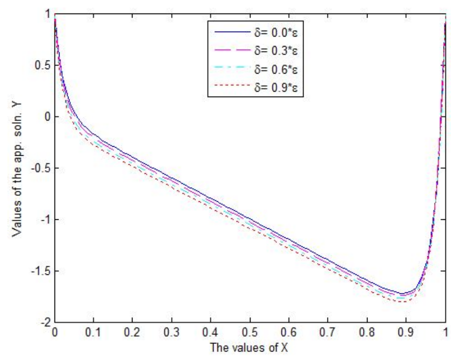

FIGURE 2. *Plot of the solutions of Example 5.2 for* $\varepsilon = 0.05$ *and* $N = 128$.

TABLE 1. Result of Example 5.1 (Maximum point wise error) for $\delta = 0.5 * \varepsilon$.

$\varepsilon_{\downarrow}/N \rightarrow$	32	64	128	256	512	1024
1.00	$1.90E-04$	$4.74E-05$	$1.19E-05$	$2.96E-06$	$7.41E-07$	$1.85E-07$
0.50	$3.59E-04$	$8.98E-05$	$2.24E-05$	$5.61E-06$	$1.40E-06$	$3.51E-07$
10^{-1}	$6.62E-03$	$1.66E-03$	$4.10E-04$	$1.03E-04$	$2.57E-05$	$6.42E-06$
10^{-2}	$1.81E-02$	$6.12E-03$	$1.99E-03$	$6.70E-04$	$2.10E-04$	$6.48E-05$
10^{-3}	$1.89E-02$	$6.38E-03$	$2.12E-03$	$6.88E-04$	$2.17E-04$	$6.61E-05$
10^{-4}	$1.90E-02$	$6.41E-03$	$2.13E-03$	$6.92E-04$	$2.18E-04$	$6.74E-05$
10^{-5}	$1.90E-02$	$6.41E-03$	$2.13E-03$	$6.92E-04$	$2.19E-04$	$6.74E-05$
10^{-6}	$1.90E-02$	$6.41E-03$	$2.13E-03$	$6.92E-04$	$2.19E-04$	$6.74E-05$
10^{-7}	$1.90E-02$	$6.41E-03$	$2.13E-03$	$6.92E-04$	$2.19E-04$	$6.74E-05$
E^N	$1.90E-02$	$6.41E-03$	$2.13E-03$	$6.92E-04$	$2.19E-04$	$6.74E-05$

TABLE 2. Result of Example 5.1 (Rate of convergence) for $\delta = 0.5 * \varepsilon$.

$\varepsilon_{\downarrow}/N \rightarrow$	32	64	128	256	512	1024
1.00	2.0000	2.0000	2.0000	2.0000	2.0000	2.0001
0.50	2.0003	2.0002	2.0000	2.0000	2.0000	2.0000
10^{-1}	2.0003	2.0121	1.9982	2.0008	2.0002	2.0000
10^{-2}	1.5677	1.6192	1.5728	1.6707	1.6989	1.7256
10^{-3}	1.5700	1.5915	1.6217	1.6628	1.7165	1.6949
10^{-4}	1.5699	1.5916	1.6212	1.6627	1.6971	1.7251
10^{-5}	1.5699	1.5916	1.6211	1.6626	1.6969	1.7249
10^{-6}	1.5699	1.5916	1.6211	1.6626	1.6969	1.7249
10^{-7}	1.5699	1.5916	1.6211	1.6626	1.6969	1.7249
R^N	1.5699	1.5916	1.6211	1.6626	1.6969	1.7249

TABLE 3. Result of Example 5.2 (Maximum point wise error) for $\delta = 0.5 * \varepsilon$.

$\varepsilon_\downarrow/N \rightarrow$	32	64	128	256	512	1024
1.00	$9.78E-04$	$2.45E-04$	$6.12E-05$	$1.53E-05$	$3.83E-06$	$5.57E-07$
0.50	$1.39E-03$	$3.48E-04$	$8.71E-05$	$2.18E-05$	$5.44E-06$	$1.36E-06$
10^{-1}	$1.16E-02$	$3.41E-03$	$8.48E-04$	$2.12E-04$	$5.29E-05$	$1.32E-05$
10^{-2}	$5.19E-02$	$1.76E-02$	$5.86E-03$	$1.89E-03$	$6.08E-04$	$1.87E-04$
10^{-3}	$5.66E-02$	$1.91E-02$	$6.33E-03$	$2.06E-03$	$6.50E-04$	$2.00E-04$
10^{-4}	$5.71E-02$	$1.92E-02$	$6.38E-03$	$2.07E-03$	$6.55E-04$	$2.02E-04$
10^{-5}	$5.71E-02$	$1.92E-02$	$6.38E-03$	$2.08E-03$	$6.56E-04$	$2.02E-04$
10^{-6}	$5.71E-02$	$1.92E-02$	$6.39E-03$	$2.08E-03$	$6.56E-04$	$2.02E-04$
10^{-7}	$5.71E-02$	$1.92E-02$	$6.39E-03$	$2.08E-03$	$6.56E-04$	$2.02E-04$
E^N	$5.71E-02$	$1.92E-02$	$6.39E-03$	$2.08E-03$	$6.56E-04$	$2.02E-04$

TABLE 4. Result of Example 5.2 (Rate of convergence) for $\delta = 0.5 * \varepsilon$.

$\varepsilon_\downarrow/N \rightarrow$	32	64	128	256	512	1024
1.00	1.9982	1.9993	1.9999	2.0000	2.0000	2.0000
0.50	1.9995	1.9999	1.9999	2.0000	2.0000	2.0000
10^{-1}	1.7644	2.0085	2.0021	2.0004	1.9999	2.0000
10^{-2}	1.5578	1.5892	1.6313	1.6387	1.6983	1.7257
10^{-3}	1.5689	1.5918	1.6212	1.6629	1.6969	1.7312
10^{-4}	1.5698	1.5916	1.6211	1.6627	1.6971	1.7251
10^{-5}	1.5699	1.5916	1.6211	1.6626	1.6969	1.7249
10^{-6}	1.5699	1.5916	1.6211	1.6626	1.6969	1.7249
10^{-7}	1.5699	1.5916	1.6211	1.6626	1.6969	1.7249
R^N	1.5699	1.5916	1.6211	1.6626	1.6969	1.7249

TABLE 5. Comparision of Maximum errors for Example 5.2 with $\delta = 0.5 * \varepsilon$.

	$N = 256$		$N = 512$	
$\varepsilon_\downarrow$	Method of [25]	Present Method	Method of [25]	Present Method
2^{-2}	$4.19E-03$	$2.07E-05$	$2.12E-03$	$5.17E-06$
2^{-4}	$2.84E-02$	$7.50E-04$	$1.47E-02$	$1.87E-04$
2^{-8}	$5.22E-012$	$2.01E-03$	$3.09E-02$	$6.35E-04$
2^{-12}	$5.27E-02$	$2.07E-03$	$3.12E-02$	$6.54E-04$
2^{-16}	$5.27E-02$	$2.08E-03$	$3.12E-02$	$6.56E-04$
2^{-20}	$5.27E-02$	$2.08E-03$	$3.12E-02$	$6.56E-04$
E^N	$5.27E-02$	$2.08E-03$	$3.12E-02$	$6.56E-04$

6. Discussion

In this article, Singularly perturbed delay differential turning point problems exhibiting twin boundary layers which contains a small delay parameter $\delta = o(\varepsilon)$

on the reaction term are considered. To tackle the delay parameter a second order Taylor's series expansion is employed. An efficient fitted finite difference scheme on an appropriate piecewise-uniform Shishkin mesh is developed for the problem. The proposed method is analyzed for stability and convergence, and it has been shown that the method is ε-uniformly convergent with an almost second order rate of convergence. Further, two numerical experiments are examined to support the theoretical results and to illustrate the effect of the small shift on the layer behavior of the solutions.
Tables 1-4 presents the computed maximum point wise error and the rate of convergence for the considered examples and the comparison of maximum error with the existing method is also given in Table 5. The results demonstrate that the method is robust i.e., converges for every ε and the maximum point wise error and the rate of convergence stabilizes as $\varepsilon \to 0$ for each appropriate N. It is also observed that the proposed method has a superior order of convergence than the existing fitted mesh method in [25] employed for the same problem. Furthermore, the numerical results clearly support the theoretical error bounds and order of convergence derived on this paper. In addition, to demonstrate the effect of the small negative shift on the behavior of the solution graphs of the considered problems are plotted in Figures 1-2 for different values of δ. It is observed that the boundary layers are maintained but layer get shifted as delay argument changes.

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Wondwosen Gebeyaw Melesse received M.Sc. from Bahir Dar University and he is a Ph.D. candidate at Bahir Dar University, Ethiopia. Since 2013 he has been teaching at Dilla University, Ethiopia. He published two paper on reputable journals. His research interests are on differential equation, numerical analysis, specially on solving Singularly Perturbed Differential Equations.

Department of Mathematics, College of Natural and Computational Sciences, Dilla University, Ethiopia.

e-mail: gwondwosen12@gmail.com

Dr. Awoke Andargie Tiruneh receive Ph.D. and Post-Doc from National Institute of Technology, Warangal,India. He is presently working in Bahir Dar University, as an Associate Professor. He is the author of more than 15 research papers published in different national and international journals. His research interests are in the areas of applied mathematics and numerical analysis including theory of perturbation. He is the member of editorial board for Ethiopian Journal of Science and Technology (EJST). He is working as a referee for several reputed journals.

Department of Mathematics, College of Sciences, Bahir Dar University, Ethiopia.
e-mail: awoke248@yahoo.com

Dr. Getachew Adamu Derese receive Ph.D. from IIT Kanpur,India. He is presently working in Bahir Dar University, as an Associate Professor. He is the author of more than 9 research papers published in different national and international journals. His research interests are in the areas of applied mathematics and numerical analysis including lubrication of bearings. He is the member of advisory board for Ethiopian Journal of Science and Technology (EJST). He is working as a referee for several reputed journals.

Department of Mathematics, College of Sciences, Bahir Dar University, Ethiopia.
e-mail: getachewsof@yahoo.com