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ON THE BOUNDS FOR THE SPECTRAL NORMS OF GEOMETRIC AND R-CIRCULANT MATRICES WITH BI-PERIODIC JACOBSTHAL NUMBERS

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Abstract. The study is about the bounds of the spectral norms of r-circulant and geometric circulant matrices with the sequences called biperiodic Jacobsthal numbers. Then we give bounds for the spectral norms of Kronecker and Hadamard products of these r-circulant matrices and geometric circulant matrices. The eigenvalues and determinant of r-circulant matrices with the bi-periodic Jacobsthal numbers are obtained.

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1. Introduction

Special integer sequences are encountered in many areas such as architucture, nature, in human body, computer programmming. The sequences have many interpretations, representations and applications in distinct areas of mathematics. One of them, Jacobsthal numbers are defined recursively by the second order linear relation such as Fibonacci numbers. In this paper we introduce the Jacobsthal representation by using two different variables a,b, construct a generalization of Jacobsthal numbers.

The circulant and r-circulant matrices were first proposed by Davis in [14]. The researchers found different properties of these matrices. It is one of the most important research subject in the field of the computation and pure mathematics. In particular, they have important position and appliation in solving coding theory, different types of partial and ordinary differential equations, numerical analysis and so on. Obviously, the matrices are determined by the parameter r and the first row elements of the matrix. When the parameter satisfies r=1, the matrix turns into the classical circulant matrix. Many scholars have studied the

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spectral norms of these matrices with famous sequences in last decade. Some great contributions for the spectral norms of r-circulant matrix and geometric circulant matrix can be found in references [1–13]. In [1], Solak has studied the spectral norms of circulant matrices with the Fibonacci and Lucas numbers. In [2], Kocer et al. obtained norms of circulant and semicirculant matrices with Horadam numbers. In [3], Shen and Cen have given upper and lower bounds for the spectral norms of r-circulant matrices with the Fibonacci and Lucas numbers. In [4], Bahsi computed the spectral norms of circulant and r-circulant matrices with the hyperharmonic numbers. Moreover, in [5], Bahsi and Solak studied norms of circulant and r-circulant matrices with the hyper-Fibonacci and hyper-Lucas numbers. In [6], Shen et al. computed the determinants and inverses of circulant matrices with Fibonacci and Lucas numbers. In [7], Yazlık and Taskara have studied eigenvalues, determinant and the spectral norms of circulant matrix involving of r-circulant matrix with the generalized k-Horadam numbers. In [8], He et al. gave the upper bound estimation of the spectral norm for r-circulant matrices with Fibonacci and Lucas numbers. In [9], Uygun computed some bounds for the norms of circulant matrices with the k-Jacobsthal and k-Jacobsthal Lucas numbers. Kızılateş and Tuglu [10] found the bounds for the spectral norms of geometric circulant matrices with the generalized Fibonacci, Lucas numbers, and hyperharmonic Fibonacci numbers. In [11], Raza et al. have also studied the norms of many special matrices with generalized Fibonacci sequences. In [12], Shi studied the spectral norms of geometric circulant matrices with the generalized k-Horadam numbers. Köme and Yazlik [13] have presented new upper and lower bounds for the spectral norms of the r-circulant matrices with biperiodic Fibonacci and biperiodic Lucas numbers.

Jacobsthal and Jacobsthal Lucas numbers are given by the recurrence relations $j_n = j_{n-1} + 2j_{n-2}$, with the initial values of $j_0 = 0, j_1 = 1$ and $c_n = c_{n-1} + 2c_{n-2}$, with the initial values of $c_0 = 2, c_1 = 1$ for $n \geq 2$, respectively in [15]. Edson, Yayenie defined a new generalization of Fibonacci sequences called bi-periodic Fibonacci sequences in [16]. The sequence arises in a natural way in the study of continued fractions of quadratic irrationals and combinatorics on words or dynamical system theory. Then Bilgici gave the properties of bi-periodic Lucas sequence in [17]. The authors studied bi-periodic Jacobsthal sequences in [18]. For any two non-zero real numbers a and b, the bi-periodic Jacobsthal sequence is defined as [18]

$$\hat{j}_0 = 0, \hat{j}_1 = 1, \hat{j}_n = \begin{cases} a\hat{j}_{n-1} + 2\hat{j}_{n-2} & \text{if } n \text{ is even} \\ b\hat{j}_{n-1} + 2\hat{j}_{n-2} & \text{if } n \text{ is odd} \end{cases} \qquad n \geq 2.$$

The Binet formula is

$$\hat{j}_m = \left(\frac{a^{1-\epsilon(m)}}{(ab)^{\lfloor \frac{m}{2} \rfloor}} \right) \frac{\alpha^m - \beta^m}{\alpha - \beta} \tag{1}$$

where α and β are the roots of the nonlinear quadratic equation for the bi-periodic Jacobsthal sequence, which is given as $x^2 - abx - 2ab = 0$, and $\lfloor a \rfloor$ is the floor function of a and $\zeta_{(n)} = n - 2\lfloor \frac{n}{2} \rfloor$ is the parity function.

For any two non-zero real numbers a and b, the bi-periodic Jacobsthal Lucas sequence is defined as [19]

$$\hat{c}_0 = 2, \hat{c}_1 = a, \hat{c}_n = \begin{cases} b\hat{c}_{n-1} + 2\hat{c}_{n-2} & \text{if } n \text{ is even} \\ a\hat{c}_{n-1} + 2\hat{c}_{n-2} & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2.$$

The Binet formula for the bi-periodic Jacobsthal Lucas sequence is

$$\hat{c}_m = \left(\frac{a^{\epsilon(m)}}{(ab)^{\lfloor \frac{m+1}{2} \rfloor}} \right) (\alpha^m + \beta^m). \tag{2}$$

Definition 1.1. Let $n \geq 2$ be an integer, r be any real or complex number. Then an r-circulant matrix C_r with order n is defined as follows:

$$C_r = \begin{bmatrix} c_0 & c_1 & c_2 & \dots & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \dots & c_{n-2} \\ rc_{n-2} & c_{n-1} & c_0 & \dots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ rc_1 & rc_2 & rc_3 & \dots & c_0 \end{bmatrix}. \tag{3}$$

From now on, we shortly denote the r- circulant matrix with $C_r = circ(c_0, c_1, \dots, c_{n-1})$.

Definition 1.2. An nxn geometric circulant matrix C_{r^*} is defined as the following

$$C_{r^*} = \begin{bmatrix} c_0 & c_1 & c_2 & \dots & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \dots & c_{n-2} \\ r^2c_{n-2} & rc_{n-1} & c_0 & \dots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{n-1}c_1 & r^{n-2}c_2 & r^{n-3}c_3 & \dots & c_0 \end{bmatrix}. \tag{4}$$

by Kızılateş and Tuğlu in [10]. For brevity, we denote the geometric circulant matrix with $C_{r^*} = circ(c_0, c_1, \dots, c_{n-1})$. If we choose $r = 1$, we get the circulant matrix.

In view of the above papers, we use the algebra methods, the properties of the r-circulant matrix and the geometric circulant matrix to estimate the upper and lower bounds for the spectral norms of these matrices involving the bi-periodic Jacobsthal numbers and bi-periodic Jacobsthal Lucas numbers. Then we give bounds for the spectral norms of Kronecker and Hadamard products of these r-circulant matrices and geometric circulant matrices. And also we give new formulas to compute the eigenvalues and determinant of r-circulant matrices with the bi-periodic Jacobsthal numbers.

Lemma 1.3. *The summation of the squares of the first n terms of bi-periodic Jacobsthal sequences is given as the following:*

$$\sum_{i=1}^{n} \left(\frac{2b}{a} \right)^{\epsilon(i+1)} \left(\frac{\hat{j}_i}{2^{\lfloor \frac{i}{2} \rfloor}} \right)^2 = \frac{1}{a} \frac{\hat{j}_m \hat{j}_{m+1}}{2^{m-1}}. \tag{5}$$

Proof. By using Binet forms of bi-periodic Jacobsthal sequences we have

$$\left(\frac{2b}{a}\right)^{\epsilon(i+1)}\left(\frac{\hat{j}_i}{2^{\lfloor\frac{i}{2}\rfloor}}\right)^2=\frac{2ab}{(\alpha-\beta)^2}\left[\left(\frac{\alpha^2}{2ab}\right)^k-\left(\frac{\beta^2}{2ab}\right)^k-2(-1)^k\right].$$

Using the properties $ab(\alpha+2)=\alpha^2$ and $ab(\beta+2)=\beta^2$, it is obtained that

$$\begin{aligned}\sum_{i=1}^{n}\left(\tfrac{2b}{a}\right)^{\epsilon(i+1)}\left(\tfrac{\hat{j}_i}{2^{\lfloor\frac{i}{2}\rfloor}}\right)^2 &= \frac{2ab}{(\alpha-\beta)^2}\left[\sum_{i=1}^{n}\left(\frac{\alpha^2}{2ab}\right)^k-\sum_{i=1}^{n}\left(\frac{\beta^2}{2ab}\right)^k-2\sum_{i=1}^{n}(-1)^k\right]\\ &= \frac{\left(\frac{\alpha^2}{2ab}\right)^{m+1}-\left(\frac{\alpha^2}{2ab}\right)}{\left(\frac{\alpha^2}{2ab}\right)-1}\\ &= \frac{\hat{j}_n\hat{j}_{n-1}}{a2^{n-2}}.\end{aligned}$$

□

Lemma 1.4. *The following property holds for the bi-periodic Jacobsthal sequences*

$$\sum_{i=1}^{m}\left(\frac{2b}{a}\right)^{\epsilon(i+1)}\left(\frac{\hat{j}_i}{|r|^i2^{\lfloor\frac{i}{2}\rfloor}}\right)^2=\frac{n|r|^{2n}}{a^2b^2+8ab}\left[\begin{array}{l}\frac{ab2\hat{c}_{2n}-|r|\hat{c}_{2n+2}+2ab|r|(a^2b^2+8ab-1)}{(1+|r|^2-\frac{|r|}{2}(ab+4))(2|r|)^n}\\ +2ab[1-(-1)^n]\end{array}\right] \tag{6}$$

Proof. The proof is made by using similar procedure with the proof of the previous lemma.

For any $A=[a_{ij}]\in M_{m,n}(C)$, the Frobenious (or Euclidean) norm of matrix A is displayed by the following equality:

$$\|A\|_E=\left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}}, \tag{7}$$

and the spectral norm of matrix A is shown as

$$\|A\|_2=\sqrt{\max_{1\leq i\leq n}\lambda_i(A^HA)}, \tag{8}$$

where A^H is the conjugate transpose of matrix A and $\lambda_i(A^HA)$ is an eigenvalue of A^HA. □

Lemma 1.5. *Suppose that $A\in M_{m,n}(C)$, then the following inequalities hold between the Euclid and spectral norms [20].*

$$\begin{aligned}&\tfrac{1}{\sqrt{n}}\|A\|_E\leq\|A\|_2\leq\|A\|_E,\\ &\|A\|_2\leq\|A\|_E\leq\sqrt{n}\|A\|_2.\end{aligned} \tag{9}$$

Lemma 1.6. *Suppose that $A,B\in M_{m,n}(C)$, then the Hadamard product of A,B is the mxn matrix of element wise products, namely [21,22,23]*

$$A\circ B=(a_{ij}b_{ij}).$$

The following property is satisfied by

$$\|A\circ B\|_2\leq\|A\|_2\|B\|_2. \tag{10}$$

$r_1(A)$, the maximum row length norm, $c_1(B)$, the maximum column length norm are given as $r_1(A) = \max_{1\le i\le n} \sqrt{\sum_{j=1}^{n} |b_{ij}|^2}$ and $c_1(C) = \max_{1\le j\le n} \sqrt{\sum_{j=1}^{n} |c_{ij}|^2}$ with the following property

$$\|A \circ B\|_2 \le r_1(A)c_1(B). \tag{11}$$

Let $A \in M_{m,n}(C)$ and $B \in M_{p,q}(C)$ be given, then the Kronecker product of A, B is defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix},$$

and has the following property [22]

$$\|A \otimes B\|_2 = \|A\|_2\|B\|_2. \tag{12}$$

2. Lower and Upper Bounds of r- Circulant Matrices Involving bi-periodic Jacobsthal Numbers

Theorem 2.1. *Let $r \in \mathbb{C}$ and $J_r = circ_r((\frac{2b}{a})^{\frac{\epsilon(1)}{2}}\hat{j}_0, (\frac{2b}{a})^{\frac{\epsilon(2)}{2}}\hat{j}_1, \dots, (\frac{2b}{a})^{\frac{\epsilon(n)}{2}}\frac{\hat{j}_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}})$ be an r- circulant matrix with bi-periodic Jacobsthal numbers, then the upper and lower bounds for the spectral norm of J_r are obtained as*

(i) *If $|r| \ge 1$, then*

$$\sqrt{\frac{\hat{j}_n\hat{j}_{n-1}}{a2^{n-2}}} \le \|J_r\|_2 \le \sqrt{(n-1)r\frac{\hat{j}_n\hat{j}_{n-1}}{a2^{n-2}}}.$$

(ii) *If $|r| < 1$, then*

$$|r|\sqrt{\frac{\hat{j}_n\hat{j}_{n-1}}{a2^{n-2}}} \le \|J_r\|_2 \le \sqrt{(n-1)\frac{\hat{j}_n\hat{j}_{n-1}}{a2^{n-2}}}.$$

Proof. The r- circulant matrix J_r is of the form

$$J_r = \begin{bmatrix} \left(\frac{2b}{a}\right)^{\frac{\epsilon(1)}{2}}\hat{j}_0 & \left(\frac{2b}{a}\right)^{\frac{\epsilon(2)}{2}}\hat{j}_1 & \left(\frac{2b}{a}\right)^{\frac{\epsilon(3)}{2}}\frac{\hat{j}_2}{2} & \dots & \left(\frac{2b}{a}\right)^{\frac{\epsilon(n)}{2}}\frac{\hat{j}_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}} \\ r\left(\frac{2b}{a}\right)^{\frac{\epsilon(n)}{2}}\frac{\hat{j}_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\epsilon(1)}{2}}\hat{j}_0 & \left(\frac{2b}{a}\right)^{\frac{\epsilon(2)}{2}}\hat{j}_1 & \dots & \left(\frac{2b}{a}\right)^{\frac{\epsilon(n-1)}{2}}\frac{\hat{j}_{n-2}}{2^{\lfloor\frac{n-2}{2}\rfloor}} \\ r\left(\frac{2b}{a}\right)^{\frac{\epsilon(n-1)}{2}}\frac{\hat{j}_{n-2}}{2^{\lfloor\frac{n-2}{2}\rfloor}} & r\left(\frac{2b}{a}\right)^{\frac{\epsilon(n-2)}{2}}\frac{\hat{j}_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\epsilon(1)}{2}}\hat{j}_0 & \dots & \left(\frac{2b}{a}\right)^{\frac{\epsilon(n-2)}{2}}\frac{\hat{j}_{n-3}}{2^{\lfloor\frac{n-3}{2}\rfloor}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r\left(\frac{2b}{a}\right)^{\frac{\epsilon(2)}{2}}\hat{j}_1 & r\left(\frac{2b}{a}\right)^{\frac{\epsilon(3)}{2}}\frac{\hat{j}_2}{2} & r\left(\frac{2b}{a}\right)^{\frac{\epsilon(4)}{2}}\frac{\hat{j}_3}{2} & \dots & \left(\frac{2b}{a}\right)^{\frac{\epsilon(1)}{2}}\hat{j}_0 \end{bmatrix}.$$

(i) For $|r| \geq 1$, by using (5), (7) we have

$$\begin{aligned}
\|J_{rJ_{r^*}}\|_F^2 &= \sum_{k=0}^{n-1}(n-k)\left(\tfrac{2b}{a}\right)^{\epsilon(k+1)}\left(\frac{\hat{j}_k}{2^{\lfloor \frac{k}{2}\rfloor}}\right)^2 + \sum_{k=1}^{n-1}k|r|^2\left(\tfrac{2b}{a}\right)^{\epsilon(k+1)}\left(\frac{\hat{j}_k}{2^{\lfloor \frac{k}{2}\rfloor}}\right)^2 \\
&\geq \sum_{k=0}^{n-1}(n-k)\left(\tfrac{2b}{a}\right)^{\epsilon(k+1)}\left(\frac{\hat{j}_k}{2^{\lfloor \frac{k}{2}\rfloor}}\right)^2 + \sum_{k=1}^{n-1}k\left(\tfrac{2b}{a}\right)^{\epsilon(k+1)}\left(\frac{\hat{j}_k}{2^{\lfloor \frac{k}{2}\rfloor}}\right)^2 \\
&= n\sum_{k=0}^{n-1}\left(\tfrac{2b}{a}\right)^{\epsilon(k+1)}\left(\frac{\hat{j}_k}{2^{\lfloor \frac{k}{2}\rfloor}}\right)^2 \\
&= n\left(\frac{\hat{j}_n\hat{j}_{n-1}}{a2^{n-2}}\right).
\end{aligned}$$

From the equality (9),

$$\|J_r\|_2 \geq \frac{\|J_r\|_F}{\sqrt{n}} \geq \sqrt{\frac{\hat{j}_n\hat{j}_{n-1}}{a2^{n-2}}}.$$

On the other hand, let $J_r = B \circ C$ where $B = [b_{ij}]$ and $C = [c_{ij}]$ are defined as

$$B = \begin{bmatrix} \hat{j}_0 & 1 & 1 & \dots & 1 \\ r & \hat{j}_0 & 1 & \dots & 1 \\ r & r & \hat{j}_0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r & r & r & \dots & \hat{j}_0 \end{bmatrix},$$

and

$$C = \begin{bmatrix}
\left(\frac{2b}{a}\right)^{\frac{\epsilon(1)}{2}}\hat{j}_0 & \left(\frac{2b}{a}\right)^{\frac{\epsilon(2)}{2}}\hat{j}_1 & \left(\frac{2b}{a}\right)^{\frac{\epsilon(3)}{2}}\frac{\hat{j}_2}{2} & \dots & \left(\frac{2b}{a}\right)^{\frac{\epsilon(n)}{2}}\frac{\hat{j}_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}} \\
\left(\frac{2b}{a}\right)^{\frac{\epsilon(n)}{2}}\frac{\hat{j}_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\epsilon(1)}{2}}\hat{j}_0 & \left(\frac{2b}{a}\right)^{\frac{\epsilon(2)}{2}}\hat{j}_1 & \dots & \left(\frac{2b}{a}\right)^{\frac{\epsilon(n-1)}{2}}\frac{\hat{j}_{n-2}}{2^{\lfloor\frac{n-2}{2}\rfloor}} \\
\left(\frac{2b}{a}\right)^{\frac{\epsilon(n-1)}{2}}\frac{\hat{j}_{n-2}}{2^{\lfloor\frac{n-2}{2}\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\epsilon(n)}{2}}\frac{\hat{j}_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\epsilon(1)}{2}}\hat{j}_0 & \dots & \left(\frac{2b}{a}\right)^{\frac{\epsilon(n-2)}{2}}\frac{\hat{j}_{n-3}}{2^{\lfloor\frac{n-3}{2}\rfloor}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\left(\frac{2b}{a}\right)^{\frac{\epsilon(2)}{2}}\hat{j}_1 & \left(\frac{2b}{a}\right)^{\frac{\epsilon(3)}{2}}\frac{\hat{j}_2}{2} & \left(\frac{2b}{a}\right)^{\frac{\epsilon(4)}{2}}\frac{\hat{j}_3}{2} & \dots & \left(\frac{2b}{a}\right)^{\frac{\epsilon(1)}{2}}\hat{j}_0
\end{bmatrix}.$$

By the maximum row and column length norm of these matrices, it is satisfied that

$$\begin{aligned}
r_1(B) &= \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|b_{ij}|^2} = \sqrt{\sum_{j=1}^{n}|b_{nj}|^2} = \sqrt{\hat{j}_0^2 + (n-1)r} = \sqrt{(n-1)r}, \\
c_1(C) &= \max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}|c_{ij}|^2} = \sqrt{\tfrac{\hat{j}_n\hat{j}_{n-1}}{a2^{n-2}}}.
\end{aligned}$$

By using (11), we obtain

$$\|J_r\|_2 \leq r_1(B)c_1(C) = \sqrt{(n-1)r\frac{\hat{j}_n\hat{j}_{n-1}}{a2^{n-2}}}.$$

The proof is completed for the first part.

(ii) For $|r| < 1$, by using (5), (7) we have

$$\begin{aligned}\|J_{r^*}\|_F^2 &= \sum_{k=0}^{n-1}(n-k)\left(\tfrac{2b}{a}\right)^{\epsilon(k+1)}\left(\frac{\hat{j}_k}{2^{\lfloor\frac{k}{2}\rfloor}}\right)^2 + \sum_{k=1}^{n-1} k|r|^2\left(\tfrac{2b}{a}\right)^{\epsilon(k+1)}\left(\frac{\hat{j}_k}{2^{\lfloor\frac{k}{2}\rfloor}}\right)^2 \\ &\geq \sum_{k=0}^{n-1}(n-k+k)|r|^2\left(\tfrac{2b}{a}\right)^{\epsilon(k+1)}\left(\frac{\hat{j}_{k+1}}{2^{\lfloor\frac{k+1}{2}\rfloor}}\right)^2 = n|r|^2\frac{\hat{j}_n\hat{j}_{n-1}}{a2^{n-2}}.\end{aligned}$$

From (9), we get

$$\|J_{r^*}\|_2 \geq \frac{\|J_{r^*}\|_F}{\sqrt{n}} \geq |r|\sqrt{\frac{\hat{j}_n\hat{j}_{n-1}}{a2^{n-2}}}.$$

On the other hand, let $J_r = B \circ C$ where B,C are given in the following forms:

$$B = \begin{bmatrix} \hat{j}_0 & 1 & 1 & \dots & 1 \\ r & \hat{j}_0 & 1 & \dots & 1 \\ r & r & \hat{j}_0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r & r & r & \dots & \hat{j}_0 \end{bmatrix},$$

and

$$C = \begin{bmatrix} \left(\frac{2b}{a}\right)^{\frac{\epsilon(1)}{2}}\hat{j}_0 & \left(\frac{2b}{a}\right)^{\frac{\epsilon(2)}{2}}\hat{j}_1 & \left(\frac{2b}{a}\right)^{\frac{\epsilon(3)}{2}}\frac{\hat{j}_2}{2} & \dots & \left(\frac{2b}{a}\right)^{\frac{\epsilon(n)}{2}}\frac{\hat{j}_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}} \\ \left(\frac{2b}{a}\right)^{\frac{\epsilon(n)}{2}}\frac{\hat{j}_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\epsilon(1)}{2}}\hat{j}_0 & \left(\frac{2b}{a}\right)^{\frac{\epsilon(2)}{2}}\hat{j}_1 & \dots & \left(\frac{2b}{a}\right)^{\frac{\epsilon(n-1)}{2}}\frac{\hat{j}_{n-2}}{2^{\lfloor\frac{n-2}{2}\rfloor}} \\ \left(\frac{2b}{a}\right)^{\frac{\epsilon(n-1)}{2}}\frac{\hat{j}_{n-2}}{2^{\lfloor\frac{n-2}{2}\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\epsilon(n)}{2}}\frac{\hat{j}_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\epsilon(1)}{2}}\hat{j}_0 & \dots & \left(\frac{2b}{a}\right)^{\frac{\epsilon(n-2)}{2}}\frac{\hat{j}_{n-3}}{2^{\lfloor\frac{n-3}{2}\rfloor}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \left(\frac{2b}{a}\right)^{\frac{\epsilon(2)}{2}}\hat{j}_1 & \left(\frac{2b}{a}\right)^{\frac{\epsilon(3)}{2}}\frac{\hat{j}_2}{2} & \left(\frac{2b}{a}\right)^{\frac{\epsilon(4)}{2}}\frac{\hat{j}_3}{2} & \dots & \left(\frac{2b}{a}\right)^{\frac{\epsilon(1)}{2}}\hat{j}_0 \end{bmatrix}.$$

By the maximum row and column length norm of these matrices, it is satisfied that

$$r_1(B) = \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}|b_{ij}|^2} = \sqrt{\left(\tfrac{2b}{a}\right)^{\frac{\epsilon(1)}{2}}\hat{j}_0^2 + (n-1)},$$

$$c_1(C) = \max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}|c_{ij}|^2} = \sqrt{\sum_{k=0}^{n-1}\left[\left(\tfrac{2b}{a}\right)^{\frac{\epsilon(n)}{2}}\frac{\hat{j}_i}{2^{\lfloor\frac{i}{2}\rfloor}}\right]^2} = \sqrt{\frac{\hat{j}_n\hat{j}_{n-1}}{a2^{n-2}}}.$$

By using (11), we obtain the second part of the proof.

$$\|J_r\|_2 \leq r_1(B)c_1(C) = \sqrt{\frac{(n-1)\hat{j}_n\hat{j}_{n-1}}{a2^{n-2}}}.$$

Therefore the proof is completed. □

Corollary 2.2. *Let* $A = B = J_r = circ_r((\frac{2b}{a})^{\frac{\epsilon(1)}{2}}\hat{j}_0, (\frac{2b}{a})^{\frac{\epsilon(2)}{2}}\hat{j}_1, \dots, (\frac{2b}{a})^{\frac{\epsilon(n)}{2}}\frac{\hat{j}_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}})$ *be an r- circulant matrix with bi-periodic Jacobsthal numbers, then the lower and* *upper bounds for the spectral norm of Kronecker product of A and B are demonstrated by*

(i) *If* $|r| \geq 1$, *then*

$$\frac{\hat{j}_n \hat{j}_{n-1}}{a2^{n-2}} \leq \|A \otimes B\|_2 \leq (n-1)r\frac{\hat{j}_n \hat{j}_{n-1}}{a2^{n-2}}.$$

(ii) *If* $|r| < 1$, *then*

$$|r|\sqrt{\frac{\hat{j}_n \hat{j}_{n-1}}{a2^{n-2}}} \leq \|A \otimes B\|_2 \leq (n-1)\frac{\hat{j}_n \hat{j}_{n-1}}{a2^{n-2}}.$$

Proof. It is clear that the proof is seen by using $\|A \otimes B\|_2 = \|A\|_2 \|B\|_2$. □

Corollary 2.3. *Let* $A = B = J_r = circ_r((\frac{2b}{a})^{\frac{\epsilon(1)}{2}}\hat{j}_0, (\frac{2b}{a})^{\frac{\epsilon(2)}{2}}\hat{j}_1, \ldots, (\frac{2b}{a})^{\frac{\epsilon(n)}{2}}\frac{\hat{j}_{n-1}}{2^{\lfloor \frac{n-1}{2} \rfloor}})$ *be an r- circulant matrix whose entries are bi-periodic Jacobsthal numbers, then the upper bounds for the spectral norm of Hadamard product of A and B are demonstrated by*

(i) *If* $|r| \geq 1$, *then*

$$\|A \circ B\|_2 \leq (n-1)r\frac{\hat{j}_n \hat{j}_{n-1}}{a2^{n-2}}.$$

(ii) *If* $|r| < 1$, *then*

$$\|A \circ B\|_2 \leq (n-1)\frac{\hat{j}_n \hat{j}_{n-1}}{a2^{n-2}}.$$

Proof. The proof is easily seen by $\|A \circ B\|_2 \leq \|A\|_2 \|B\|_2$. □

Lemma 2.4. *] Let* $r \in \mathbb{C}$ *and* $J_r = circ_r((\frac{2b}{a})^{\frac{\epsilon(1)}{2}}\hat{j}_0, (\frac{2b}{a})^{\frac{\epsilon(2)}{2}}\hat{j}_1, \ldots, (\frac{2b}{a})^{\frac{\epsilon(n)}{2}}\frac{\hat{j}_{n-1}}{2^{\lfloor \frac{n-1}{2} \rfloor}})$ *be an r- circulant matrix with bi-periodic Jacobsthal numbers. Then the eigenvalues are computed as*

$$\lambda_i(J_r) = \sum_{k=0}^{n-1} \left(\frac{2b}{a}\right)^{\frac{\epsilon(k+1)}{2}} \frac{\hat{j}_k}{2^{\lfloor \frac{k}{2} \rfloor}} r^{\frac{k}{n}} w^{jk},$$

where $w = e^{-\frac{2\pi i}{n}}, i = \sqrt{-1}, j = 0, 1, ..., n-1$.

Theorem 2.5. *Let* $r \in \mathbb{C}$ *and* $J_r = circ_r((\frac{2b}{a})^{\frac{\epsilon(1)}{2}}\hat{j}_0, (\frac{2b}{a})^{\frac{\epsilon(2)}{2}}\hat{j}_1, \ldots, (\frac{2b}{a})^{\frac{\epsilon(n)}{2}}\frac{\hat{j}_{n-1}}{2^{\lfloor \frac{n-1}{2} \rfloor}})$ *be an r- circulant matrix with bi-periodic Jacobsthal numbers. Then the eigenvalues are computed as*

$$\lambda_j(J_r) = \frac{r^{\frac{1}{n}} w^j ((2ab)^{\xi(n)} r \hat{j}_{n-1} + 1) + r\sqrt{(2ab)}\hat{j}_n}{(r^{\frac{2}{n}} w^{2j} - 1 + \sqrt{ab/2} r^{\frac{1}{n}} w^j)}$$

where $w = e^{-\frac{2\pi i}{n}}, i = \sqrt{-1}, j = 0, 1, ..., n-1$.

Proof. By Binet formula, for n is even, we obtain that

$$\begin{aligned}
&\sum_{k=0}^{n-1}\left(\tfrac{2b}{a}\right)^{\frac{\epsilon(k+1)}{2}}\frac{\hat{j}_k}{2^{\lfloor\frac{k}{2}\rfloor}}r^{\frac{k}{n}}w^{jk}\\
=\;&\frac{\sqrt{2ab}}{\alpha-\beta}\sum_{k=0}^{n-1}\frac{\alpha^k-\beta^k}{(2ab)^{\lfloor\frac{k}{2}\rfloor}}r^{\frac{k}{n}}w^{jk}\\
=\;&\frac{\sqrt{2ab}}{\alpha-\beta}\left[\frac{\frac{(\alpha r^{\frac{1}{n}}w^j)^n}{(2ab)^{\lfloor\frac{n}{2}\rfloor}}-1}{\frac{\alpha r^{\frac{1}{n}}w^j}{\sqrt{(2ab)}}-1}-\frac{\frac{(\beta r^{\frac{1}{n}}w^j)^n}{(2ab)^{\lfloor\frac{n}{2}\rfloor}-1}}{\frac{\beta r^{\frac{1}{n}}w^j}{\sqrt{(2ab)}}-1}\right]\\
=\;&\frac{\sqrt{2ab}}{(\alpha-\beta)\sqrt{(2ab)^{n-1}}}\left[\frac{r(\alpha w^j)^n-\sqrt{(2ab)^n}}{\alpha r^{\frac{1}{n}}w^j-\sqrt{(2ab)}}-\frac{r(\beta w^j)^n-\sqrt{(2ab)^n}}{\beta r^{\frac{1}{n}}w^j-\sqrt{(2ab)}}\right].
\end{aligned}$$

Then

$$\begin{aligned}
=\;&\frac{1}{(\alpha-\beta)\sqrt{(2ab)^{n-2}}}\left[\frac{\begin{array}{l}-2abr^{1+\frac{1}{n}}w^{jn+j}(\alpha^{n-1}-\beta^{n-1})\\ -rw^{jn}\sqrt{(2ab)}(\alpha^n-\beta^n)\\ -\sqrt{(2ab)^n}r^{\frac{1}{n}}w^j(\alpha-\beta)\end{array}}{-2abr^{\frac{2}{n}}w^{2j}+2ab-\sqrt{(2ab)}abr^{\frac{1}{n}}w^j}\right]\\
=\;&\frac{r^{1+\frac{1}{n}}w^j\hat{j}_{n-1}+r\sqrt{(2ab)}\hat{j}_n+r^{\frac{1}{n}}w^j}{(r^{\frac{2}{n}}w^{2j}-1+\sqrt{ab/2}r^{\frac{1}{n}}w^j)}\\
=\;&\frac{r^{\frac{1}{n}}w^j(\hat{j}_{n-1}+1)+r\sqrt{(2ab)}\hat{j}_n}{(r^{\frac{2}{n}}w^{2j}-1+\sqrt{ab/2}r^{\frac{1}{n}}w^j)}.
\end{aligned}$$

Similarly for n is odd, we have

$$\begin{aligned}
\sum_{k=0}^{n-1}\left(\tfrac{2b}{a}\right)^{\frac{\epsilon(k+1)}{2}}\frac{\hat{j}_k}{2^{\lfloor\frac{k}{2}\rfloor}}r^{\frac{k}{n}}w^{jk}&=\frac{2abr^{1+\frac{1}{n}}w^{jn+j}\hat{j}_{n-1}+rw^{jn}\sqrt{(2ab)}\hat{j}_n+r^{\frac{1}{n}}w^j}{(r^{\frac{2}{n}}w^{2j}-1+\sqrt{ab/2}r^{\frac{1}{n}}w^j)}\\
&=\frac{r^{\frac{1}{n}}w^j(2abr\hat{j}_{n-1}+1)+r\sqrt{(2ab)}\hat{j}_n}{(r^{\frac{2}{n}}w^{2j}-1+\sqrt{ab/2}r^{\frac{1}{n}}w^j)}.
\end{aligned}$$

By combining the results, the proof is completed. □

Theorem 2.6. *The determinant of*

$$J_r=circ_r\left(\left(\frac{2b}{a}\right)^{\frac{\epsilon(1)}{2}}\hat{j}_0,\left(\frac{2b}{a}\right)^{\frac{\epsilon(2)}{2}}\hat{j}_1,\ldots,\left(\frac{2b}{a}\right)^{\frac{\epsilon(n)}{2}}\frac{\hat{j}_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}}\right)$$

is formulated by

$$det(J_r)=\frac{(r\sqrt{(2ab)}\hat{j}_n)^n-r((2ab)^{\xi(n)}r\hat{j}_{n-1}+1)^n}{(\sqrt{(2ab)^n}-\alpha^n r)(\sqrt{(2ab)^n}-\beta^n r)}.$$

Proof. We know that $det(J_r)=\prod_{j=0}^{n-1}\lambda_j(J_r)$. By the Theorem 2.6, it is obtained that

$$det(J_r)=\prod_{j=0}^{n-1}\frac{r^{\frac{1}{n}}w^j((2ab)^{\xi(n)}r\hat{j}_{n-1}+1)+r\sqrt{(2ab)}\hat{j}_n}{(\alpha r^{\frac{1}{n}}w^j-\sqrt{(2ab)})(\beta r^{\frac{1}{n}}w^j-\sqrt{(2ab)})}$$

By using the property $\prod_{j=0}^{n-1}(x-yw^j)=x^n-y^n$ the proof is easily completed.

$$det(J_r)=\frac{(r\sqrt{(2ab)}\hat{j}_n)^n-r((2ab)^{\xi(n)}r\hat{j}_{n-1}+1)^n}{(\sqrt{(2ab^n)}-\alpha^n r)(\sqrt{(2ab^n)}-\beta^n r)}.$$

□

3. Lower and Upper Bounds of Geometric Circulant Matrices Involving bi-periodic Jacobsthal Numbers

Theorem 3.1. *Let $r\in\mathbb{C}$ and $J_{r^*}=circ_r((\frac{2b}{a})^{\frac{\epsilon(1)}{2}}\hat{j}_0,(\frac{2b}{a})^{\frac{\epsilon(2)}{2}}\hat{j}_1,\ldots,(\frac{2b}{a})^{\frac{\epsilon(n)}{2}}\frac{\hat{j}_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}})$ be a geometric circulant matrix with bi-periodic Jacobsthal numbers, then the upper and lower bounds for the spectral norm of J_{r^*} are obtained as*

(i) *If $|r|\geq 1$, then*

$$\sqrt{\frac{\hat{j}_n\hat{j}_{n-1}}{a2^{n-2}}}\leq\|J_{r^*}\|_2\leq\sqrt{\frac{\hat{j}_n\hat{j}_{n-1}}{a2^{n-2}}.\frac{1-|r|^{2n}}{1-|r|^2}}.$$

(ii) *If $|r|<1$, then*

$$\frac{n|r|^{2n}}{a^2b^2+8ab}\left[\frac{ab(2\hat{c}_{2n}-|r|\hat{c}_{2n+2}+2ab(|r|(a^2b^2+8ab-1)))}{(1+|r|^2-\frac{|r|}{2}(ab+4))(2|r|)^n}+2ab[1-(-1)^m]\right]\leq\|J_{r^*}\|_2$$

$$\|J_{r^*}\|_2\leq\sqrt{(n-1)\frac{\hat{j}_n\hat{j}_{n-1}}{a2^{n-2}}}.$$

Proof. The geometric circulant matrix J_{r^*} is of the form

$$J_{r^*}=\begin{bmatrix}\left(\frac{2b}{a}\right)^{\frac{\epsilon(1)}{2}}\hat{j}_0 & \left(\frac{2b}{a}\right)^{\frac{\epsilon(2)}{2}}\hat{j}_1 & \left(\frac{2b}{a}\right)^{\frac{\epsilon(3)}{2}}\frac{\hat{j}_2}{2} & \cdots & \left(\frac{2b}{a}\right)^{\frac{\epsilon(n)}{2}}\frac{\hat{j}_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}}\\ r\left(\frac{2b}{a}\right)^{\frac{\epsilon(n)}{2}}\frac{\hat{j}_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\epsilon(1)}{2}}\hat{j}_0 & \left(\frac{2b}{a}\right)^{\frac{\epsilon(2)}{2}}\hat{j}_1 & \cdots & \left(\frac{2b}{a}\right)^{\frac{\epsilon(n-1)}{2}}\frac{\hat{j}_{n-2}}{2^{\lfloor\frac{n-2}{2}\rfloor}}\\ r^2\left(\frac{2b}{a}\right)^{\frac{\epsilon(n-1)}{2}}\frac{\hat{j}_{n-2}}{2^{\lfloor\frac{n-2}{2}\rfloor}} & r\left(\frac{2b}{a}\right)^{\frac{\epsilon(n)}{2}}\frac{\hat{j}_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\epsilon(1)}{2}}\hat{j}_0 & \cdots & \left(\frac{2b}{a}\right)^{\frac{\epsilon(n-2)}{2}}\frac{\hat{j}_{n-3}}{2^{\lfloor\frac{n-3}{2}\rfloor}}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ r^{n-1}\left(\frac{2b}{a}\right)^{\frac{\epsilon(2)}{2}}\hat{j}_1 & r^{n-2}\left(\frac{2b}{a}\right)^{\frac{\epsilon(3)}{2}}\frac{\hat{j}_2}{2} & r^{n-3}\left(\frac{2b}{a}\right)^{\frac{\epsilon(4)}{2}}\frac{\hat{j}_3}{2} & \cdots & \left(\frac{2b}{a}\right)^{\frac{\epsilon(1)}{2}}\hat{j}_0\end{bmatrix}.$$

(i) For $|r|\geq 1$, by using the definiton of Frebinous norm, we have

$$\begin{aligned}\|J_{r^*}\|_E^2 &= \sum_{k=0}^{n-1}(n-k)\left(\frac{2b}{a}\right)^{\epsilon(k+1)}\left(\frac{\hat{j}_k}{2^{\lfloor\frac{k}{2}\rfloor}}\right)^2+\sum_{k=1}^{n-1}k|r^{n-k}|^2\left(\frac{2b}{a}\right)^{\epsilon(k+1)}\left(\frac{\hat{j}_k}{2^{\lfloor\frac{k}{2}\rfloor}}\right)^2\\ &\geq \sum_{k=0}^{n-1}(n-k)\left(\frac{2b}{a}\right)^{\epsilon(k+1)}\left(\frac{\hat{j}_k}{2^{\lfloor\frac{k}{2}\rfloor}}\right)^2+\sum_{k=1}^{n-1}k\left(\frac{2b}{a}\right)^{\epsilon(k+1)}\left(\frac{\hat{j}_k}{2^{\lfloor\frac{k}{2}\rfloor}}\right)^2\\ &= n\sum_{k=0}^{n-1}\left(\frac{2b}{a}\right)^{\epsilon(k+1)}\left(\frac{\hat{j}_k}{2^{\lfloor\frac{k}{2}\rfloor}}\right)^2\\ &= n\left(\frac{\hat{j}_n\hat{j}_{n-1}}{a2^{n-2}}\right).\end{aligned}$$

From the equality (9),

$$\sqrt{\left(\frac{\hat{j}_n\hat{j}_{n-1}}{a2^{n-2}}\right)} \leq \|J_{r^*}\|_2.$$

On the other hand, let the matrices B and C be presented by

$$B = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ r & 1 & 1 & \dots & 1 \\ r^2 & r & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{n-1} & r^{n-2} & r^{n-3} & \dots & 1 \end{bmatrix},$$

and

$$C = \begin{bmatrix} \left(\frac{2b}{a}\right)^{\frac{\epsilon(1)}{2}} \hat{j}_0 & \left(\frac{2b}{a}\right)^{\frac{\epsilon(2)}{2}} \hat{j}_1 & \left(\frac{2b}{a}\right)^{\frac{\epsilon(3)}{2}} \frac{\hat{j}_2}{2} & \dots & \left(\frac{2b}{a}\right)^{\frac{\epsilon(n)}{2}} \frac{\hat{j}_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}} \\ \left(\frac{2b}{a}\right)^{\frac{\epsilon(n)}{2}} \frac{\hat{j}_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\epsilon(1)}{2}} \hat{j}_0 & \left(\frac{2b}{a}\right)^{\frac{\epsilon(2)}{2}} \hat{j}_1 & \dots & \left(\frac{2b}{a}\right)^{\frac{\epsilon(n-1)}{2}} \frac{\hat{j}_{n-2}}{2^{\lfloor\frac{n-2}{2}\rfloor}} \\ \left(\frac{2b}{a}\right)^{\frac{\epsilon(n-1)}{2}} \frac{\hat{j}_{n-2}}{2^{\lfloor\frac{n-2}{2}\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\epsilon(n)}{2}} \frac{\hat{j}_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\epsilon(1)}{2}} \hat{j}_0 & \dots & \left(\frac{2b}{a}\right)^{\frac{\epsilon(n-2)}{2}} \frac{\hat{j}_{n-3}}{2^{\lfloor\frac{n-3}{2}\rfloor}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \left(\frac{2b}{a}\right)^{\frac{\epsilon(2)}{2}} \hat{j}_1 & \left(\frac{2b}{a}\right)^{\frac{\epsilon(3)}{2}} \frac{\hat{j}_2}{2} & \left(\frac{2b}{a}\right)^{\frac{\epsilon(4)}{2}} \frac{\hat{j}_3}{2} & \dots & \left(\frac{2b}{a}\right)^{\frac{\epsilon(1)}{2}} \hat{j}_0 \end{bmatrix}.$$

where $J_{r^*} = B \circ C$. The maximum row and column length norm of these matrices are presented by

$$\begin{aligned} r_1(B) &= \max_{1\leq i\leq n} \sqrt{\sum_{j=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |b_{nj}|^2} = \sqrt{\tfrac{1-|r|^{2n}}{1-|r|^2}}, \\ c_1(C) &= \max_{1\leq j\leq n} \sqrt{\sum_{i=1}^{n} |c_{ij}|^2} = \sqrt{\tfrac{\hat{j}_n\hat{j}_{n-1}}{a2^{n-2}}}. \end{aligned}$$

By using (11), we obtain

$$\|J_{r^*}\|_2 \leq r_1(B)c_1(C) = \sqrt{\frac{\hat{j}_n\hat{j}_{n-1}}{a2^{n-2}}.\frac{1-|r|^{2n}}{1-|r|^2}}.$$

(ii) For $|r| < 1$,

$$\begin{aligned} \|J_{r^*}\|_E^2 &= \sum_{k=0}^{n-1}(n-k)\left(\tfrac{2b}{a}\right)^{\epsilon(k+1)}\left(\tfrac{\hat{j}_k}{2^{\lfloor\frac{k}{2}\rfloor}}\right)^2 + \sum_{k=1}^{n-1} k|r^{n-k}|^2\left(\tfrac{2b}{a}\right)^{\epsilon(k+1)}\left(\tfrac{\hat{j}_k}{2^{\lfloor\frac{k}{2}\rfloor}}\right)^2 \\ &\geq \sum_{k=0}^{n-1}|r^{n-k}|^2\left(\tfrac{2b}{a}\right)^{\epsilon(k+1)}\left(\tfrac{\hat{j}_k}{2^{\lfloor\frac{k}{2}\rfloor}}\right)^2 \\ &= n|r|^{2n}\sum_{k=0}^{n-1}\left(\tfrac{2b}{a}\right)^{\epsilon(k+1)}\left(\frac{\hat{j}_k}{2^{\lfloor\frac{k}{2}\rfloor}|r|^k}\right)^2 \\ &= \tfrac{n|r|^{2n}}{a^2b^2+8ab}\left[\tfrac{ab(2\hat{c}_{2n}-|r|\hat{c}_{2n+2}+2ab(|r|(a^2b^2+8ab-1)))}{(1+|r|^2-\frac{|r|}{2}(ab+4))(2|r|)^n} + 2ab[1-(-1)^n]\right] \end{aligned}$$

For the matrices B and C as mentioned above, we have

$$B = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ r & 1 & 1 & \dots & 1 \\ r^2 & r & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{n-1} & r^{n-2} & r^{n-3} & \dots & 1 \end{bmatrix},$$

and

$$C = \begin{bmatrix} \left(\frac{2b}{a}\right)^{\frac{\epsilon(1)}{2}} \hat{j}_0 & \left(\frac{2b}{a}\right)^{\frac{\epsilon(2)}{2}} \hat{j}_1 & \left(\frac{2b}{a}\right)^{\frac{\epsilon(3)}{2}} \frac{\hat{j}_2}{2} & \dots & \left(\frac{2b}{a}\right)^{\frac{\epsilon(n)}{2}} \frac{\hat{j}_{n-1}}{2^{\lfloor \frac{n-1}{2} \rfloor}} \\ \left(\frac{2b}{a}\right)^{\frac{\epsilon(n)}{2}} \frac{\hat{j}_{n-1}}{2^{\lfloor \frac{n-1}{2} \rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\epsilon(1)}{2}} \hat{j}_0 & \left(\frac{2b}{a}\right)^{\frac{\epsilon(2)}{2}} \hat{j}_1 & \dots & \left(\frac{2b}{a}\right)^{\frac{\epsilon(n-1)}{2}} \frac{\hat{j}_{n-2}}{2^{\lfloor \frac{n-2}{2} \rfloor}} \\ \left(\frac{2b}{a}\right)^{\frac{\epsilon(n-1)}{2}} \frac{\hat{j}_{n-2}}{2^{\lfloor \frac{n-2}{2} \rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\epsilon(n)}{2}} \frac{\hat{j}_{n-1}}{2^{\lfloor \frac{n-1}{2} \rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\epsilon(1)}{2}} \hat{j}_0 & \dots & \left(\frac{2b}{a}\right)^{\frac{\epsilon(n-2)}{2}} \frac{\hat{j}_{n-3}}{2^{\lfloor \frac{n-3}{2} \rfloor}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \left(\frac{2b}{a}\right)^{\frac{\epsilon(2)}{2}} \hat{j}_1 & \left(\frac{2b}{a}\right)^{\frac{\epsilon(3)}{2}} \frac{\hat{j}_2}{2} & \left(\frac{2b}{a}\right)^{\frac{\epsilon(4)}{2}} \frac{\hat{j}_3}{2} & \dots & \left(\frac{2b}{a}\right)^{\frac{\epsilon(1)}{2}} \hat{j}_0 \end{bmatrix}.$$

In this case $J_{r^*} = B \circ C$ and it is obtained that

$$\begin{aligned} r_1(B) &= \max_{1 \le i \le n} \sqrt{\sum_{j=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |b_{1j}|^2} = \sqrt{n}, \\ c_1(C) &= \max_{1 \le j \le n} \sqrt{\sum_{i=1}^{n} |c_{ij}|^2} = \sqrt{\frac{\hat{j}_n \hat{j}_{n-1}}{a 2^{n-2}}}. \end{aligned}$$

By using (11), we obtain

$$\|J_{r^*}\|_2 \le r_1(B) c_1(C) = \sqrt{n \frac{\hat{j}_n \hat{j}_{n-1}}{a 2^{n-2}}}.$$

□

Corollary 3.2. *Let* $A = B = J_{r^*} = circ_r((\frac{2b}{a})^{\frac{\epsilon(1)}{2}} \hat{j}_0, (\frac{2b}{a})^{\frac{\epsilon(2)}{2}} \hat{j}_1, \dots, (\frac{2b}{a})^{\frac{\epsilon(n)}{2}} \frac{\hat{j}_{n-1}}{2^{\lfloor \frac{n-1}{2} \rfloor}})$ *be a geometric circulant matrix with bi-periodic Jacobsthal numbers, then the lower and upper bounds for spectral norm of Kronecker product of* A *and* B *are demonstrated by*

(i) *If* $|r| \ge 1$, *then*

$$\frac{\hat{j}_n \hat{j}_{n-1}}{a 2^{n-2}} \le \|A \otimes B\|_2 \le (n-1) r \frac{\hat{j}_n \hat{j}_{n-1}}{a 2^{n-2}} . \frac{1 - |r|^{2n}}{1 - |r|^2}.$$

(ii) *If* $|r| < 1$, *then*

$$\begin{aligned} &\left\{ \frac{n|r|^{2n}}{a^2 b^2 + 8ab} \left[\frac{ab(2\hat{c}_{2n} - |r|\hat{c}_{2n+2} + 2ab(|r|(a^2 b^2 + 8ab - 1)))}{(1 + |r|^2 - \frac{|r|}{2}(ab+4))(2|r|)^n} + 2ab[1 - (-1)^m] \right] \right\} \\ \le \quad & \|A \otimes B\|_2 \le \frac{\hat{j}_n \hat{j}_{n-1}}{a 2^{n-2}}. \end{aligned}$$

Corollary 3.3. *Let* $A = B = J_{r^*} = circ_r((\frac{2b}{a})^{\frac{\epsilon(1)}{2}}\hat{\hat{j}}_0, (\frac{2b}{a})^{\frac{\epsilon(2)}{2}}\hat{\hat{j}}_1, \ldots, (\frac{2b}{a})^{\frac{\epsilon(n)}{2}}\frac{\hat{\hat{j}}_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}})$ *be a geometric circulant matrix with bi-periodic Jacobsthal numbers, then the upper bounds for spectral norm of Hadamard product of* A *and* B *are demonstrated by*

(i) *If* $|r| \geq 1$*, then*

$$\|A \circ B\|_2 \leq \frac{\hat{\hat{j}}_n \hat{\hat{j}}_{n-1}}{a2^{n-2}} . \frac{1-|r|^{2n}}{1-|r|^2}.$$

(ii) *If* $|r| < 1$*, then*

$$\|A \circ B\|_2 \leq n\frac{\hat{\hat{j}}_n \hat{\hat{j}}_{n-1}}{a2^{n-2}}.$$

Proof. The proof is easily seen by $\|A \circ B\|_2 \leq \|A\|_2\|B\|_2$. □

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