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A MATHEMATICAL MODEL OF A PREY-PREDATOR TYPE FISHERY IN THE PRESENCE OF TOXICITY WITH FUZZY OPTIMAL HARVESTING

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Abstract. In this paper, we have presented a multispecies prey-predator harvesting system based on Lotka–Voltera model with two competing species which are affected not only by harvesting but also by the presence of a predator, the third species. We also assume that the two competing fish species releases a toxic substance to each other. We derive the condition for global stability of the system using a suitable Lyapunov function. The possibility of existence of bionomic equilibrium is considered. The optimal harvest policy is studied and the solution is derived under imprecise inflation in fuzzy environment using Pontryagin's maximal principle. Finally some numerical examples are discussed to illustrate the model.

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1. Introduction

Recently, the effects of toxicants on ecological communities is an dynamic field of research due to the global raise of harmful phytoplankton blooms. Hallam and Clark [1], Hallam et al. [2], Hallam and De Luna [3], Freedman and Shukla [4] has started working with ecotoxicological problems. After that many researchers such as Chattopadhyay [5], Shukla and Dubey [6], Dubey and Hussain [7] and others elaborately emphasized on the studies of the ecotoxicology mathematical modeling. Though, the majority of these models deals with general single species or two-species ecological communities without any special importance on either terrestrial or aquatic environments.

The multispecies fisheries or marine aquaculture is affected by the damaging effects of toxicants from both environmental and economical point of view.

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Industries are producing a vast amount of toxicants and chemicals such as arsenic, lead, cadmium, zinc, copper, iron, mercury, etc. and released in lakes, rivers and oceans. The waste materials of industries polluted the water of lakes, rivers and oceans, which affecting the species living therein ([8], [9]). Fish, birds and mammals that feed on contaminated sea water, their life are affected by the undesirable effects of this pollution. Therefore, the effect of toxicant to the environment is causing many species to extinct and several others are on the verge of extinction. Again, there are many species in the ocean, which produces a toxin (toxin producing phytoplankton (TPP)) and toxin released by them may affect the growth of the other species significantly.

Maynard Smith [10] presented the effects of toxic substances in a two species Lotka–Volterra competitive system by considering that each species produces a substance toxic to the other only when the other is present. After that Chattopadhyay et al. [11] presented a mathematical model based on field observations in the Talsari Digha region of the Bay of Bengal in West Bengal, India and showed that toxin producing plankton may act as a biological control for planktonic blooms. Bandyopadhyay et al. [12], Abbas et al. [13] modified the Maynard-Smith [10] to a delay differential equation model. Recently, Pal and Mahapatra [14] have presented two new delay mathematical models (toxic inhibitory and toxic stimulatory) for allelopathy in the presence of two phytoplankton species.

Now we are facing several problems due to the shortage of biological resources. The main reason behind that the extensive and unregulated harvesting of marine fish, which lead to the extinction of several species. Also the modern technology in fishing power, rate of increase world population and lack of knowledge of the benefits of the exploited species among the people are all causes of exploitation of different species of fishes. Therefore to protect the exploitation of different species of fishes, optimal harvesting policy is very essential. Usually, one objective in studying marine multi-species problems is to find the conditions/constraints for bionomic equilibrium of the species and also determine the optimum harvesting policy of the species in order to maximize the present value of the revenues earned from them maintaining the ecological balance amongst the species. In 1976 Clark [15] put foundation stone in this field of work. Clark [15] presented the problem of harvesting only one of the two competing species in the model of Gause [16]. In the next year, Silvert and Smith [17] presented a similar problem. Later Mesterton-Gibbons [18], Ragozin and Brown [19], Pal et al. [20] and others studied the two species prey–predator fishery models for optimal harvesting of both the species. There are very few harvesting models with three species—prey, predator system. Kar and Chaudhuri [21] studied a prey–predator combined harvesting model of two competing species in presence of a predator.

Again the inflation and discount rates representing the time value of money are uncertain and fuzzy in nature. Therefore, the instantaneous annual rate of discount which is the difference of two fuzzy quantity inflation and discount rates is also fuzzy in nature. However, there are very few harvesting models exist with the fuzzy instantaneous annual rate of discount. Sadhukhan et al. [22] presented the optimal harvest policy under imprecise inflation in fuzzy environment using Pontryagin's maximal principle. Recently, Pal and Mahapatra [23] considered optimal harvesting policy by considering instantaneous annual rate of discount under fuzziness. However to our knowledge, yet no attempt has been made to study the optimal harvesting policy with fuzzy instantaneous annual rate of discount of two competing prey species with a predator where competing prey species releases a substance toxic to the other species as a biological measure of deterring the competitor from sharing the food resource. The major assumptions in the existing works and the current work have been summarised in Table 1.

Papers	Number of species	Effect of toxicity	Harvesting	Environment
Kar and Chaudhuri [21]	3	No	Yes	Crisp
Sadhukhan et al. [22]	3	No	Yes	Fuzzy
Abbas et al. [13]	2	Yes	No	Crisp
Pal et al. [20]	2	No	Yes	Interval
Pal and Mahapatra [23]	3	No	Yes	Interval and fuzzy
Mandal et al. [24]	2	Yes	No	Stochastic
Pal et al. [25]	2	No	Yes	Fuzzy
Pal and Mahapatra [26]	2	Yes	No	Interval
Current paper	3	Yes	Yes	Crisp and fuzzy

In the present paper, we discuss nonselective harvesting of two prey one predator fishery model, each prey species obeys the logistic law of growth. We also assume that the two prey fish species compete with each other for using a common source of food and each species releases a substance toxic to the other species as a biological measure of deterring the competitor from sharing the food resource. The predator species is also affected by consuming the toxic release by the two prey species. The species only belonging to the communities of algae and planktons releasing toxicant. It is the first time to develop a three species bioeconomic model of harvesting where the competing fish species has toxin producing interspecific reaction and predator species indirectly infected by toxic substances. The local and the global stabilities of the dynamical system for the model are examined and the existence of a bionomic equilibrium is investigated. The instantaneous annual rate of discount (δ) [19] is the difference of two imprecise quantity inflation and discount rates [27] which representing the time value of money. For this reason, here we consider δ as fuzzy in nature and presented by triangular fuzzy numbers due to intuitive, easy to use, computationally simple, and useful in promoting representation. Then optimal control problem is formulated for maximum return of revenue and solved for optimum harvesting of the species using Pontryagin's maximal principle. Lastly, numerical examples are validate to illustrate the model.

2. Basic Concept of Fuzzy Set

Zadeh [28] first introduced fuzzy sets as a mathematical way of representing the vagueness.

Definition 1. Fuzzy set: A fuzzy set $\widetilde{A}$ in a universe of discourse X is defined as the following set of pairs $\widetilde{A} = \left\{\left(x, \mu_{\widetilde{A}}(x)\right) : x \in X\right\}$. The mapping $\mu_{\widetilde{A}} : X \to [0,1]$ is called the membership function of the fuzzy set $\widetilde{A}$ and $\mu_{\widetilde{A}}(x)$ is called the membership value or degree of membership of $x \in X$ in the fuzzy set $\widetilde{A}$.

Definition 2. Triangular fuzzy number: A triangular fuzzy number (TFN) $\widetilde{A} \equiv (a_1, a_2, a_3)$ is fuzzy set of the real line $\mathbb{R}$ characterized by the membership function $\mu_{\widetilde{A}} : \mathbb{R} \to [0,1]$ as follows

$$\mu_{\widetilde{A}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, a_1 \le x \le a_2, \\ \frac{a_3-x}{a_3-a_2}, a_2 \le x \le a_3, \\ 0, \text{ otherwise} \end{cases}$$

Definition 3. α-cut of fuzzy number: The α-cut of a fuzzy number $\widetilde{A}$ is a crisp set and it is defined by $A_\alpha = \left\{x : \mu_{\widetilde{A}}(x) \ge \alpha\right\}$, $\alpha \in (0,1]$. For $\alpha = 0$ the support of $\widetilde{A}$ is defined as $A_0 = Supp\left(\widetilde{A}\right) = \overline{\left\{x \in \mathbb{R}, \mu_{\widetilde{A}}(x) > 0\right\}}$.

As per definition of TFN the α-cut is a bounded closed internal $[a_l(\alpha), a_r(\alpha)]$, where $a_l(\alpha) = \inf\left\{\mu_{\widetilde{A}}(x) \ge \alpha\right\} = a_1 + \alpha(a_2 - a_1)$ and $a_r(\alpha) = \sup\left\{\mu_{\widetilde{A}}(x) \ge \alpha\right\}$ $= a_3 - \alpha(a_3 - a_2)$.

3. Solution methodology of optimization problem

Weighted sum method

In weighted sum method [29], a utility function $Y_i(J_i)$ is defined for each objective depending on the importance of J_i compared to the other objective functions. Then a total or overall utility function Y is defined, for example, as:

$$Y(x) = \sum_{i=L,R} Y_i(J_i(x)). \tag{1}$$

The solution vector x^* is then found by maximizing the total utility $Y(x)$ subject to the underlying constraints.

We may take a suitable form of the Eq. (1) for maximization formulation as:

$$\begin{gathered} Y(x) = \sum_{i=L,R} w_i J_i(x), \\ \text{subject to } \sum_{i=L,R} w_i = 1 \text{ and } 0 < w_L, w_R < 1. \end{gathered} \tag{2}$$

Here w_L and w_R are the weights of the objective functions. Since the maximum of the above problem does not change if all the weights are multiplied by a constant, it is the usual practice to choose weights such that their sum is one.

4. Formulation of the Model

We assume two fish species which compete with each other for the use of a common resource and both of them are subjected to continuous harvesting. In addition to competition between the two fish species, each species produces a substance toxic to the other only when the other is present. There is a predator (for example a whale) feeding on both of them. It is assumed that the predator population is not harvested (for example whale harvesting has been prohibited). Thus the interaction between the harvesting agency and the predator is through the third party, namely, the prey. Also the prey is directly infected by toxic substance while the predator feeding on this infected prey is indirectly affected by the toxic substance. Since we are not making a case study in respect of a specific prey-predator community, we have opted for the logistic growth function for both the prey species and for simplicity, the feeding rate of the predator species is assumed to increase linearly with prey density. Hence, governing equations of the system can be written as

$$\begin{aligned}
\frac{dx_1}{dt} &= r_1 x_1\left(1-\frac{x_1}{k_1}\right) - \alpha_{12}x_1x_2 - \alpha_{13}x_1x_3 - \gamma_1 x_1^2 x_2 - q_1 E x_1,\\
\frac{dx_2}{dt} &= r_2 x_2\left(1-\frac{x_2}{k_2}\right) - \alpha_{21}x_1x_2 - \alpha_{23}x_2x_3 - \gamma_2 x_1 x_2^2 - q_2 E x_2,\\
\frac{dx_3}{dt} &= \alpha_{31}x_1x_3 + \alpha_{32}x_2x_3 - (1+\gamma_1+\gamma_2)x_3^2,
\end{aligned} \tag{3}$$

with initial densities $x_1(0) > 0$, $x_2(0) > 0$, $x_3(0) > 0$. Where $x_1(t)$, $x_2(t)$ and $x_3(t)$ denote the population densities of the two competing species and predator species at any time t respectively. r_1, r_2, α_{12}, α_{13}, α_{21}, α_{23}, α_{31}, α_{32}, γ_1, γ_2, $k_1$and k_2 are all positive constants. Here, r_1, r_2 represent the biotic potentials and k_1, k_2 are the carrying capacities of the two prey species. The two fish species compete for the use of an external resource as food which helps each species to grow, as per the logistic law of growth, in the absence of the other species. α_{12}, α_{21} are the coefficients of interspecific competition between the species. α_{13}, α_{23} are the predation coefficients and α_{31}, α_{32} are the conversion parameters. γ_1 and γ_2 are the co-efficients of toxicity. E is the harvesting effort. q_1, q_2 are the catchability coefficients of x_1 and x_2 species respectively.

5. Boundedness of the System

Boundedness of a system guarantees its validity. The following theorem ensures the boundedness of the system (3).

Theorem 1. Every solution of system (3) with initial conditions $x_1(0) > 0$, $x_2(0) > 0$, $x_3(0) > 0$ is positive and bounded for all $t > 0$.

Proof. Since the right hand side of system (3) is completely continuous and locally Lipschitzian on C (space of continuous functions), the solution $(x_1(t), x_2(t), x_3(t))$ of (3) with initial conditions $x_1(0) > 0$, $x_2(0) > 0$, $x_3(0) > 0$ exists and is unique on $[0,\xi)$, where $0 < \xi < +\infty$. From the first equation of

(3), we have

$$x_1(t) = x_1(0)\left[\exp\int_0^t \left\{ r_1\left(1-\frac{x_1(s)}{k_1}\right) - \alpha_{12}x_2(s) - \alpha_{13}x_3(s) - \gamma_1 x_1(s)\,x_2(s) - q_1E \right\} ds\right] > 0. \tag{4}$$

From the second equation of (3), we have

$$x_2(t) = x_2(0)\left[\exp\int_0^t \left\{ r_2\left(1-\frac{x_2(s)}{k_2}\right) - \alpha_{21}x_1(s) - \alpha_{23}x_3(s) - \gamma_2 x_1(s)\,x_2(s) - q_2E \right\} ds\right] > 0.$$

Similarly, from the third equation of (3), we have

$$x_3(t) = x_3(0)\left[\exp\int_0^t \{\alpha_{31}x_1(s) + \alpha_{32}x_2(s) - (1+\gamma_1+\gamma_2)x_3(s)\}\, ds\right] > 0.$$

From the first two equations of (3), we have

$$\frac{dx_i}{dt} \le r_i x_i\left(1-\frac{x_i}{k_i}\right),\ i = 1, 2,$$

which implies $\lim_{t\to\infty}\sup x_i(t) \le k_i$. Therefore, from the third equation of (3), we have:

$$\frac{dx_3}{dt} \le r_3x_3 - \gamma x_3^2, \tag{5}$$

where $r_3 = \alpha_{31}k_1 + \alpha_{32}k_2$. Inequation (5) can be written as

$$\frac{dx_3}{dt} \le r_3x_3\left(1-\frac{x_3}{k_3}\right), \tag{6}$$

where $k_3 = \frac{r_3}{\gamma}$. From (6): we have $\lim_{t\to\infty}\sup x_3(t) \le k_3$. Therefore, the theorem is proved. □

6. The Steady States and Their Stability

The steady states of the system (3) are $P_1(0, \overline{x}_2, \overline{x}_3)$, $P_2\left(\overline{\overline{x}}_1, 0, \overline{\overline{x}}_3\right)$ and $P_3(x_1^*, x_2^*, x_3^*)$ where

$$\overline{x}_2 = \frac{\gamma(r_2 - q_2E)}{\left(\gamma\frac{r_2}{k_2} + \alpha_{23}\alpha_{32}\right)},\ \overline{x}_3 = \frac{\alpha_{32}(r_2 - q_2E)}{\left(\gamma\frac{r_2}{k_2} + \alpha_{23}\alpha_{32}\right)}$$

$$\text{and}\quad \overline{\overline{x}}_1 = \frac{\gamma(r_1 - q_1E)}{\left(\gamma\frac{r_1}{k_1} + \alpha_{13}\alpha_{31}\right)},\ \overline{\overline{x}}_3 = \frac{\alpha_{31}(r_1 - q_1E)}{\left(\gamma\frac{r_1}{k_1} + \alpha_{13}\alpha_{31}\right)}$$

where $\gamma = (1 + \gamma_1 + \gamma_2)$. We assume here that the interior equilibrium point (x_1^*, x_2^*, x_3^*) exists. The equilibrium point P_1 exists if $E < \frac{r_2}{q_2}$ and P_2 exists if $E < \frac{r_1}{q_1}$.

6.1. Local Stability Analysis. The variational matrix corresponding to $P_1\,(0, \overline{x}_2, \overline{x}_3)$ is given by

$$V\,(0, \overline{x}_2, \overline{x}_3) = \begin{bmatrix} r_1 - \alpha_{12}\overline{x}_2 - \alpha_{13}\overline{x}_3 - q_1 E & 0 & 0 \\ -\alpha_{21}\overline{x}_2 - \gamma_2\overline{x}_2^2 & -\frac{r_2}{k_2}\overline{x}_2 & -\alpha_{23}\overline{x}_2 \\ \alpha_{31}\overline{x}_3 & \alpha_{32}\overline{x}_3 & -\gamma\overline{x}_3 \end{bmatrix}$$

The characteristic equation of the variational matrix $V\,(0, \overline{x}_2, \overline{x}_3)$ is given by

$$\begin{aligned} &(r_1 - \alpha_{12}\overline{x}_2 - \alpha_{13}\overline{x}_3 - q_1 E - \lambda) \\ &\times \left\{ \lambda^2 + \lambda\left(\frac{r_2}{k_2}\overline{x}_2 + \gamma\overline{x}_3\right) + \left(\gamma\frac{r_2}{k_2} + \alpha_{23}\alpha_{32}\right)\overline{x}_2\overline{x}_3 \right\} = 0. \end{aligned} \tag{7}$$

One root of the equation (7) i.e., one of the eigenvalues of the variational matrix $V\,(0, \overline{x}_2, \overline{x}_3)$ is given by $r_1 - \alpha_{12}\overline{x}_2 - \alpha_{13}\overline{x}_3 - q_1 E$. This eigenvalue is negative or positive according to whether $\frac{r_1}{q_1}$ is less or greater than $\frac{(\alpha_{12}\overline{x}_2 + \alpha_{13}\overline{x}_3)}{q_1} + E$. The other two eigenvalues are given by the roots of the following quadratic equation:

$$\lambda^2 + \lambda\left(\frac{r_2}{k_2}\overline{x}_2 + \gamma\overline{x}_3\right) + \left(\gamma\frac{r_2}{k_2} + \alpha_{23}\alpha_{32}\right)\overline{x}_2\overline{x}_3 = 0. \tag{8}$$

The sum of the roots of the equation (8) $= -\left(\frac{r_2}{k_2}\overline{x}_2 + \gamma\overline{x}_3\right)$which is always negative and the product of the roots $= \left(\gamma\frac{r_2}{k_2} + \alpha_{23}\alpha_{32}\right)\overline{x}_2\overline{x}_3$, which is always positive.Therefore, the roots of (8) are real and negative or complex conjugates having negative real parts. Thus P_1 is asymptotically stable only if

$$\frac{r_1}{q_1} < \frac{(\alpha_{12}\overline{x}_2 + \alpha_{13}\overline{x}_3)}{q_1} + E$$

We have already found that the steady state P_1 exists if $E < \frac{r_2}{k_2}$. Hence the condition for asymptotic stability of P_1becomes

$$\frac{r_1}{q_1} - \frac{(\alpha_{12}\overline{x}_2 + \alpha_{13}\overline{x}_3)}{q_1} < E < \frac{r_2}{q_2} \tag{9}$$

Similarly, the axial equilibrium $P_2\left(\overline{\overline{x}}_1, 0, \overline{\overline{x}}_3\right)$ is locally asymptotically stable if $\frac{r_2}{q_2} - \frac{\left(\alpha_{21}\overline{\overline{x}}_1 + \alpha_{23}\overline{\overline{x}}_3\right)}{q_2} < E < \frac{r_1}{q_1}$. Now, the characteristic equation of the variational matrix of system (3) at $P_3\,(x_1^*, x_2^*, x_3^*)$ is given by

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0 \tag{10}$$

where
$A_1 = -a_{11} - a_{22} - a_{33}$,
$A_2 = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{23}a_{32} - a_{31}a_{13} - a_{12}a_{21}$,
$A_3 = a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31} - a_{11}a_{22}a_{33} - a_{13}a_{21}a_{32} - a_{12}a_{23}a_{31}$,

and $a_{11} = -\left(\frac{r_1 x_1^*}{k_1} + \gamma_1 x_1^* x_2^*\right)$, $a_{12} = -\left(\alpha_{12} + \gamma_1 x_1^*\right) x_1^*, a_{13} = -\alpha_{13} x_1^*$, $a_{21} = -\left(\alpha_{21} + \gamma_2 x_2^*\right) x_2^*$, $a_{22} = -\left(\frac{r_2 x_2^*}{k_2} + \gamma_2 x_1^* x_2^*\right)$, $a_{23} = -\alpha_{23} x_2^*$, $a_{31} = \alpha_{31} x_3^*$, $a_{32} = \alpha_{32} x_3^*$ and $a_{33} = -\gamma x_3^*$

Here, by the Routh–Hurwitz criterion ([30]) it follows that all eigenvalues of the characteristic equation (10) has negative real part if and only if

$$A_1 > 0,\ A_3 > 0 \text{ and } A_1 A_2 - A_3 > 0 \tag{11}$$

and then $P_3\left(x_1^*, x_2^*, x_3^*\right)$ is locally asymptotically stable. Hence we have the following result:

Theorem 2. The interior equilibrium $P_3\left(x_1^*, x_2^*, x_3^*\right)$ of the system (3) is locally asymptotically stable if condition (11) is satisfied.

6.2. Global Stability Analysis. In this section, we shall prove the global stability of the system (3) by constructing a suitable Lyapunov function

Theorem 3. The interior equilibrium point $P_3\left(x_1^*, x_2^*, x_3^*\right)$ is globally asymptotically stable if

$$(i)\left\{\alpha_{12} + \alpha_{21} + \gamma_1 x_1^* + \gamma_2 x_2^*\right\}^2 > \frac{4 r_1 r_2}{k_1 k_2}\ (ii)\ \alpha_{13} = \alpha_{31}\ (iii)\ \alpha_{23} = \alpha_{32}$$

Proof. Let us consider a suitable Lyapunov function:

$$V\left(x_1, x_2, x_3\right) = \left(x_1 - x_1^*\right) - x_1^* \log\left(\frac{x_1}{x_1^*}\right) + \left(x_2 - x_2^*\right) - x_2^* \log\left(\frac{x_2}{x_2^*}\right) + \left(x_3 - x_3^*\right) - x_3^* \log\left(\frac{x_3}{x_3^*}\right).$$

It can be easily verified that the function V is zero at the equilibrium $\left(x_1^*, x_2^*, x_3^*\right)$ and is positive for all other positive values of x_1^*, x_2^* and x_3^*. The time derivative of V along with the solutions of Eq. (3) is

$$\begin{aligned}\frac{dV}{dt} &= \frac{x_1 - x_1^*}{x_1}\frac{dx_1}{dt} + \frac{x_2 - x_2^*}{x_2}\frac{dx_2}{dt} + \frac{x_2 - x_3^*}{x_3}\frac{dx_3}{dt} \\ &= -\Bigg[\left(x_1 - x_1^*\right)^2\left(\frac{r_1}{k_1} + \gamma_1 x_2\right) + \left(\alpha_{12} + \alpha_{21} + \gamma_1 x_1^* + \gamma_2 x_2^*\right) \\ &\quad \times \left(x_1 - x_1^*\right)\left(x_2 - x_2^*\right) + \left(x_2 - x_2^*\right)^2\left(\frac{r_2}{k_2} + \gamma_2 x_1\right) \\ &\quad + \left(\alpha_{23} - \alpha_{32}\right)\left(x_2 - x_2^*\right)\left(x_3 - x_3^*\right) + \gamma\left(x_3 - x_3^*\right)^2\Bigg]\end{aligned}$$

The right hand side is a quadratic form in the variables $(x_1 - x_1^*)$ and $(x_2 - x_2^*)$ and $(x_3 - x_3^*)$ which is negative definite if the matrix

$$\begin{bmatrix} \frac{r_1}{k_1} + \gamma_1 x_2 & \frac{(\alpha_{12}+\alpha_{21}+\gamma_1 x_1^*+\gamma_2 x_2^*)}{2} & \frac{(\alpha_{13}-\alpha_{31})}{2} \\ \frac{(\alpha_{12}+\alpha_{21}+\gamma_1 x_1^*+\gamma_2 x_2^*)}{2} & \frac{r_2}{k_2} + \gamma_2 x_1 & \frac{(\alpha_{23}-\alpha_{32})}{2} \\ \frac{(\alpha_{13}-\alpha_{31})}{2} & \frac{(\alpha_{23}-\alpha_{32})}{2} & \gamma \end{bmatrix}$$

is positive definite. Thus $\frac{dV}{dt} \leq 0$ if $\alpha_{13} = \alpha_{31}$, $\alpha_{23} = \alpha_{32}$ and $4\left(\frac{r_1}{k_1} + \gamma_1 x_2\right)\left(\frac{r_2}{k_2} + \gamma_2 x_1\right) \geq (\alpha_{12} + \alpha_{21} + \gamma_1 x_1^* + \gamma_2 x_2^*)^2$. This inequality is of the form $Ax_1x_2 + Bx_1 + Cx_2 + D \geq 0$, where $A = 4\gamma_1\gamma_2$, $B = \frac{4r_1\gamma_2}{k_1}$, $C = \frac{4r_2\gamma_1}{k_2}$ and $D = \frac{4r_1r_2}{k_1k_2} - (\alpha_{12} + \alpha_{21} + \gamma_1 x_1^* + \gamma_2 x_2^*)^2$. Comparing with the general equation of second degree, it can be easily shown that $Ax_1x_2+Bx_1+Cx_2+D = 0$ represent a rectangular hyperbola with asymptotes parallel to the axes. Now for $x_1 = 0$ we have:

$$x_2 = \frac{(\alpha_{12} + \alpha_{21} + \gamma_1 x_1^* + \gamma_2 x_2^*)^2 - \frac{4r_1r_2}{k_1k_2}}{\frac{4r_2\gamma_1}{k_2}}$$

and for $x_2 = 0$, we have

$$x_1 = \frac{(\alpha_{12} + \alpha_{21} + \gamma_1 x_1^* + \gamma_2 x_2^*)^2 - \frac{4r_1r_2}{k_1k_2}}{\frac{4r_1\gamma_2}{k_1}}$$

Therefore, the curve will have a branch in the positive quadrant if: $(\alpha_{12} + \alpha_{21} + \gamma_1 x_1^* + \gamma_2 x_2^*)^2 > \frac{4r_1r_2}{k_1k_2}$. Hence the equilibrium point $P_3\,(x_1^*, x_2^*, x_3^*)$ is globally asymptotically stable if the conditions $(i)\,\{\alpha_{12} + \alpha_{21} + \gamma_1 x_1^* + \gamma_2 x_2^*\}^2 > \frac{4r_1r_2}{k_1k_2}$ $(ii)\,\alpha_{13} = \alpha_{31}$ $(iii)\,\alpha_{23} = \alpha_{32}$ hold. This completes the proof. □

7. Bionomic Aspect of the Model

The term bionomic equilibrium is a combination of the concepts of biological equilibrium as well as economic equilibrium. As we already saw, a biological equilibrium is given by $\dot{x_1} = 0$, $\dot{x_2} = 0$ and $\dot{x_3} = 0$. The economic equilibrium is said to be achieved when TR (the total revenue obtained by selling the harvested biomass) equals TC (the total cost for the effort devoted to harvesting).

Let c is the constant fishing cost per unit effort, p_1 constant price per unit biomass of the first prey species, and p_1 is the constant price per unit biomass of the second prey species.

The economic rent (net revenue) at any time is given by

$$\pi\,(x_1, x_2, x_3, E) = TR - TC = (p_1q_1x_1 + p_2q_2x_2 - c)\,E. \tag{12}$$

Now,

$$\dot{x_1} = 0 \Rightarrow x_1 = 0 \text{ or } E = \frac{r_1}{q_1} - \frac{r_1}{k_1q_1}x_1 - \frac{\alpha_{12}}{q_1}x_2 - \frac{\alpha_{13}}{q_1}x_3 - \frac{\gamma_1}{q_1}x_1x_2$$

$$\dot{x_2} = 0 \Rightarrow x_2 = 0 \text{ or } E = \frac{r_2}{q_2} - \frac{r_2}{k_2 q_2} x_2 - \frac{\alpha_{21}}{q_2} x_1 - \frac{\alpha_{23}}{q_2} x_3 - \frac{\gamma_2}{q_2} x_1 x_2$$

$$\dot{x_3} = 0 \Rightarrow x_3 = 0 \text{ or } x_3 = \frac{1}{\gamma}\left(\alpha_{31} x_1 + \alpha_{32} x_2\right)$$

Hence the nontrivial biological equilibrium solution occurs at a point on the line

$$\begin{cases} \left(\frac{r_1}{k_1 q_1} - \frac{\alpha_{21}}{q_2}\right) x_1 - \left(\frac{r_2}{k_2 q_2} - \frac{\alpha_{12}}{q_1}\right) x_2 \\ + \left(\frac{\gamma_1}{q_1} - \frac{\gamma_2}{q_2}\right) x_1 x_2 + \left(\frac{\alpha_{13}}{q_1} - \frac{\alpha_{23}}{q_2}\right) x_3 + \left(\frac{r_2}{q_2} - \frac{r_1}{q_1}\right) = 0 \\ \\ \alpha_{31} x_1 + \alpha_{32} x_2 - \gamma x_3 = 0 \end{cases} \tag{13}$$

where $0 \le x_1 \le k_1$ and $0 \le x_2 \le k_2$.

The equilibrium line (13) meets the plane $x_1 = 0$ at $(0, \widetilde{x}_2, \widetilde{x}_3)$ where

$$\widetilde{x}_2 = \frac{\frac{r_2}{q_2} - \frac{r_1}{q_1}}{\left(\frac{r_2}{k_2 q_2} + \frac{\alpha_{23}\alpha_{32}}{\gamma q_2}\right) - \left(\frac{\alpha_{12}}{q_1} + \frac{\alpha_{13}\alpha_{32}}{\gamma q_1}\right)} \text{ and } \widetilde{x}_3 = \frac{\alpha_{32}}{\gamma} \widetilde{x}_2$$

provided either

$$(a)\ \frac{r_2}{q_2} > \max\left\{\frac{k_2 \alpha_{12}}{q_1}, \frac{r_1}{q_1}\right\} \text{ and } \frac{\alpha_{23}}{q_2} > \frac{\alpha_{13}}{q_1}$$

$$(b)\ \frac{r_2}{q_2} < \min\left\{\frac{k_2 \alpha_{12}}{q_1}, \frac{r_1}{q_1}\right\} \text{and } \frac{\alpha_{23}}{q_2} < \frac{\alpha_{13}}{q_1}$$

Similarly, (13) meets the plane $x_2 = 0$ at $(\widetilde{x}_1, 0, \widetilde{x}_3)$ where

$$\widetilde{x}_1 = \frac{\frac{r_2}{q_2} - \frac{r_1}{q_1}}{\left(\frac{\alpha_{21}}{q_2} + \frac{\alpha_{23}\alpha_{31}}{\gamma q_2}\right) - \left(\frac{r_1}{k_1 q_1} + \frac{\alpha_{13}\alpha_{31}}{\gamma q_1}\right)} \text{ and } \widetilde{x}_3 = \frac{\alpha_{31}}{\gamma} \widetilde{x}_1$$

provided either

$$(c)\ \frac{r_1}{q_1} > \max\left\{\frac{k_1 \alpha_{21}}{q_2}, \frac{r_2}{q_2}\right\} \text{ and } \frac{\alpha_{13}}{q_1} > \frac{\alpha_{23}}{q_2} \text{ or}$$

$$(d)\ \frac{r_1}{q_1} < \min\left\{\frac{k_1 \alpha_{21}}{q_2}, \frac{r_2}{q_2}\right\} \text{ and } \frac{\alpha_{13}}{q_1} < \frac{\alpha_{23}}{q_2}$$

The bionomic equilibrium of the open-access fishery is determined by Eq. (13) together with the condition:

$$\pi\left(x_1, x_2, x_3, E\right) = \left(p_1 q_1 x_1 + p_2 q_2 x_2 - c\right) E = 0, \tag{14}$$

we refer to Eq. (14) as the zero profit line. Therefore, the bionomic equilibrium $R\left(x_{1\infty}, x_{2\infty}, x_{3\infty}\right)$ will be the point of intersection (if it exists) of (13) and the zero profit line (14).

Eliminating x_2 and x_3 from Eqs. (13) and (14), we get:

$$A_4 x_1^2 + B_2 x_1 + C_2 = 0, \tag{15}$$

where

$$A_4 = \left(\frac{\gamma_2}{q_2} - \frac{\gamma_1}{q_1}\right)\frac{p_1 q_1}{p_2 q_2},$$

$$B_2 = \frac{p_1 q_1}{p_2 q_2}\left(\frac{r_2}{k_2 q_2} - \frac{\alpha_{12}}{q_1}\right) - \left\{\frac{\alpha_{21}}{q_2} - \frac{r_1}{k_1 q_1} + \frac{c}{p_2 q_2}\left(\frac{\gamma_2}{q_2} - \frac{\gamma_1}{q_1}\right) + \left(\frac{\alpha_{23}}{q_2} - \frac{\alpha_{13}}{q_1}\right)\left(\frac{\alpha_{31}}{\gamma} - \frac{\alpha_{32} p_1 q_1}{\gamma p_2 q_2}\right)\right\}$$

and

$$C_2 = \frac{c}{p_2 q_2}\left(\frac{\alpha_{12}}{q_1} - \frac{r_2}{k_2 q_2}\right) + \left(\frac{r_2}{q_2} - \frac{r_1}{q_1}\right) + \frac{c\alpha_{32}}{\gamma p_2 q_2}\left(\frac{\alpha_{13}}{q_1} - \frac{\alpha_{23}}{q_2}\right)$$

In Eq. (15), we have sum of the roots $= -\left(\frac{B_2}{A_4}\right)$ and product of the roots $= -\left(\frac{C_2}{A_4}\right)$.

Now the following cases may arise.

$\frac{\gamma_2}{q_2} > \frac{\gamma_1}{q_1}$

In this case $A_4 > 0$. We have one positive root when $C_2 < 0$. Then we must have:

$$\frac{k_2 \alpha_{12}}{q_2} < \frac{r_2}{q_2} < \frac{r_1}{q_1} \text{ and } \frac{\alpha_{13}}{q_1} < \frac{\alpha_{23}}{q_2}$$

$\frac{\gamma_2}{q_2} < \frac{\gamma_1}{q_1}$

In this case $A_4 > 0$. We have one positive root when $C_2 > 0$. Then we must have: $\frac{k_2 \alpha_{12}}{q_2} < \frac{r_2}{q_2} < \frac{r_1}{q_1}$ and $\frac{\alpha_{13}}{q_1} < \frac{\alpha_{23}}{q_2}$

In both cases, $p_2 q_2 x_{2\infty} = c - p_1 q_1 x_{1\infty}$. Therefore,

$$x_{2\infty} = \frac{c - p_1 q_1 x_{1\infty}}{p_2 q_2} > 0, \text{ provided } x_{1\infty} < \frac{c}{p_1 q_1} \text{ and } x_{3\infty} = \frac{\alpha_{31} x_{1\infty} + \alpha_{32} x_{2\infty}}{\gamma}$$

8. Optimal Harvesting Policy under Fuzziness

Let $\widetilde{k}$ and $\widetilde{r}$ be the inflation and discount rates representing the time value of money and these are fuzzy in nature (see Refs [22], [23], [31]). The present value J of continuous time-stream of revenues is given by

$$\widetilde{J} = \int_0^\infty e^{-\widetilde{\delta} t}\left\{p_1 q_1 x_1 + p_2 q_2 x_2 - c\right\} E(t)\, dt \tag{16}$$

where $\widetilde{\delta} = \widetilde{r} - \widetilde{k}$ is the fuzzy net discount rate of inflation and considered as triangular fuzzy number i.e., $\widetilde{\delta} = (\delta_1, \delta_2, \delta_3)$. Our problem is to maximise $\widetilde{J}$ subject to the state equations (3) by invoking Pontryagin's maximal principle ([32]). The control variable $E(t)$ is subjected to the constraints $0 \le E(t) \le E_{\max}$, so that $V_t = [0, E_{\max}]$ is the control set. Following Maiti and Maiti [27], Sadhukhan et al. [22] and Pal and Mahapatra [23] the fuzzy number $\widetilde{\delta} = (\delta_1, \delta_2, \delta_3)$ is expressed as an interval number $[\delta_L, \delta_R]$ where $\delta_L = \delta_1 + \alpha(\delta_2 - \delta_1)$,

$\delta_R = \delta_3 - \alpha(\delta_3 - \delta_2)$ and $0 \le \alpha \le 1$. Now, the corresponding integral (16) which is to be maximized is expressed as

$$Max\,[J_L, J_R] = \int_0^\infty e^{-[\delta_L,\delta_R]t}\,\{p_1 q_1 x_1 + p_2 q_2 x_2 - c\}\,E(t)\,dt, \tag{17}$$

where

$$\begin{aligned} J_L &= \int_0^\infty e^{-\delta_R t}\,\{p_1 q_1 x_1 + p_2 q_2 x_2 - c\}\,E(t)\,dt, \\ J_R &= \int_0^\infty e^{-\delta_L t}\,\{p_1 q_1 x_1 + p_2 q_2 x_2 - c\}\,E(t)\,dt. \end{aligned} \tag{18}$$

We can write

$$MaxJ = Max\,[J_L, J_R] = w_1 J_L + w_2 J_R \tag{19}$$

where w_1, w_2 are two weights such that $w_1 + w_2 = 1$ and $w_1, w_2 \ge 0$.The Hamiltonian is given by

$$\begin{aligned} H = {} & \left(w_1 e^{-\delta_R t} + w_2 e^{-\delta_L t}\right)(p_1 q_1 x_1 + p_2 q_2 x_2 - c)\,E \\ & + \lambda_1\left(r_1 x_1\left(1 - \frac{x_1}{k_1}\right) - \alpha_{12}x_1x_2 - \alpha_{13}x_1x_3 - \gamma_1 x_1^2 x_2 - q_1 E x_1\right) \\ & + \lambda_2\left(r_2 x_2\left(1 - \frac{x_2}{k_2}\right) - \alpha_{21}x_1x_2 - \alpha_{23}x_2x_3 - \gamma_2 x_1 x_2^2 - q_2 E x_2\right) \\ & + \lambda_3\left(\alpha_{31}x_1x_3 + \alpha_{32}x_2x_3 - \gamma x_3^2\right), \end{aligned} \tag{20}$$

where $\lambda_i = \lambda_i(t)\,(i = 1, 2, 3)$ are the adjoint variables. The adjoint equations are

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x_1},\ \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial x_2},\ \frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial x_3} \tag{21}$$

Therefore,

$$\begin{aligned} \frac{d\lambda_1}{dt} = -\Bigg\{ & p_1 q_1\left(w_1 e^{-\delta_R t} + w_2 e^{-\delta_L t}\right)E \\ & + \lambda_1\left(r_1 - \frac{2x_1}{k_1} - \alpha_{12}x_2 - 2\gamma_1 x_1 x_2 - q_1 E\right) \\ & - \lambda_2\left(\alpha_{21}x_2 + \gamma_2 x_2^2\right) + \alpha_{31}x_3\lambda_3 \Bigg\} \end{aligned} \tag{22}$$

$$\begin{aligned} \frac{d\lambda_2}{dt} = -\Bigg\{ & p_2 q_2\left(w_1 e^{-\delta_R t} + w_2 e^{-\delta_L t}\right)E - \lambda_1\left(\alpha_{12}x_1 + \gamma_1 x_1^2\right) \\ & + \lambda_2\left(r_2 - \frac{2x_2}{k_2} - \alpha_{21}x_1 - \alpha_{23}x_3 - 2\gamma_2 x_1 x_2 - q_2 E\right) + \alpha_{32}x_3\lambda_3 \Bigg\} \end{aligned} \tag{23}$$

$$\frac{d\lambda_3}{dt} = -\left\{-\alpha_{13}x_1\lambda_1 - \alpha_{23}x_2\lambda_2 + \lambda_3\left(\alpha_{31}x_1 + \alpha_{32}x_2 - 2\gamma x_3\right)\right\}. \tag{24}$$

At the equilibria (22)–(24) become

$$\begin{aligned}\frac{d\lambda_1}{dt} &= \lambda_1\left(\frac{r_1}{k_1}+\gamma_1 x_1\right)x_2+\lambda_2\left(\alpha_{21}+\gamma_2 x_2\right)x_2\\ &\quad-\lambda_3\alpha_{31}x_3-p_1q_1\left(w_1e^{-\delta_R t}+w_2e^{-\delta_L t}\right)E,\end{aligned} \tag{25}$$

$$\begin{aligned}\frac{d\lambda_2}{dt} &= \lambda_1\left(\alpha_{12}+\gamma_1 x_1\right)x_1+\lambda_2\left(\frac{r_2}{k_2}+\gamma_2 x_1\right)x_2\\ &\quad-\lambda_3\alpha_{32}x_3-p_2q_2\left(w_1e^{-\delta_R t}+w_2e^{-\delta_L t}\right)E,\end{aligned} \tag{26}$$

$$\frac{d\lambda_3}{dt}=\lambda_1\alpha_{13}x_1+\lambda_2\alpha_{23}x_2+\lambda_3\gamma x_3. \tag{27}$$

Now, by eliminating λ_1 and λ_2 from (25)–(27), we get a reduced differential equation for λ_3 as

$$\left(a_0D^3+a_1D^2+a_2D+a_3\right)\lambda_3=M_{3L}e^{-\delta_R t}+M_{3R}e^{-\delta_L t}, \tag{28}$$

where $D\equiv\frac{d}{dt}$, $a_0=1$, $a_1=-\left(\frac{r_1x_1}{k_1}+\frac{r_2x_2}{k_2}+\left(\gamma_1+\gamma_2\right)x_1x_2\right)$,

$$\begin{aligned}a_2 &= \left\{\left(\tfrac{r_1}{k_1}+\gamma_1x_2\right)\left(\tfrac{r_2}{k_2}+\gamma_2x_1\right)-\left(\alpha_{21}+\gamma_1x_1\right)\right\}x_1x_2\\ &\quad+\left\{\gamma\left(\tfrac{r_1}{k_1}+\gamma_1x_2\right)+\alpha_{13}\alpha_{31}\right\}x_1x_3+\alpha_{23}\alpha_{32}x_2x_3,\end{aligned}$$

$$\begin{aligned}a_3 = \Big[&\gamma\left(\tfrac{r_1}{k_1}+\gamma_1x_2\right)\left(\tfrac{r_2}{k_2}+\gamma_2x_1\right)+\alpha_{23}\alpha_{32}\left(\tfrac{r_1}{k_1}+\gamma_1x_2\right)\\ &-\gamma\left(\alpha_{21}+\gamma_2x_2\right)\left(\alpha_{12}+\gamma_1x_1\right)-\alpha_{13}\alpha_{32}\left(\alpha_{21}+\gamma_2x_2\right)\\ &-\alpha_{23}\alpha_{31}\left(\alpha_{12}+\gamma_1x_1\right)+\alpha_{13}\alpha_{31}\left(\tfrac{r_2}{k_2}+\gamma_2x_1\right)\Big]x_1x_2x_3.\end{aligned}$$

$$\begin{aligned}M_{3L} &= w_1E\Big[p_1q_1\{\alpha_{23}\delta_Rx_2+\{-\alpha_{13}\left(\alpha_{21}+\gamma_2x_2\right)+\left(\tfrac{r_1}{k_1}+\gamma_1x_2\right)\alpha_{23}\}x_1x_2\}\\ &+p_2q_2\{\alpha_{13}\delta_Rx_1+\{-\alpha_{23}\left(\alpha_{12}+\gamma_1x_1\right)+\left(\tfrac{r_2}{k_2}+\gamma_2x_1\right)\alpha_{13}\}x_1x_2\}\Big],\\ M_{3R} &= w_2E\Big[p_1q_1\{\alpha_{23}\delta_Lx_2+\{-\alpha_{13}\left(\alpha_{21}+\gamma_2x_2\right)+\left(\tfrac{r_1}{k_1}+\gamma_1x_2\right)\alpha_{23}\}x_1x_2\}\\ &+p_2q_2\{\alpha_{13}\delta_Lx_1+\{-\alpha_{23}\left(\alpha_{12}+\gamma_1x_1\right)+\left(\tfrac{r_2}{k_2}+\gamma_2x_1\right)\alpha_{13}\}x_1x_2\}\Big]\end{aligned}$$

The complete solution of (28) is

$$\lambda_3=A_1e^{m_1t}+A_2e^{m_2t}+A_3e^{m_3t}+\frac{M_{3L}}{N_L}e^{-\delta_R t}+\frac{M_{3R}}{N_R}e^{-\delta_L t}, \tag{29}$$

where A_i $(i=1,2,3)$ are arbitrary constants and $m_i(i=1,2,3)$ are the roots of the auxiliary equations $a_0m^3+a_1m^2+a_2m+a_3=0$ and $N_L=-\left(a_0\delta_R^3-a_1\delta_R^2+a_2\delta_R-a_3\right)\neq 0$, $N_R=-\left(a_0\delta_L^3-a_1\delta_L^2+a_2\delta_L-a_3\right)\neq 0$. It is clear from (29) that λ_3 is bounded if and only if $m_i<0$ $(i=1,2,3)$ or the A_i's are identically equal to zero. It being very difficult to check whether $m_i<0$, we take $A_i=0$ $(i=1,2,3)$. Then

$$\lambda_3=\frac{M_{3L}}{N_L}e^{-\delta_R t}+\frac{M_{3R}}{N_R}e^{-\delta_L t}.$$

Similarly, we get

$$\lambda_2 = \frac{M_{2L}}{N_L} e^{-\delta_R t} + \frac{M_{2R}}{N_R} e^{-\delta_L t} \text{ and } \lambda_1 = \frac{M_{1L}}{N_L} e^{-\delta_R t} + \frac{M_{1R}}{N_R} e^{-\delta_L t},$$

where $M_{2L} = w_1 E[p_1 q_1 \{x_1(\alpha_{12} + \gamma_1 x_1)\delta_R + (\gamma((\alpha_{12} + \gamma_1 x_1) + \alpha_{13}\alpha_{23})x_1 x_3)\} - p_2 q_2 \{\delta_R^2 + (\gamma x_3 + x_1(\frac{r_1}{k_1} + \gamma_1 x_2))\delta_R + (\gamma(\frac{r_1}{k_1} + \gamma_1 x_2) + \alpha_{13}\alpha_{23})x_1 x_3\}$, $M_{2L} = w_2 E[p_1 q_1 \{x_1(\alpha_{12} + \gamma_1 x_1)\delta_L + (\gamma((\alpha_{12} + \gamma_1 x_1) + \alpha_{13}\alpha_{23})x_1 x_3)\} - p_2 q_2 \{\delta_L^2 + (\gamma x_3 + x_1(\frac{r_1}{k_1} + \gamma_1 x_2))\delta_L + (\gamma(\frac{r_1}{k_1} + \gamma_1 x_2) + \alpha_{13}\alpha_{23})x_1 x_3\}$, $M_{1L} = w_1 E[p_2 q_2 \{x_2(\alpha_{21} + \gamma_2 x_2)\delta_R + (\gamma((\alpha_{21} + \gamma_2 x_2) + \alpha_{23}\alpha_{31})x_2 x_3)\} - p_1 q_1 \{\delta_R^2 + (\gamma x_3 + x_2(\frac{r_2}{k_2} + \gamma_2 x_1))\delta_R + (\gamma(\frac{r_2}{k_2} + \gamma_2 x_1) + \alpha_{23}\alpha_{31})x_2 x_3\}$ and $M_{1R} = w_1 E[p_2 q_2 \{x_2(\alpha_{21} + \gamma_2 x_2)\delta_L + (\gamma((\alpha_{21} + \gamma_2 x_2) + \alpha_{23}\alpha_{31})x_2 x_3)\} - p_1 q_1 \{\delta_L^2 + (\gamma x_3 + x_2(\frac{r_2}{k_2} + \gamma_2 x_1))\delta_L + (\gamma(\frac{r_2}{k_2} + \gamma_2 x_1) + \alpha_{23}\alpha_{31})x_2 x_3\}$.

We find the shadow prices $\lambda_i e^{\delta_L t}$ and $\lambda_i e^{\delta_R t}$ $(i = 1, 2, 3)$ of the three species remain bounded as $t \to \infty$ and hence satisfy the transversality condition at ∞ [15]. The Hamiltonian in (20) must be maximised for $E \in [0, E_{\max}]$. Assuming that the control constraints $0 \le E \le E_{\max}$ are not binding (that is, the optimal equilibrium does not occur either at $E = 0$ or $E = E_{\max}$) so we consider the singular control

$$\begin{aligned} &\tfrac{\partial H}{\partial E} = \left(w_1 e^{-\delta_R t} + w_2 e^{-\delta_L t}\right)(p_1 q_1 x_1 + p_2 q_2 x_2 - c) - \lambda_1 q_1 x_1 - \lambda_2 q_2 x_2 = 0, \\ &\Rightarrow \left(w_1 e^{-\delta_R t} + w_2 e^{-\delta_L t}\right)(p_1 q_1 x_1 + p_2 q_2 x_2 - c) = \lambda_1 q_1 x_1 + \lambda_2 q_2 x_2, \end{aligned} \tag{30}$$

or

$$\left(w_1 e^{-\delta_R t} + w_2 e^{-\delta_L t}\right) \frac{d\pi}{dE} = \lambda_1 q_1 x_1 + \lambda_2 q_2 x_2.$$

This indicates that the total user cost of harvest per unit effort must be equal to the discounted value of the future profit at the steady state effort level [15]. Substituting λ_1 and λ_2 into (30) we get

$$M_L e^{-\delta_R t} + M_R e^{-\delta_L t} = c\left(w_1 e^{-\delta_R t} + w_2 e^{-\delta_L t}\right), \tag{31}$$

where $M_L = \left(p_1 w_1 - \frac{M_{1L}}{N_L}\right) q_1 x_1 + \left(p_2 w_1 - \frac{M_{2L}}{N_L}\right) q_2 x_2$ and
$M_R = \left(p_1 w_2 - \frac{M_{1R}}{N_R}\right) q_1 x_1 + \left(p_2 w_2 - \frac{M_{2R}}{N_R}\right) q_2 x_2$.

From equation (31), we have

$$\begin{aligned} &e^{-\delta_L t}\left(M_L e^{-(\delta_R - \delta_L)t} + M_R\right) = c e^{-\delta_L t}\left(w_1 e^{-(\delta_R - \delta_L)t} + w_2\right) \\ &\Rightarrow \left(M_L e^{-(\delta_R - \delta_L)t} + M_R\right) = c\left(w_1 e^{-(\delta_R - \delta_L)t} + w_2\right) \end{aligned} \tag{32}$$

when $[\delta_L, \delta_R] \to \infty \Rightarrow \delta_L \to \infty$, $\delta_R \to \infty$ and $(\delta_R - \delta_L)$ is tending to ∞. So when $\widetilde{\delta} \to \infty$ then (32) becomes

$$M_R = c w_2 \tag{33}$$

The value of E at the interior equilibrium is given by

$$E = \frac{r_1}{q_1} - \frac{r_1 x_1}{q_1 k_1} - \frac{\alpha_{12}}{q_1} x_2 - \frac{\alpha_{13}}{q_1} x_3 - \frac{\gamma_1}{q_1} x_1 x_2 = \frac{r_2}{q_2} - \frac{r_2 x_2}{q_2 k_2} - \frac{\alpha_{21}}{q_2} x_1 - \frac{\alpha_{23}}{q_2} x_3 - \frac{\gamma_2}{q_2} x_1 x_2 \tag{34}$$

substituting the value of E in the expressions for M_{1L}, M_{1R}, M_{2L}, M_{2R} then solving (33) and (3) to obtain the optimal equilibrium solution $x_1 = x_{1\widetilde{\delta}}$, $x_1 = x_{1\widetilde{\delta}}$, $x_3 = x_{3\widetilde{\delta}}$ for a given value of $\widetilde{\delta}$. Since $\frac{M_{iR}}{N_R} = O\left(\delta_L^{-1}\right)$, $i = 1, 2$. So $\frac{M_{iR}}{N_R} \to 0$ as $\delta_L \to \infty$. Therefore equ. (32) becomes

$$p_1 w_2 q_1 x_1 + p_2 w_2 q_2 x_2 = c w_2, \Rightarrow p_1 q_1 x_1 + p_2 q_2 x_2 = c. \tag{35}$$

Therefore we have,

$$\pi\left(x_{1\infty}, x_{2\infty}, x_{3\infty}, E_\infty\right) = 0,$$

This indicates that, an infinite inflation rate leads to complete dissipation of economic revenue. Again,

$$M_R - c w_2 = \left(p_1 w_2 - \frac{M_{1R}}{N_R}\right) q_1 x_1 + \left(p_2 w_2 - \frac{M_{2R}}{N_R}\right) q_2 x_2 - c w_2 = 0$$
$$\Rightarrow \pi = \frac{1}{w_2 N_R}\left[M_{1R} q_1 x_1 + M_{2R} q_2 x_2\right]$$

where we note that each of M_{1R}, M_{2R} are of $O\left(\delta_L^2\right)$ and N_R is of $O\left(\delta_L^3\right)$ so that π is a decreasing function of $\delta_L\,(\geq 0)$. We therefore conclude $\delta_L = 0$ leads to the maximization of π .

9. Numerical Simulation

Numeric results of the proposed model are verified in the following four cases (i) $\gamma_1 < \gamma_2$ (ii) $\gamma_1 > \gamma_2$ $(iii)\,\gamma_1 = \gamma_2 \neq 0$ $(iv)\,\gamma_1 = \gamma_2 = 0$. In order to illustrate steady states of the model, bionomic equilibria and optimal harvesting at equilibrium, we present two numerical examples. Parameters of the system (3) are taken as follows: $r_1 = 3.2$, $r_2 = 2.5$, $k_1 = 100$, $k_2 = 80$, $\alpha_{12} = 0.006$, $a_{13} = 0.04$, $a_{21} = 0.002$, $a_{23} = 0.03$, $a_{31} = 0.12$, $a_{32} = 0.31$, $E = 15$, $q_1 = 0.02$, $q_2 = 0.02$, $\gamma_1 = 0.008$, $\gamma_2 = 0.004$ $(\gamma_1 > \gamma_2)$, $\gamma_1 = 0.004$, $\gamma_2 = 0.008$ $(\gamma_1 < \gamma_2)$, $\gamma_1 = 0.004$, $\gamma_2 = 0.008$ $(\gamma_1 = \gamma_2 \neq 0)$ in appropriate units.

The non-trivial steady states and their nature of the proposed prey–predator model for different effects of toxicity is presented in Table 2.

Table 2: Equilibrium point, eigenvalues and their nature of system (3)

Toxicity coefficient	Equilibrium point	Eigenvalues	Nature
$\gamma_1 > \gamma_2$	$(9.25, 27.74, 9.59)$	$-27.93, -1.52, -0.28$	stable
$\gamma_1 < \gamma_2$	$(53.23, 4.08, 7.56)$	$-7.03, -4.15, -0.91$	stable
$\gamma_1 = \gamma_2 \neq 0$	$(27.96, 7.74, 5.66)$	$-0.49, -4.9 \pm 0.7i$	stable
$\gamma_1 = \gamma_2 = 0$	$(55.51, 46.59, 21.10)$	$-20.30, -2.58, -1.44$	stable

From Table 2, we observe that if the toxicity coefficient $\gamma_i\,(i = 1, 2)$ of one competing species is greater than the other species, the corresponding steady state level of that species is lower than other species as expected. Also, when $\gamma_i\,(i = 1, 2)$ are equal but not equal to zero, the steady state level of first prey species is much more greater than the second prey species and the predator species. Again, in the absence of toxic substances (i.e., $\gamma_1 = 0, \gamma_2 = 0$) the steady state level of all the species are increases which is biologically relevant. It is also found from Table 2 that in any case the eigenvalues of the characteristic polynomial of system (3) are always negative. Hence, by Theorem 5.3 the interior equilibrium point (x_1^*, x_2^*, x_3^*) is always stable.

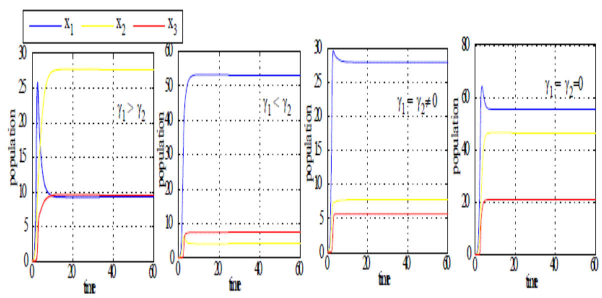

FIGURE 1. Time series plot of the three species population (x_1, x_2, x_3) with different kinds of toxicity and initial condition $(x_1(0), x_2(0), x_3(0)) = (0.05, 0.05, 0.05)$

From Fig. 1, we notice that in any case ($\gamma_1 > \gamma_2, \gamma_1 < \gamma_2, \gamma_1 = \gamma_2 \neq 0, \gamma_1 = \gamma_2 = 0$) as the time increases population converges to their equilibrium. The phase-space trajectories corresponding to the stabilities of the populations for different initial conditions is presented in Fig. 2.

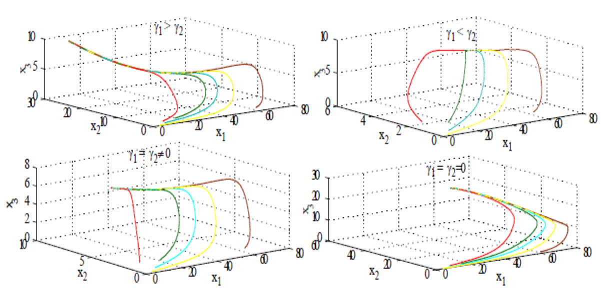

FIGURE 2. Phase portrait of the system (3) with different effect of toxicity beginning with different initial levels

The trajectories in Fig. 2 specify that the steady state (x_1^*, x_2^*, x_3^*), in any case, is globally asymptotically stable for different initial populations. Taking fixed the coefficient γ_1 in certain level, the variation of the three species with respect to γ_2 is depicted in Fig 3.

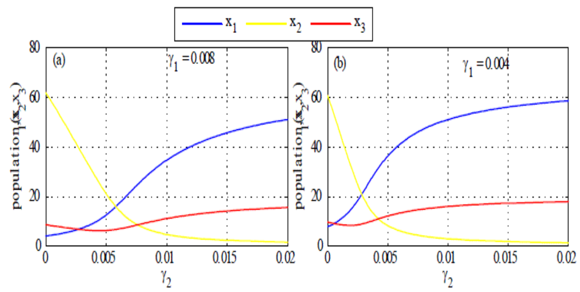

FIGURE 3. Variations of species with γ_2 when γ_1 fixed at (a) 0.008 and (b) 0.004

Fig. 3 shows that, as γ_2 increases with fixed $\gamma_1$at 0.008 or 0.004 the second prey species population (x_2) goes to extinct, i.e. the system approaches to the steady state level $P_2\left(\overline{\overline{x}}_1, 0, \overline{\overline{x}}_3\right)$. Similarly if converse case is happened, i.e., taking $\gamma_2 = 0.008$ or 0.004 then the variation of the three species with respect to γ_1 is presented through Fig 4.

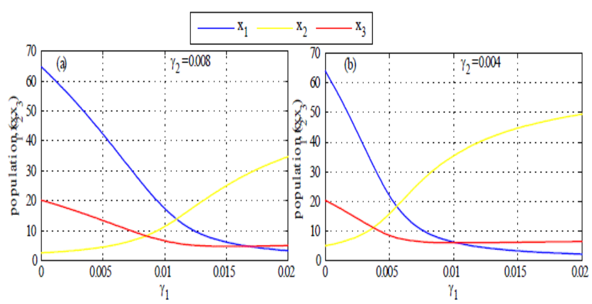

FIGURE 4. Variations of species with γ_1 when γ_2 fixed at (a) 0.008 and (b) 0.004

From Fig. 4 we observe that, as γ_1 increases the first prey species (x_1) gradually decreases and goes to extinct, i.e. the system approaches to the steady state level $P_1\left(0, \overline{x}_2, \overline{x}_3\right)$. Again the dynamical behaviour of the system (3) for different kinds of toxicity with the given values of the parameters is presented through Fig. 5

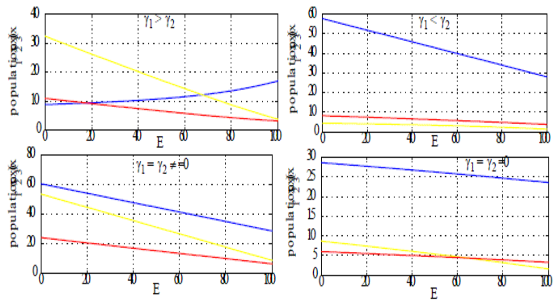

FIGURE 5. Dynamical behaviour of the three species population (x_1, x_2, x_3) with respect to E with given values of biological parameters and $q_1 = 0.02$, $q_2 = 0.02$ for different effects of toxicity

Figure 5 shows the dynamical behaviour of the three species population (x_1, x_2, x_3) with respect to the harvesting effort E with given values of the biological parameters and the initial condition $(x_1(0), x_2(0), x_3(0)) = (0.05, 0.05, 0.05)$. This figure shows that the equilibrium point changes for different values of E.

Taking the same values of the parameters together with $p_1 = 4$, $p_2 = 6$, $c = 3$, the bionomic equilibrium $R(x_{1\infty}, x_{2\infty}, x_{3\infty})$ for different kinds of toxicant effort are presented in Table 3. The nontrivial steady state is the same as in Table 2.

Table 3: Bionomic equilibrium for different efforts of toxicity of system (3)

Toxicity coefficient	$R(x_{1\infty}, x_{2\infty}, x_{3\infty})$
$\gamma_1 > \gamma_2$	$(10.68, 17.80, 6.74)$
$\gamma_1 < \gamma_2$	$(34.07, 2.29, 4.74)$
$\gamma_1 = \gamma_2 \neq 0$	$(27.29, 6.80, 5.30)$
$\gamma_1 = \gamma_2 = 0$	$(27.27, 6.81, 5.38)$

From the Tables 2 and 3 as presented above, we may note the following points: (i) For $\gamma_1 > \gamma_2$, the bionomic population level for the first species $x_{1\infty} = 10.68$ is higher than the corresponding steady state level $x_1^* = 9.25$. The bionomic population level for the second species $x_{2\infty} = 17.80$ is lower than the corresponding steady state level $x_2^* = 27.74$. The bionomic population level for the third species $x_{3\infty} = 6.74$ is lower than the corresponding steady state level $x_3^* = 9.59$ (ii) For $\gamma_1 < \gamma_2$, the bionomic population level for the first species $x_{1\infty} = 34.07$ is lower than the corresponding steady state level $x_1^* = 53.23$. The bionomic population level for the second species $x_{2\infty} = 2.29$ is lower than the corresponding steady state level $x_2^* = 4.08$. The bionomic population level for the third species $x_{3\infty} = 4.74$ is lower than the corresponding steady state level $x_3^* = 7.56$ (iii) For $\gamma_1 = \gamma_2 \neq 0$, the bionomic population level for the first species $x_{1\infty} = 27.29$ is lower than the corresponding steady state level $x_1^* = 53.23$. The bionomic population level for the second species $x_{2\infty} = 6.80$ is lower than the corresponding steady state level $x_2^* = 4.08$. The bionomic population level for the third species $x_{3\infty} = 5.30$ is lower than the corresponding steady state level $x_3^* = 7.56$ (iv) For $\gamma_1 = \gamma_2 = 0$, the bionomic population level for the first species $x_{1\infty} = 27.27$ is much more lower than the corresponding steady state level $x_1^* = 55.51$. Similarly, the bionomic population level for the second species $x_{2\infty} = 6.81$ is much more lower than the corresponding steady state level $x_2^* = 46.59$. Again, the bionomic population level for the third species $x_{3\infty} = 5.38$ is also much more lower than the corresponding steady state level $x_3^* = 21.10$ (v) Intensities of releasing toxins by the two species alter the steady state and bionomic equilibrium levels of the system (3).

To illustrate optimal equilibrium with fuzzy instantaneous annual rate of discount we consider the values of the parameters as follows: $r_1 = 2.4$, $r_2 = 2.5$, $k_1 = 80$, $k_2 = 60$, $\alpha_{12} = 0.006$, $a_{13} = 0.004$, $a_{21} = 0.002$, $a_{23} = 0.03$, $a_{31} = 0.12$, $a_{32} = 0.31$, $q_1 = 0.5$, $q_2 = 0.5$, $p_1 = 2$, $p_2 = 8$, $c = 10$, $\widetilde{\delta} = (0.08, 0.09, 0.10)$, $\gamma_1 = 0.0008$, $\gamma_2 = 0.0004$ $(\gamma_1 > \gamma_2)$, $\gamma_1 = 0.0004$, $\gamma_2 = 0.0008$ $(\gamma_1 < \gamma_2)$, $\gamma_1 = 0.0004$, $\gamma_2 = 0.0008$ $(\gamma_1 = \gamma_2 \neq 0)$ in appropriate units.

The optimal equilibrium and corresponding harvesting efforts of the system (3) for different kinds of toxicant effort for different combinations α, w_1 and w_2 are presented in Tables 4, 5, 6 and 7, respectively. The computational work has been done on a PC with Intel (R) Core (TM) i3-3210 3.20 GHz Processor in windows environment.

Table 4: Optimal harvesting steady state of prey and predator system (3) for $\gamma_1 > \gamma_2$

		$\alpha = 0$		$\alpha = 0.5$		$\alpha = 0.9$	
w_1	w_2	$(x_{1\widetilde{\delta}}, x_{2\widetilde{\delta}}, x_{3\widetilde{\delta}})$	$E_{\widetilde{\delta}}$	$(x_{1\widetilde{\delta}}, x_{2\widetilde{\delta}}, x_{3\widetilde{\delta}})$	$E_{\widetilde{\delta}}$	$(x_{1\widetilde{\delta}}, x_{2\widetilde{\delta}}, x_{3\widetilde{\delta}})$	$E_{\widetilde{\delta}}$
0.2	0.8	(4.86, 5.29, 2.22)	4.38	(4.80, 5.25, 2.20)	4.39	(4.74, 5.22, 2.18)	4.40
0.4	0.6	(4.77, 5.23, 2.19)	4.39	(4.75, 5.22, 2.18)	4.39	(4.74, 5.21, 2.18)	4.40
0.5	0.5	(4.72, 5.20, 2.18)	4.39	(4.73, 5.21, 2.18)	4.40	(4.73, 5.21, 2.18)	4.40
0.6	0.4	(4.67, 5.17, 2.16)	4.40	(4.71, 5.19, 2.17)	4.40	(4.73, 5.20, 2.18)	4.40
0.8	0.2	(4.58, 5.11, 2.13)	4.41	(4.66, 5.16, 2.16)	4.40	(4.72, 5.20, 2.17)	4.40

Table 5: Optimal harvesting steady state of prey and predator system (3) for $\gamma_1 < \gamma_2$

		$\alpha = 0$		$\alpha = 0.5$		$\alpha = 0.9$	
w_1	w_2	$(x_{1\widetilde{\delta}}, x_{2\widetilde{\delta}}, x_{3\widetilde{\delta}})$	$E_{\widetilde{\delta}}$	$(x_{1\widetilde{\delta}}, x_{2\widetilde{\delta}}, x_{3\widetilde{\delta}})$	$E_{\widetilde{\delta}}$	$(x_{1\widetilde{\delta}}, x_{2\widetilde{\delta}}, x_{3\widetilde{\delta}})$	$E_{\widetilde{\delta}}$
0.2	0.8	(5.29, 5.05, 2.20)	4.38	(5.22, 5.02, 2.18)	4.39	(5.16, 4.99, 2.16)	4.39
0.4	0.6	(5.18, 5, 2.17)	4.39	(5.16, 4.99, 2.16)	4.39	(5.14, 4.98, 2.16)	4.39
0.5	0.5	(5.13, 4.97, 2.15)	4.39	(5.14, 4.96, 2.15)	4.39	(5.14, 4.98, 2.16)	4.39
0.6	0.4	(5.08, 4.95, 2.14)	4.40	(5.11, 4.96, 2.15)	4.40	(5.13, 4.98, 2.16)	4.40
0.8	0.2	(4.97, 4.89, 2.11)	4.41	(5.06, 4.94, 2.13)	4.40	(5.12, 4.97, 2.15)	4.40

Table 6: Optimal harvesting steady state of prey and predator system (3) for $\gamma_1 = \gamma_2 \neq 0$

		$\alpha = 0$		$\alpha = 0.5$		$\alpha = 0.9$	
w_1	w_2	$(x_{1\tilde{\delta}}, x_{2\tilde{\delta}}, x_{3\tilde{\delta}})$	$E_{\tilde{\delta}}$	$(x_{1\tilde{\delta}}, x_{2\tilde{\delta}}, x_{3\tilde{\delta}})$	$E_{\tilde{\delta}}$	$(x_{1\tilde{\delta}}, x_{2\tilde{\delta}}, x_{3\tilde{\delta}})$	$E_{\tilde{\delta}}$
0.2	0.8	$(4.99, 5.13, 2.18)$	4.38	$(4.92, 5.09, 2.16)$	4.38	$(5.16, 4.99, 2.16)$	4.39
0.4	0.6	$(4.89, 5.07, 2.16)$	4.39	$(4.87, 5.06, 2.15)$	4.39	$(5.14, 4.98, 2.16)$	4.39
0.5	0.5	$(4.84, 5.04, 2.14)$	4.39	$(4.85, 5.05, 2.14)$	4.39	$(5.14, 4.98, 2.16)$	4.39
0.6	0.4	$(4.79, 5.02, 2.13)$	4.40	$(4.82, 5.03, 2.14)$	4.39	$(5.13, 4.98, 2.16)$	4.40
0.8	0.2	$(4.69, 4.96, 2.10)$	4.40	$(4.78, 5, 2.12)$	4.40	$(5.12, 4.97, 2.15)$	4.40

Table 7: Optimal harvesting steady state of prey and predator system (3) for $\gamma_1 = \gamma_2 = 0$

		$\alpha = 0$		$\alpha = 0.5$		$\alpha = 0.9$	
w_1	w_2	$(x_{1\tilde{\delta}}, x_{2\tilde{\delta}}, x_{3\tilde{\delta}})$	$E_{\tilde{\delta}}$	$(x_{1\tilde{\delta}}, x_{2\tilde{\delta}}, x_{3\tilde{\delta}})$	$E_{\tilde{\delta}}$	$(x_{1\tilde{\delta}}, x_{2\tilde{\delta}}, x_{3\tilde{\delta}})$	$E_{\tilde{\delta}}$
0.2	0.8	$(5.38, 5.35, 2.3)$	4.39	$(5.30, 5.30, 2.28)$	4.40	$(5.24, 5.27, 2.26)$	4.40
0.4	0.6	$(5.27, 5.28, 2.27)$	4.40	$(5.25, 5.27, 2.26)$	4.40	$(5.23, 5.26, 2.26)$	4.41
0.5	0.5	$(5.21, 5.25, 2.25)$	4.41	$(5.22, 5.26, 2.26)$	4.41	$(5.22, 5.26, 2.26)$	4.41
0.6	0.4	$(5.16, 5.22, 2.24)$	4.41	$(5.19, 5.24, 2.25)$	4.41	$(5.22, 5.26, 2.26)$	4.41
0.8	0.2	$(5.05, 5.16, 2.21)$	4.42	$(5.14, 5.21, 2.23)$	4.41	$(5.21, 5.25, 2.25)$	4.41

From Tables 4, 5, 6 and 7, we observe that the feasible optimal equilibrium exist in all cases for different combinations of α, w_1 and w_2. It is also observed that at optimal equilibrium, the population density of the prey and predator species in the absence of toxicity is always higher than the population density of the prey and predator species in the presence of toxicity. We also obtain from the above tables, in any kinds of effect of toxicity for a particular value of α (say $0, 0.5, 0.9$) and increasing values of w_1, the optimal level of both prey and predator species gradually decreases (see Figure 6 to Figure 9) as well as the optimal efforts employed gradually increases.

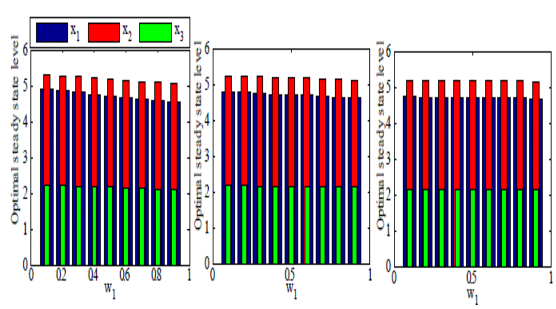

FIGURE 6. Optimal level of prey, predator species vs. w_1 for different values of α with $\gamma_1 > \gamma_2$

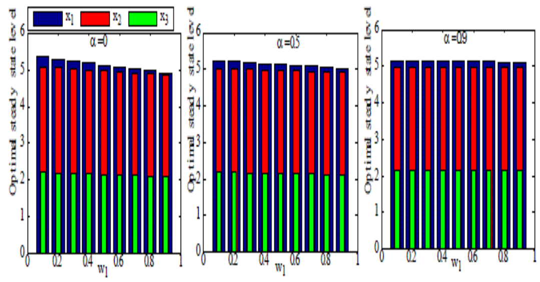

FIGURE 7. Optimal level of prey, predator species vs. w_1 for different values of α with $\gamma_1 < \gamma_2$

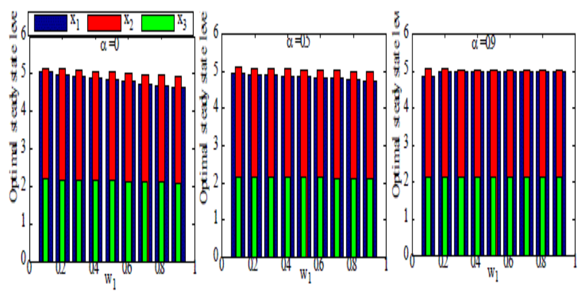

FIGURE 8. Optimal level of prey, predator species vs. w_1 for different values of α with $\gamma_1 \neq \gamma_2$

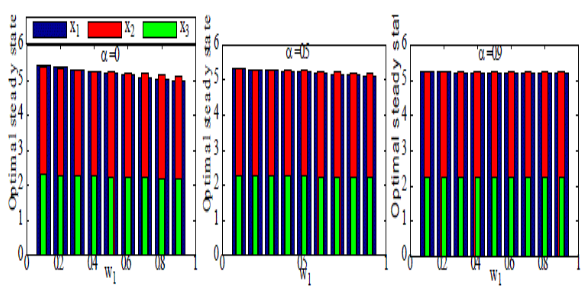

FIGURE 9. Optimal level of prey, predator species vs. w_1 for different values of α with $\gamma_1 = \gamma_2 = 0$

So, we can conclude that the fuzzy net discount rate $\left(\widetilde{\delta}\right)$ plays a significant role for studying the optimal harvesting policy of our proposed model (3)

10. Discussion

In this paper, we have developed a three species competition model consisting of two prey and one predator with the effect of toxic substances and harvesting of the two prey species. Both the prey species are assumed toxic and release toxic substances. Release of toxic substances within the surrounding environment by the first prey species is harmful for the second prey species as well as the predator species and vice versa. Although the authors unable to identify the specific three species (two prey and one predator) prey-predator system in which both the prey species release toxic substances to each other, they considered it to be a very plausible form of interaction between marine fish species competing for the use of a common food supply. It seems to be quite unlikely that toxicant releasing species are limited to the communities of algae and planktons only. Studies on interacting marine fish species are still inadequate. Several pomacentrid fishes which feed on benthic algae surely adopt different techniques, biotic as well as abiotic, to deter other members of their feeding guild.

We have discussed the existence and stability of various equilibrium points of our proposed system (Theorem 6.1 to Theorem 6.4). We have also analysed the bionomic equilibrium of the proposed harvesting model. It is observed that the exploited system may have a stable bionomic equilibrium with positive population levels for all the species (see Table 3).

The most important part of this paper is to set up an optimal control problem with fuzzy inflation and discount and the harvesting effort $E(t)$ as the control variable so as to maximise the fuzzy objective functional $\widetilde{J}$ given in (16). Optimal steady state solution is computed for a data set for different effects of toxicity (see Table 4 to Table 7).

The important mathematical results for the dynamical behaviour of the toxicant affected three species prey-predator model with harvesting are also numerically verified. The graphical representation of a variety of solutions (Fig. 1 to Fig. 5) of system (3) are depicted by using MATLAB software.

There is still some works to do in the proposed model (3) such as we would consider maturity of both the species to release toxic substances within the surrounding environment. These modifications make the model more interesting and realistic. We leave this for future consideration.

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