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AN ANNIHILATOR CONDITION ON MAXIMAL IDEALS OF COMPOSITE HURWITZ RINGS†

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Abstract. Let $R \subseteq T$ be an ascending chain of commutative rings with identity and $\mathrm{H}(R,T)$ (resp., $\mathrm{h}(R,T)$) the composite Hurwitz series ring (resp., composite Hurwitz polynomial ring). In this article, we give some conditions for the rings $\mathrm{H}(R,T)$ and $\mathrm{h}(R,T)$ to be PS-rings.

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1. Introduction

1.1. Composite Hurwitz rings. Let R be a commutative ring with identity and let $\mathrm{H}(R)$ be the set of formal expressions of the type $f = \sum_{i=0}^{\infty} a_i X^i$, where $a_i \in R$ for all $i \geq 0$. Define addition and $*$-product on $\mathrm{H}(R)$ as follows: for $f = \sum_{i=0}^{\infty} a_i X^i, g = \sum_{i=0}^{\infty} b_i X^i \in \mathrm{H}(R)$,

$$f + g = \sum_{i=0}^{\infty} (a_i + b_i) X^i \text{ and } f * g = \sum_{n=0}^{\infty} c_n X^n,$$

where $c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$. Then $\mathrm{H}(R)$ becomes a commutative ring with identity containing R under these two operations. The ring $\mathrm{H}(R)$ is called the *Hurwitz series ring* over R. The *Hurwitz polynomial ring* $\mathrm{h}(R)$ is the subring of $\mathrm{H}(R)$ consisting of formal expressions of the form $f = \sum_{i=0}^{n} a_i X^i$.

Let $R \subseteq T$ be an extension of commutative rings with identity. Let $\mathrm{H}(R,T) = \{f \in \mathrm{H}(T) \,|\, \text{the constant term of } f \text{ belongs to } R\}$ and $\mathrm{h}(R,T) = \{f \in \mathrm{h}(T) \,|\, \text{the constant term of } f \text{ belongs to } R\}$. Then $\mathrm{H}(R,T)$ and $\mathrm{h}(R,T)$ are commutative rings with identity satisfying $\mathrm{H}(R) \subseteq \mathrm{H}(R,T) \subseteq \mathrm{H}(T)$ and $\mathrm{h}(R) \subseteq \mathrm{h}(R,T) \subseteq$

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h(T). The rings H(R,T) and h(R,T) are called the *composite Hurwitz series ring* and the *composite Hurwitz polynomial ring*, respectively. Note that if $R \subsetneq T$, then H(R,T) (resp., h(R,T)) gives algebraic properties of Hurwitz series (resp., Hurwitz polynomial) type rings strictly between two Hurwitz series rings H(R) and H(T) (resp., Hurwitz polynomial rings h(R) and h(T)). Also, it is easy to see that H(R,T) (resp., h(R,T)) is a pullback of R and H(T) (resp., h(T)).

The readers can refer to [1] for the Hurwitz rings and to [4] for the composite Hurwitz rings.

1.2. PS-rings. Let R be a commutative ring with identity and let char(R) be the characteristic of R. For an ideal I of R, set $\text{ann}_R(I) = \{r \in R \,|\, rI = (0)\}$. Then $\text{ann}_R(I)$ is an ideal of R. Recall that R is a *PS-ring* if for each maximal ideal M of R, $\text{ann}_R(M) = (e)$ for some idempotent element e of R. In [1], Benhissi and Koja studied when the Hurwitz rings H(R) and h(R) are PS-rings. In fact, they showed that H(R) is always a PS-ring [1, Theorem 6.4] and if char$(R) > 0$, then h(R) is a PS-ring [1, Corollary 6.5]. They also proved that if R is a PS-ring which is a torsion-free $\mathbb{Z}$-module, then h(R) is a PS-ring [1, Theorem 6.7].

Let $R \subseteq T$ be an extension of commutative rings with identity. In this article, we study when the composite Hurwitz rings H(R,T) and h(R,T) are PS-rings. More precisely, we show that H(R,T) is always a PS-ring and if char$(R) > 0$, then h(R,T) is a PS-ring. We also prove that if T is a torsion-free $\mathbb{Z}$-module, then h(R,T) is a PS-ring.

2. Basic results

We start this section with a simple result.

Proposition 2.1. *Let $R \subseteq T$ be an extension of commutative rings with identity and let D be either* H(R,T) *or* h(R,T). *Let I be an ideal of R and $\pi : D \to R$ the canonical epimorphism. Then the following assertions hold.*

(1) *$D/\pi^{-1}(I)$ is isomorphic to R/I.*
(2) *$\pi^{-1}(I)$ is a prime ideal of D if and only if I is a prime ideal of R.*
(3) *$\pi^{-1}(I)$ is a maximal ideal of D if and only if I is a maximal ideal of R.*

Proof. (1) Let $\phi : R \to R/I$ be the canonical epimorphism and let $\psi = \phi \circ \pi$. Then ψ is a ring epimorphism. Note that $f \in \text{Ker}(\psi)$ if and only if $f \in \pi^{-1}(I)$. Hence $\text{Ker}(\psi) = \pi^{-1}(I)$. Thus $D/\pi^{-1}(I)$ is isomorphic to R/I.

(2) and (3) These equivalences follow directly from (1). □

Let $R \subseteq T$ be an extension of commutative rings with identity and let m be a nonnegative integer. In order to avoid the confusion, if f is an element of either H(R,T) or h(R,T), then we denote the mth power of f by $f^{(m)}$.

Lemma 2.2. *Let $R \subseteq T$ be an extension of commutative rings with identity and let D be either* H(R,T) *or* h(R,T). *Let $f = \sum_{i \geq 0} a_i X^i \in D$. If* char$(R) = m > 0$, *then the following assertions hold.*

(1) $f^{(m)} = a_0^m$.
(2) *f is a unit in D if and only if a_0 is a unit in R.*

Proof. (1) Note that $f \in \mathrm{H}(T)$ and $\mathrm{char}(T) = m$; so $f^{(m)} = a_0^m$ [2, Proposition 3.2].

(2) If a_0 is a unit in R, then by (1), $f * f^{(m-1)} * (a_0^{-1})^m = 1$. Hence $f^{-1} = f^{(m-1)} * (a_0^{-1})^m \in D$. Thus f is a unit in D. The converse is obvious. □

Let R be a commutative ring with identity. Then $\mathrm{Max}(R)$ means the set of maximal ideals of R. We next investigate the structure of maximal ideals of the composite Hurwitz series ring and the composite Hurwitz polynomial ring.

Proposition 2.3. *Let $R \subseteq T$ be an extension of commutative rings with identity. Then the following assertions hold.*

(1) *If $\pi : \mathrm{H}(R,T) \to R$ is the canonical epimorphism, then $\mathrm{Max}(\mathrm{H}(R,T)) = \{\pi^{-1}(M) \mid M \in \mathrm{Max}(R)\}$.*
(2) *If $\mathrm{char}(R) > 0$ and $\pi : \mathrm{h}(R,T) \to R$ is the canonical epimorphism, then $\mathrm{Max}(\mathrm{h}(R,T)) = \{\pi^{-1}(M) \mid M \in \mathrm{Max}(R)\}$.*

Proof. (1) By Proposition 2.1(3), it is enough to show that for any maximal ideal $\mathcal{M}$ of $\mathrm{H}(R,T)$, there exists a maximal ideal M of R such that $\mathcal{M} = \pi^{-1}(M)$. Since π is a ring epimorphism, $\pi(\mathcal{M})$ is an ideal of R. Let $M = \pi(\mathcal{M})$. If $M = R$, then $1 \in M$; so we can find an element $f \in \mathcal{M}$ such that $\pi(f) = 1$. Therefore f is a unit in $\mathrm{H}(R,T)$ [5, Lemma 2.2(1)]. This is a contradiction to the fact that $\mathcal{M} \subsetneq \mathrm{H}(R,T)$. Hence M is a proper ideal of R. Since $\mathcal{M} \subseteq \pi^{-1}(M) \subsetneq \mathrm{H}(R,T)$ and $\mathcal{M}$ is a maximal ideal of $\mathrm{H}(R,T)$, $\mathcal{M} = \pi^{-1}(M)$. Note that by Proposition 2.1(3), M is a maximal ideal of R. Thus the proof is done.

(2) The result can be obtained by combining Lemma 2.2(2) with a similar argument as in the proof of (1). □

Let R be a commutative ring with identity. Then $\mathrm{Spec}(R)$ stands for the set of prime ideals of R. We are closing this section with a study of the prime spectrum of the composite Hurwitz series ring and the composite Hurwitz polynomial ring.

Proposition 2.4. *Let $R \subseteq T$ be an extension of commutative rings with identity and let D be either $\mathrm{H}(R,T)$ or $\mathrm{h}(R,T)$. Let $\pi : D \to R$ be the canonical epimorphism and $\phi : \mathrm{Spec}(R) \to \mathrm{Spec}(D)$ the map given by $\phi(P) = \pi^{-1}(P)$. Then the following assertions hold.*

(1) *ϕ is an order-preserving injection.*
(2) *If $P \subsetneq Q$ are consecutive prime ideals of R, then $\phi(P) \subsetneq \phi(Q)$ are consecutive prime ideals of D.*
(3) *If $\mathrm{char}(R) > 0$, then ϕ is an order-preserving bijection.*

Proof. (1) Clearly, ϕ is well-defined and one-to-one. If $P_1 \subseteq P_2$ are prime ideals of R, then it is obvious that $\pi^{-1}(P_1) \subseteq \pi^{-1}(P_2)$ are prime ideals of D. Hence $\phi(P_1) \subseteq \phi(P_2)$. Thus ϕ is order-preserving.

(2) Let $A \in \text{Spec}(D)$ such that $\pi^{-1}(P) \subsetneq A \subseteq \pi^{-1}(Q)$. Then there exists an element $f \in A \setminus \pi^{-1}(P)$. Since $f - f(0) \in \pi^{-1}(0) \subsetneq A$, $f(0) \in A \setminus P$; so $f(0) \in A \cap R$. Therefore $P = \pi^{-1}(P) \cap R \subsetneq A \cap R \subseteq \pi^{-1}(Q) \cap R = Q$. Since $P \subsetneq Q$ are consecutive prime ideals of R, $Q = A \cap R$. Let $g \in \pi^{-1}(Q)$. Then $g(0) \in Q \subseteq A$. Note that $g - g(0) \in \pi^{-1}(0) \subsetneq A$; so $g \in A$. Hence $A = \pi^{-1}(Q)$. Thus $\pi^{-1}(P) \subsetneq \pi^{-1}(Q)$ are consecutive prime ideals of D.

(3) Suppose that $\text{char}(R) = m > 0$. Then by (1), it suffices to show that ϕ is onto. Let $Q \in \text{Spec}(D)$ and let $f \in Q$. Then by Lemma 2.2(1), $(\pi(f))^m = f^{(m)} \in Q \cap R$; so $\pi(f) \in Q \cap R$. Hence $f \in \pi^{-1}(Q \cap R)$. For the reverse containment, let $g \in \pi^{-1}(Q \cap R)$. Then $\pi(g) \in Q \cap R$; so by Lemma 2.2(1), $g^{(m)} = (\pi(g))^m \in Q$. Hence $g \in Q$. Thus $\phi(Q \cap R) = \pi^{-1}(Q \cap R) = Q$. □

3. Main results

3.1. When the composite Hurwitz series ring is a PS-ring. In this subsection, we study when the composite Hurwitz series ring is a PS-ring.

Lemma 3.1. *Let $R \subseteq T$ be an extension of commutative rings with identity and $\mathcal{M}$ a maximal ideal of $\text{H}(R,T)$. If $\text{ann}_{\text{H}(R,T)}(\mathcal{M}) = r * \text{H}(R,T)$ for some $r \in R$, then $r = 0$.*

Proof. Note that by Proposition 2.3(1), $X \in \mathcal{M}$; so $r * X = 0$. Thus $r = 0$. □

Proposition 3.2. *If $R \subseteq T$ is an extension of commutative rings with identity, then the following assertions are equivalent.*

(1) $\text{H}(R,T)$ *is a PS-ring.*

(2) $\text{ann}_{\text{H}(R,T)}(\mathcal{M}) = (0)$ *for all maximal ideals $\mathcal{M}$ of $\text{H}(R,T)$.*

Proof. (1) ⇒ (2) Suppose that $\text{H}(R,T)$ is a PS-ring and let $\mathcal{M}$ be a maximal ideal of $\text{H}(R,T)$. Then $\text{ann}_{\text{H}(R,T)}(\mathcal{M}) = r*\text{H}(R,T)$ for some idempotent element r of R [3, Lemma 4.1]. Hence $r = 0$ by Lemma 3.1. Thus $\text{ann}_{\text{H}(R,T)}(\mathcal{M}) = (0)$.

(2) ⇒ (1) This implication is obvious. □

Lemma 3.3. *Let $R \subseteq T$ be an extension of commutative rings with identity, $\pi : \text{H}(R,T) \to R$ the canonical epimorphism, $\mathcal{M}$ a maximal ideal of $\text{H}(R,T)$ and M the maximal ideal of R with $\mathcal{M} = \pi^{-1}(M)$. Then the following assertions are equivalent.*

(1) $f = \sum_{i=0}^{\infty} a_i X^i \in \text{ann}_{\text{H}(R,T)}(\mathcal{M})$.

(2) *For all $i \geq 0$, $a_i X^i \in \text{ann}_{\text{H}(R,T)}(\mathcal{M})$.*

(3) $a_0 = 0$ *and for all $i, j \geq 1$, $a_i \in \text{ann}_T(MT)$ and $\binom{i+j}{i} a_i = 0$.*

Proof. (1) ⇒ (2) Let $g = \sum_{j=0}^{\infty} b_j X^j \in \mathcal{M}$. Then $b_0 \in M \subseteq \mathcal{M}$ and $b_j X^j \in \mathcal{M}$ for all $j \geq 1$. Since $f \in \text{ann}_{\text{H}(R,T)}(\mathcal{M})$, $f * b_j X^j = 0$ for all $j \geq 0$; so $a_i X^i * b_j X^j = 0$ for all $i, j \geq 0$. Hence $a_i X^i * g = 0$ for all $i \geq 0$. Thus $a_i X^i \in \text{ann}_{\text{H}(R,T)}(\mathcal{M})$ for all $i \geq 0$.

(2) ⇒ (3) Note that by Proposition 2.3(1), $X \in \mathcal{M}$; so $a_0 * X = 0$. Thus $a_0 = 0$. Let $b \in M$. Then $b \in \mathcal{M}$; so $a_i X^i * b = 0$ for all $i \geq 1$. Hence $a_i b = 0$ for

all $i \geq 1$. Thus $a_i \in \mathrm{ann}_T(MT)$ for all $i \geq 1$. Note that by Proposition 2.3(1), $X^j \in \mathcal{M}$ for all $j \geq 1$; so $\binom{i+j}{i} a_i X^{i+j} = a_i X^i * X^j = 0$ for all $i, j \geq 1$. Thus $\binom{i+j}{i} a_i = 0$ for all $i, j \geq 1$.

(3) $\Rightarrow$ (1) Let $g = \sum_{j=0}^{\infty} b_j X^j \in \mathcal{M}$. Then $b_0 \in M$; so by the assumption, $a_i X^i * b_0 = 0$ for all $i \geq 0$. Also, by the hypothesis, $a_i X^i * b_j X^j = \binom{i+j}{i} a_i b_j X^{i+j} = 0$ for all $i \geq 0$ and $j \geq 1$. Hence $f * g = 0$. Thus $f \in \mathrm{ann}_{\mathrm{H}(R,T)}(\mathcal{M})$. □

We give the main result in this subsection.

Theorem 3.4. *If $R \subseteq T$ is an extension of commutative rings with identity, then $\mathrm{ann}_{\mathrm{H}(R,T)}(\mathcal{M}) = (0)$ for all $\mathcal{M} \in \mathrm{Max}(\mathrm{H}(R,T))$. In particular, $\mathrm{H}(R,T)$ is a PS-ring.*

Proof. Let $\mathcal{M}$ be a maximal ideal of $\mathrm{H}(R,T)$ and let $f = \sum_{i=0}^{\infty} a_i X^i$ be an element of $\mathrm{ann}_{\mathrm{H}(R,T)}(\mathcal{M})$. Then by Lemma 3.3, $a_0 = 0$ and $\binom{i+j}{i} a_i = 0$ for all $i, j \geq 1$. Fix an integer $i \geq 1$ and let $p_1, \dots, p_n$ be all the prime integers dividing $\binom{i+1}{i}$. Then $\binom{i+p_k^{i+1}}{i} \equiv 1 \pmod{p_k}$ for all $k = 1, \dots, n$ [6, Lemma 2(3)]; so $\binom{i+1}{i}, \binom{i+p_1^{i+1}}{i}, \dots, \binom{i+p_n^{i+1}}{i}$ are relatively prime. Therefore there exist $u_0, \dots, u_n \in \mathbb{Z}$ such that $u_0\binom{i+1}{i} + u_1\binom{i+p_1^{i+1}}{i} + \cdots + u_n\binom{i+p_n^{i+1}}{i} = 1$. By multiplying both sides by a_i, $a_i = 0$. Hence $f = 0$. Thus $\mathrm{ann}_{\mathrm{H}(R,T)}(\mathcal{M}) = (0)$.

The last statement follows directly from Proposition 3.2. □

By applying Theorem 3.4 to the case $R = T$, we recover

Corollary 3.5. ([1, Theorem 6.4]) *If R is a commutative ring with identity, then $\mathrm{H}(R)$ is always a PS-ring.*

3.2. When the composite Hurwitz polynomial ring is a PS-ring. In this subsection, we characterize when the composite Hurwitz polynomial ring is a PS-ring.

Lemma 3.6. *Let $R \subseteq T$ be an extension of commutative rings with identity and $\mathcal{M}$ a maximal ideal of $\mathrm{h}(R,T)$. If $\mathrm{char}(R) > 0$ and $\mathrm{ann}_{\mathrm{h}(R,T)}(\mathcal{M}) = r * \mathrm{h}(R,T)$ for some $r \in R$, then $r = 0$.*

Proof. Note that by Proposition 2.3(2), $X \in \mathcal{M}$; so $r * X = 0$. Thus $r = 0$. □

Proposition 3.7. *Let $R \subseteq T$ be an extension of commutative rings with identity. If $\mathrm{char}(R) > 0$, then the following assertions are equivalent.*

(1) *$\mathrm{h}(R,T)$ is a PS-ring.*
(2) *$\mathrm{ann}_{\mathrm{h}(R,T)}(\mathcal{M}) = (0)$ for all $\mathcal{M} \in \mathrm{Max}(\mathrm{h}(R,T))$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $\mathrm{h}(R,T)$ is a PS-ring and let $\mathcal{M}$ be a maximal ideal of $\mathrm{h}(R,T)$. Then $\mathrm{ann}_{\mathrm{h}(R,T)}(\mathcal{M}) = r * \mathrm{h}(R,T)$ for some idempotent element r of R [3, Lemma 4.1]. Hence $r = 0$ by Lemma 3.6. Thus $\mathrm{ann}_{\mathrm{h}(R,T)}(\mathcal{M}) = (0)$.

(2) $\Rightarrow$ (1) This implication is clear. □

Lemma 3.8. *Let $R \subseteq T$ be an extension of commutative rings with identity and suppose that* $\mathrm{char}(R) > 0$. *Let $\pi : \mathrm{h}(R,T) \to R$ be the canonical epimorphism, $\mathcal{M}$ a maximal ideal of* $\mathrm{h}(R,T)$ *and M the maximal ideal of R with $\mathcal{M} = \pi^{-1}(M)$. Then the following assertions are equivalent.*

(1) $f = \sum_{i=0}^{n} a_i X^i \in \mathrm{ann}_{\mathrm{h}(R,T)}(\mathcal{M})$.

(2) *For all $i = 0, \ldots, n$, $a_i X^i \in \mathrm{ann}_{\mathrm{h}(R,T)}(\mathcal{M})$.*

(3) *$a_0 = 0$ and for all $i = 1, \ldots, n$ and $j \geq 1$, $a_i \in \mathrm{ann}_T(MT)$ and $\binom{i+j}{i} a_i = 0$.*

Proof. (1) $\Rightarrow$ (2) Let $g = \sum_{j=0}^{m} b_j X^j \in \mathcal{M}$. Then $b_0 \in M \subseteq \mathcal{M}$ and $b_j X^j \in \mathcal{M}$ for all $j \in \{1, \ldots, m\}$. Since $f \in \mathrm{ann}_{\mathrm{h}(R,T)}(\mathcal{M})$, $f * b_j X^j = 0$ for all $j \in \{0, \ldots, m\}$; so $a_i X^i * b_j X^j = 0$ for all $i \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, m\}$. Hence $a_i X^i * g = 0$ for all $i \in \{0, \ldots, n\}$. Thus $a_i X^i \in \mathrm{ann}_{\mathrm{h}(R,T)}(\mathcal{M})$ for all $i \in \{0, \ldots, n\}$.

(2) $\Rightarrow$ (3) Note that by Proposition 2.3(2), $X \in \mathcal{M}$; so $a_0 * X = 0$. Thus $a_0 = 0$. Let $b \in M$. Then $b \in \mathcal{M}$; so $a_i X^i * b = 0$ for all $i = 1, \ldots, n$. Hence $a_i b = 0$ for all $i = 1, \ldots, n$. Thus $a_i \in \mathrm{ann}_T(MT)$ for all $i = 1, \ldots, n$. Note that by Proposition 2.3(2), $X^j \in \mathcal{M}$ for all $j \geq 1$; so $\binom{i+j}{i} a_i X^{i+j} = a_i X^i * X^j = 0$ for all $i = 1, \ldots, n$ and $j \geq 1$. Thus $\binom{i+j}{i} a_i = 0$ for all $i = 1, \ldots, n$ and $j \geq 1$.

(3) $\Rightarrow$ (1) Let $g = \sum_{j=0}^{m} b_j X^j \in \mathcal{M}$. Then $b_0 \in M$; so by the assumption, $a_i X^i * b_0 = 0$ for all $i = 0, \ldots, n$. Also, by the hypothesis, $a_i X^i * b_j X^j = \binom{i+j}{i} a_i b_j X^{i+j} = 0$ for all $i = 0, \ldots, n$ and $j = 1, \ldots, m$. Hence $f * g = 0$. Thus $f \in \mathrm{ann}_{\mathrm{h}(R,T)}(\mathcal{M})$. □

We next give two main results in this subsection.

Theorem 3.9. *Let $R \subseteq T$ be an extension of commutative rings with identity. If* $\mathrm{char}(R) > 0$, *then* $\mathrm{ann}_{\mathrm{h}(R,T)}(\mathcal{M}) = (0)$ *for all* $\mathcal{M} \in \mathrm{Max}(\mathrm{h}(R,T))$. *In particular,* $\mathrm{h}(R,T)$ *is a PS-ring.*

Proof. Let $\mathcal{M}$ be a maximal ideal of $\mathrm{h}(R,T)$ and let $f = \sum_{i=0}^{n} a_i X^i$ be an element of $\mathrm{ann}_{\mathrm{h}(R,T)}(\mathcal{M})$. Then by Lemma 3.8, $a_0 = 0$ and $\binom{i+j}{i} a_i = 0$ for all $i = 1, \ldots, n$ and $j \geq 1$. Let $i \in \{1, \ldots, n\}$ be fixed and let $p_1, \ldots, p_m$ be all the prime integers dividing $\binom{i+1}{i}$. Then $\binom{i+p_k^{i+1}}{i} \equiv 1 \pmod{p_k}$ for all $k = 1, \ldots, m$ [6, Lemma 2(3)]; so $\binom{i+1}{i}, \binom{i+p_1^{i+1}}{i}, \ldots, \binom{i+p_m^{i+1}}{i}$ are relatively prime. Therefore there exist $u_0, \ldots, u_m \in \mathbb{Z}$ such that $u_0 \binom{i+1}{i} + u_1 \binom{i+p_1^{i+1}}{i} + \cdots + u_m \binom{i+p_m^{i+1}}{i} = 1$. By multiplying both sides by a_i, $a_i = 0$. Hence $f = 0$. Thus $\mathrm{ann}_{\mathrm{h}(R,T)}(\mathcal{M}) = (0)$.

The last statement follows directly from Proposition 3.7. □

Corollary 3.10. ([1, Corollary 6.5]) *Let R be a commutative ring with identity. If* $\mathrm{char}(R) > 0$, *then* $\mathrm{h}(R)$ *is a PS-ring.*

Proof. The result follows from Theorem 3.9. □

Let $R \subseteq T$ be an extension of commutative rings with identity and $f = \sum_{i \geq 0} a_i X^i \in \mathrm{h}(R,T)$. By the *order* of f, we shall mean the nonnegative integer

n such that $a_i = 0$ for all $i < n$ and $a_n \neq 0$. If $f = 0$, then the order of f is defined to be ∞. We denote the order of f by $\text{ord}(f)$.

Theorem 3.11. *Let $R \subseteq T$ be an extension of commutative rings with identity. If T is a torsion-free $\mathbb{Z}$-module, then $\text{ann}_{\text{h}(R,T)}(\mathcal{M}) = (0)$ for all $\mathcal{M} \in \text{Max}(\text{h}(R,T))$. In particular, $\text{h}(R,T)$ is a PS-ring.*

Proof. Let $\mathcal{M}$ be a maximal ideal of $\text{h}(R,T)$ and let I be the ideal of T generated by the set $\{a_{\text{ord}(f)} \mid f = \sum_{i\geq 0} a_i X^i \in \mathcal{M} \setminus (0)\}$. Suppose that $I \subsetneq T$. Then we can find an element $a \in T \setminus I$; so $aX \in \text{h}(R,T) \setminus \mathcal{M}$. Since $\mathcal{M}$ is a maximal ideal of $\text{h}(R,T)$, $\mathcal{M} + (aX * \text{h}(R,T)) = \text{h}(R,T)$; so there exist $f \in \mathcal{M}$ and $g \in \text{h}(R,T)$ such that $f + aX * g = 1$. Therefore $1 = f(0) \in I$. This is absurd. Hence $I = T$. Now, we can find $t_1, \ldots, t_n \in T$ and $f_1, \ldots, f_n \in \mathcal{M}$ such that $t_1 a_1 + \cdots + t_n a_n = 1$, where for each $i \in \{1, \ldots, n\}$, a_i is the coefficient of $X^{\text{ord}(f_i)}$ in f_i. Suppose that $\text{ann}_{\text{h}(R,T)}(\mathcal{M}) \neq (0)$. Then there exists a nonzero element $g = \sum_{j=0}^{m} b_j X^j \in \text{ann}_{\text{h}(R,T)}(\mathcal{M})$; so $f_i * g = 0$ for all $i = 1, \ldots, n$. Therefore for each $i = 1, \ldots, n$, the coefficient of $X^{\text{ord}(f_i)+\text{ord}(g)}$ in $f_i * g$ is $\binom{\text{ord}(f_i)+\text{ord}(g)}{\text{ord}(f_i)} a_i b_{\text{ord}(g)} = 0$. Since T is a torsion-free $\mathbb{Z}$-module, $a_i b_{\text{ord}(g)} = 0$ for all $i = 1, \ldots, n$. Hence $b_{\text{ord}(g)} = b_{\text{ord}(g)}(t_1 a_1 + \cdots + t_n a_n) = 0$. This contradicts the fact that $b_{\text{ord}(g)} \neq 0$. Thus $\text{ann}_{\text{h}(R,T)}(\mathcal{M}) = (0)$. This means that $\text{h}(R,T)$ is a PS-ring. □

Let R be a commutative ring with identity. In [1, Theorem 6.7], it was shown that if R is both a PS-ring and a torsion-free $\mathbb{Z}$-module, then $\text{h}(R)$ is a PS-ring. By applying Theorem 3.11 to the case $R = T$, we can remove the condition "R is a PS-ring".

Corollary 3.12. (cf. [1, Theorem 6.7]) *Let R be a commutative ring with identity. If R is a torsion-free $\mathbb{Z}$-module, then $\text{h}(R)$ is a PS-ring.*

We are closing this article with the following question.

Question 3.13. If $R \subseteq T$ is an extension of commutative rings with identity, then is $\text{h}(R,T)$ always a PS-ring?

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