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THE FLAT EXTENSION OF NONSINGULAR EMBRY MOMENT MATRICES $E(3)$

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Abstract. Let $\gamma^{(n)} \equiv \{\gamma_{ij}\}\,(0 \le i+j \le 2n,\ |i-j| \le n)$ be a sequence in the complex number set $\mathbb{C}$ and let $E\,(n)$ be the Embry truncated moment matrices corresponding from $\gamma^{(n)}$. For an odd number n, it is known that $\gamma^{(n)}$ has a rank $E\,(n)$-atomic representing measure if and only if $E(n) \ge 0$ and $E(n)$ admits a flat extension $E(n+1)$. In this paper we suggest a related problem: if $E(n)$ is positive and nonsingular, does $E(n)$ have a flat extension $E(n+1)$? and give a negative answer in the case of $E(3)$. And we obtain some necessary conditions for positive and nonsingular matrix $E\,(3)$, and also its sufficient conditions.

1. Introduction and preliminaries

The notion of the moment problem was introduced by Stieltjes in 1894 and it is known as a very common problem in physics and engineering nowadays. At present many scientists and mathematicians have studied it in several topics such as Stieltjes, Hamburger, Hausdorff and Toeplitz truncated moment problems, etc. ([1]). In particular, in the 1990s, R. Curto and L. Fialkow introduced and have developed the truncated complex moment problem related to flat extensions ([2–5]). In 2003, one introduced the Embry truncated complex moment problem (as defined below) which is closely related to Embry characterization for subnormal operators as a modification from Curto-Fialkow moment matrices. The authors in [6] showed that the Embry truncated complex moment problem is solvable with matrix flat extensions. And the flatness for some special cases were studied ([7–10]).

For $n \in \mathbb{N}$, let $m = m(n) := (\left[\frac{n}{2}\right] + 1)(\left[\frac{n+1}{2}\right] + 1)$. For $A \in \mathcal{M}_m(\mathbb{C})$ (the algebra of $m \times m$ complex matrices), we denote the successive rows and columns

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according to the following ordering:

$$\underbrace{1}_{(1)}, \underbrace{Z}_{(1)}, \underbrace{Z^2, \bar{Z}Z}_{(2)}, \underbrace{Z^3, \bar{Z}Z^2}_{(2)}, \underbrace{Z^4, \bar{Z}Z^3, \bar{Z}^2Z^2}_{(3)}, \ldots. \tag{1.1}$$

For a collection of complex numbers

$$\gamma^{(n)} \equiv \{\gamma_{ij}\} \ \ (0 \le i+j \le 2n, \ |i-j| \le n) \ \text{ with } \gamma_{00} > 0 \text{ and } \gamma_{ji} = \bar{\gamma}_{ij}, \tag{1.2}$$

we define the moment matrix $E(n) \equiv E(n)(\gamma^{(n)})$ in $\mathcal{M}_m(\mathbb{C})$ as follows:

$$E(n)_{(k,l)(i,j)} := \gamma_{l+i,j+k}.$$

For example, if $n = 3$, i.e.,

$$\gamma^{(3)} := \left\{ \begin{array}{l} \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}, \gamma_{03}, \gamma_{12}, \gamma_{21}, \gamma_{30}, \\ \gamma_{13}, \gamma_{22}, \gamma_{31}, \gamma_{14}, \gamma_{23}, \gamma_{32}, \gamma_{41}, \gamma_{24}, \gamma_{33}, \gamma_{42} \end{array} \right\},$$

then we obtain the moment matrix

$$E(3) = \begin{array}{c} \begin{array}{cccccc} 1 & Z & Z^2 & \bar{Z}Z & Z^3 & \bar{Z}Z^2 \end{array} \\ \left[\begin{array}{cccccc} \gamma_{00} & \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} \\ \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} \\ \gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{31} & \gamma_{23} & \gamma_{32} \\ \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{22} & \gamma_{14} & \gamma_{23} \\ \gamma_{30} & \gamma_{31} & \gamma_{32} & \gamma_{41} & \gamma_{33} & \gamma_{42} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{32} & \gamma_{24} & \gamma_{33} \end{array} \right] \end{array}.$$

Embry truncated complex moment problem entails finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that

$$\gamma_{ij} = \int \bar{z}^i z^j \, d\mu(z) \qquad (0 \le i+j \le 2n, \, |i-j| \le n);$$

μ is called a *representing measure* for $\gamma^{(n)}$ as in (1.2).

Recall that Embry quadratic moment problem for $n = 1$ was solved in [8], Embry quartic moment problem for $n = 2$ was solved in the singular case in [7], and the nonsingular case was solved in [10]. In this paper, we consider Embry moment problem for $n = 3$.

For a positive matrix A, an *extension* of A is a block matrix of the form

$$\tilde{A} := \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}, \tag{1.3}$$

where B^* is the conjugate transpose of B. That $\tilde{A}$ is a *flat extension* of A means rank $\tilde{A}$ = rank A.

We now recall a theorem concerning Embry moment matrices $E(n)$ which gives a motivation for our discussion.

Theorem 1.1 ([6, Theorem 3.10]). *Let $\gamma^{(n)} \equiv \{\gamma_{ij}\}$ $(0 \le i+j \le 2n, \, |i-j| \le n)$ be given.*

(i) *If n is even number, then $\gamma^{(n)}$ has a rank $E(n)$-atomic representing measure if and only if $E(n) \ge 0$ and $E(n)$ admits a flat extension $E(n+2)$.*

(ii) *If n is odd number, then $\gamma^{(n)}$ has a rank $E(n)$-atomic representing measure if and only if $E(n) \geq 0$ and $E(n)$ admits a flat extension $E(n+1)$.*

In terms of Theorem 1.1, it is worthwhile to find a flat extension of $E(n)$, and the following question arises.

Nonsingular Embry truncated complex moment problem. *If $E(n)$ is positive and nonsingular, does $E(n)$ have a flat extension $E(n+1)$?*

But we show that the answer of this problem is negative, see Theorem 2.1 in Section 2 below. Thus we detect conditions for the positivity and nonsingularity of Embry moment matrix $E(3)$ via flatness property in Section 2. For this purpose, we recall some terminology related this study below.

If $A \geq 0$ and $AW = B$, i.e., $\operatorname{Ran} B \subseteq \operatorname{Ran} A$, there is a unique flat extension of the form (1.3), which is denoted by $[A; B]$. For $E(n) \geq 0$, we want to construct a positive flat extension of the form $E(n+1) = [E(n); B(n)]$. Let $C := (c_{ij})_{1 \leq i,j \leq [\frac{n+1}{2}]+1}$ in $[E(n); B(n)]$. For $n = 3$,

$$C = B(3)^* E(3)^{-1} B(3) = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix},$$

where

$$B(3) = \begin{bmatrix} \gamma_{04} & \gamma_{14} & \gamma_{24} & \gamma_{15} & \gamma_{34} & \gamma_{25} \\ \gamma_{13} & \gamma_{23} & \gamma_{33} & \gamma_{24} & \gamma_{43} & \gamma_{34} \\ \gamma_{22} & \gamma_{32} & \gamma_{42} & \gamma_{33} & \gamma_{52} & \gamma_{43} \end{bmatrix}^{\mathrm{T}}.$$

If $E(3) > 0$ (i.e., $E(3)$ is positive and invertible), then $[E(3); B(3)]$ is a flat extension of $E(3)$ if and only if $c_{11} = c_{22} = c_{33}$ and $c_{21} = c_{32}$, in which contains variables $\gamma_{04}, \gamma_{15}, \gamma_{34}$, and γ_{25}, has a solution.

Some of the calculations in this article were obtained throughout computer experiments using the software tool *Scientific WorkPlace* ([11]).

Notations. The superscript "T" stands for matrix transposition; $\mathbb{R}^+$ denotes the set of positive real numbers. In symmetric block matrices, we use an asterisk ($\star$) to present a term that is induced by symmetry.

2. Flat extension of nonsingular Embry moment matrices $E(3)$

2.1. The basic case

We consider

$$E(3) = \begin{bmatrix} 1 & 0 & 0 & a & 0 & 0 \\ 0 & a & 0 & 0 & 0 & b \\ 0 & 0 & b & 0 & 0 & 0 \\ a & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 \\ 0 & b & 0 & 0 & 0 & c \end{bmatrix} \quad \text{with } a, b, c \in \mathbb{R}^+. \tag{2.1}$$

Theorem 2.1. *$E(3)$ as in (2.1) is positive and invertible if and only if $b > a^2$ and $ac > b^2$, but it has no flat extension $E(4)$. Therefore, $\gamma^{(3)}$ does not admit a 6-atomic representing measure.*

Proof. In fact, if $E(3)$ has a flat extension $E(4)$, then we obtain

(1) $\gamma_{25}\gamma_{43} = \gamma_{34}^2$,

(2) $|\gamma_{34}|^2 = \frac{c}{(b-a^2)b} + |\gamma_{25}|^2$.

From (1), we know that $\gamma_{34} = \bar{\gamma}_{43} = 0$ or $|\gamma_{34}|^2 = |\gamma_{25}|^2$. But this is contradict to (2). Thus we have our conclusion. □

2.2. Case of γ_{01}

We consider

$$E(3) = \begin{bmatrix} 1 & \gamma_{01} & 0 & a & 0 & 0 \\ \gamma_{10} & a & 0 & 0 & 0 & b \\ 0 & 0 & b & 0 & 0 & 0 \\ a & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 \\ 0 & b & 0 & 0 & 0 & c \end{bmatrix} \text{ with } a, b, c \in \mathbb{R}^+, \text{ and } \gamma_{01} = \bar{\gamma}_{10} \neq 0. \tag{2.2}$$

Proposition 2.2. *$E(3)$ as in (2.2) is positive and invertible if and only if*

$$ac > b^2 \quad \text{and} \quad w := bc|\gamma_{01}|^2 - \left(b - a^2\right)\left(ac - b^2\right) < 0.$$

We know that $E(3)$ as in (2.2) has a flat extension $E(4)$ if the following conditions hold (here we take $\gamma_{25} = \bar{\gamma}_{52} = 0$):

(1) $-\gamma_{34}^2(a^2 - b)b + c\gamma_{10}\gamma_{43}(a\gamma_{15} - b\gamma_{04}) = c\gamma_{01}\gamma_{34}(ac - b^2)$.

(2) $\frac{a^2c-ab^2}{w}(\gamma_{04}\gamma_{15} + \gamma_{40}\gamma_{51}) = \frac{c^2}{b} + |\gamma_{34}|^2 \frac{a^3-ab+b|\gamma_{01}|^2}{w} - |\gamma_{04}|^2 \frac{b^3-abc}{w} - |\gamma_{15}|^2 \frac{b^2-ac+c|\gamma_{01}|^2}{w}$.

(3) $|\gamma_{34}|^2 = -\frac{c(ac-b^2)^3}{bw} + bc\frac{ac-b^2}{w}(\gamma_{10}\gamma_{34} + \gamma_{01}\gamma_{43})$.

Let $\gamma_{34} = u + iv$ and $\gamma_{01} = s + ti$. Then by (3) we have

$$u^2 + v^2 = -\frac{c(ac - b^2)^3}{bw} + 2bc\frac{ac - b^2}{w}(su + tv).$$

Let $v = -u$. Then we have

$$2u^2 - 2bc\frac{ac - b^2}{w}(s - t)u + \frac{c(ac - b^2)^3}{bw} = 0.$$

We can obtain

$$u = \frac{bc(ac - b^2)}{2w}(s - t) \pm \frac{\left(ac - b^2\right)}{2w}\sqrt{\left(-2c\frac{ac - b^2}{b}\right)w + b^2c^2\left(s - t\right)^2}.$$

Next, we let $\gamma_{04} = p + iq$ and $\gamma_{15} = m + in$. Then by (1) we have

(4) $-a\left(s + t\right)m + a\left(s - t\right)n + b\left(s + t\right)p + b\left(t - s\right)q = X$,

(5) $ac\left(t - s\right)m - ac\left(s + t\right)n + bc\left(s - t\right)p + bc\left(s + t\right)q = Y$, where

$$X = -\left(s + t\right)\left(ac - b^2\right),$$

$$Y = 2b\left(a^2 - b\right)u + c\left(s - t\right)\left(ac - b^2\right).$$

By (4) and (5), we have

$$m = \frac{t\left(Y - Xc\right) - s\left(Y + Xc\right)}{2ac\left(s^2 + t^2\right)} + \frac{b}{a}p = \mu + \frac{b}{a}p,$$
$$n = -\frac{s\left(Y - Xc\right) + t\left(Y + Xc\right)}{2ac\left(s^2 + t^2\right)} + \frac{b}{a}q = \lambda + \frac{b}{a}q.$$

Substituting γ_{04} and γ_{15} with m and n to (2), we obtain

$$\begin{aligned} &a^2\left(2b\left(bs^2+bt^2-ab+a^3\right)u^2+a^2\left(\lambda^2+\mu^2\right)\left(-cs^2-ct^2+ac-b^2\right)\right) \\ &= bw\left(bp^2+bq^2+2ap\mu+2aq\lambda\right)-a^2c^2w. \end{aligned} \tag{2.3}$$

The equation (2.3) is solvable if

$$2b\left(bs^2 + bt^2 - ab + a^3\right)u^2 + a^2\left(ac - b^2\right)\left(\mu^2 + \lambda^2\right) + c^2w \leq 0.$$

Finally, we obtain the following result.

Theorem 2.3. *Assume that* $ac > b^2$ *and* $w := bc\left|\gamma_{01}\right|^2 - \left(b - a^2\right)\left(ac - b^2\right) < 0$. *If*

$$2b\left(b\left|\gamma_{01}\right|^2 - a\left(b - a^2\right)\right)u^2 + a^2\left(ac - b^2\right)\left(\mu^2 + \lambda^2\right) + c^2w \leq 0, \tag{2.4}$$

then $E(3)$ *as in* (2.2) *is positive and invertible, furthermore it has a flat extension* $E(4)$.

We give an illustrating example for Theorem 2.3 below.

Example 2.4. Let $a = 1, b = 2, c = 8$ and $\gamma_{01} = \frac{1+i}{4}$. That is

$$E(3) = \begin{bmatrix} 1 & \frac{1+i}{4} & 0 & 1 & 0 & 0 \\ \frac{1-i}{4} & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 2 & 0 & 0 & 0 & 8 \end{bmatrix}. \tag{2.5}$$

Then the condition (2.4) of Theorem 2.3 is satisfied. Thus $E(3)$ as in (2.5) is positive and invertible, furthermore it has a flat extension $E(4)$. Indeed, we can find an extension $E(4)$ of $E(3)$ as following:

$$E(4) = \begin{bmatrix} 1 & \frac{1+i}{4} & 0 & 1 & 0 & 0 & -2-4i & 0 & 2 \\ \frac{1-i}{4} & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 8 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 8 & 0 & 8-8i & 8+8i & 0 \\ 0 & 2 & 0 & 0 & 0 & 8 & 0 & 8-8i & 8+8i \\ \star & \star & \star & \star & \star & \star & 96 & -16+48i & -16-48i \\ \star & \star & \star & \star & \star & \star & -16-48i & 96 & -16+48i \\ \star & \star & \star & \star & \star & \star & -16+48i & -16-48i & 96 \end{bmatrix}.$$

2.3. Case of γ_{02}

We consider

$$E(3)=\begin{bmatrix} 1 & 0 & \gamma_{02} & a & 0 & 0 \\ 0 & a & 0 & 0 & 0 & b \\ \gamma_{20} & 0 & b & 0 & 0 & 0 \\ a & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 \\ 0 & b & 0 & 0 & 0 & c \end{bmatrix} \text{ with } a,b,c\in\mathbb{R}^{+} \text{ and } \gamma_{02}=\bar{\gamma}_{20}\neq 0. \tag{2.6}$$

Proposition 2.5. *$E(3)$ as in (2.6) is positive and invertible if and only if $ac>b^2$ and $|\gamma_{02}|^2<b-a^2$.*

2.3.1. *Sufficient condition for $E(3)$ has no flat extension $E(4)$.* Let $w=a^2-b+|\gamma_{02}|^2$. Then by Proposition 2.5 we know that $w<0$. By direct computation, we have

$$\begin{aligned}
c_{11} &= \frac{-b^2|\gamma_{04}|^2+ab\left(\gamma_{04}\gamma_{51}+\gamma_{40}\gamma_{15}\right)+\left(|\gamma_{02}|^2-b\right)|\gamma_{15}|^2}{bw} \\
&\quad+\frac{1}{c}|\gamma_{34}|^2+a\frac{|\gamma_{25}|^2}{ac-b^2}, \\
c_{12} &= \frac{1}{c}\gamma_{43}^2-c\gamma_{02}\frac{a\gamma_{51}-b\gamma_{40}}{wb}+a\frac{\gamma_{34}\gamma_{52}}{ac-b^2}, \\
c_{13} &= \frac{b\gamma_{40}\left(ac-b^2\right)+\gamma_{51}\left(ab^2-bc+c|\gamma_{02}|^2\right)}{bw}+\gamma_{43}\gamma_{52}\frac{2ac-b^2}{c\left(ac-b^2\right)}, \\
c_{22} &= |\gamma_{34}|^2\frac{2ac-b^2}{c\left(ac-b^2\right)}-c^2\frac{b-a^2}{bw}, \\
c_{23} &= \frac{1}{c}\gamma_{34}\gamma_{52}+a\frac{\gamma_{43}^2}{ac-b^2}-c\gamma_{20}\frac{ac-b^2}{bw}, \\
c_{33} &= \frac{-bc^2-b^4+2ab^2c+c^2|\gamma_{02}|^2}{bw}+\frac{1}{c}|\gamma_{25}|^2+\frac{a}{ac-b^2}|\gamma_{34}|^2.
\end{aligned}$$

Thus, $E(3)$ as in (2.6) has a flat extension $E(4)$ if and only if the following conditions hold:

(1) $\gamma_{04}\left(a\gamma_{51}-b\gamma_{40}\right)-\gamma_{15}\left(\gamma_{51}\frac{b-|\gamma_{02}|^2}{b}-a\gamma_{40}\right)+aw\frac{|\gamma_{25}|^2}{ac-b^2}=-c^2\frac{b-a^2}{b}+aw\frac{|\gamma_{34}|^2}{ac-b^2}$,

(2) $\frac{w}{c}|\gamma_{34}|^2-c^2\frac{b-a^2}{b}=c\left(ab-c\frac{b-|\gamma_{02}|^2}{b}\right)-b\left(b^2-ac\right)+\frac{w}{c}|\gamma_{25}|^2$,

(3) $\frac{w}{c}\gamma_{34}^2+c\gamma_{20}\gamma_{04}+aw\gamma_{25}\frac{\gamma_{43}}{ac-b^2}-ac\gamma_{20}\frac{\gamma_{15}}{b}=c\gamma_{02}\frac{b^2-ac}{b}+\frac{w}{c}\gamma_{25}\gamma_{43}+aw\frac{\gamma_{34}^2}{ac-b^2}$.

From (2), we have

$$|\gamma_{34}|^2-|\gamma_{25}|^2=\frac{c}{bw}\left(c^2|\gamma_{02}|^2-\left(ac-b^2\right)^2\right).$$

By (1), we have

$$\begin{aligned}&\gamma_{04}\left(a\gamma_{51}-b\gamma_{40}\right)-\gamma_{15}\left(\gamma_{51}\frac{b-\left|\gamma_{02}\right|^2}{b}-a\gamma_{40}\right)\\&=c\left((-c+ab)+\frac{ac^2\left|\gamma_{02}\right|^2}{b\left(ac-b^2\right)}\right):=Z.\end{aligned}$$

Let $\gamma_{04}=p+iq$, $\gamma_{15}=m+in$, $\gamma_{34}=u+iv$, $\gamma_{25}=x+iy$ and $\gamma_{02}=s+it$. Then from (3) we have

$$\begin{aligned}ac^2s\left(-ac+b^2\right)m+ac^2t\left(-ac+b^2\right)n+bc^2t\left(ac-b^2\right)q=A,\\ac^2t\left(ac-b^2\right)m+ac^2s\left(-ac+b^2\right)n+bc^2s\left(ac-b^2\right)q=B,\end{aligned}$$

where

$$\begin{aligned}A&=bc^2s\left(ac-b^2\right)p-\left(c^2\left(ac-b^2\right)^2s+b^3w\left(ux+vy-u^2+v^2\right)\right),\\B&=bc^2t\left(ac-b^2\right)p-\left(c^2\left(ac-b^2\right)^2t+b^3w\left(-2uv+uy-vx\right)\right).\end{aligned}$$

Thus

$$\begin{aligned}m&=-\frac{As-Bt}{ac^2\left(s^2+t^2\right)\left(ac-b^2\right)}=\lambda,\\n&=\frac{b}{a}q+\frac{At+Bs}{ac^2\left(s^2+t^2\right)\left(-ac+b^2\right)}=\frac{b}{a}q+\mu.\end{aligned}$$

Here, $\mu^2+\lambda^2\neq 0$. If $\mu^2+\lambda^2=0$, we know that $s^2+t^2=0$, i.e., $\gamma_{02}=0$. Thus, we can assume that $\lambda\neq 0$.

From (1), we have $a^2bZ=P\left(p,q\right)$, where

$$\begin{aligned}P\left(p,q\right)&=b^2wq^2+2ab\mu wq\\&\quad+\left(-a^2b^2\right)p^2+2a^3b\lambda p+a^2\left(\lambda^2+\mu^2\right)\left(-b+s^2+t^2\right)\\&=-a^2b^2\left(p-\frac{a\lambda}{b}\right)^2+wb^2\left(q+\frac{a\mu}{b}\right)^2+a^2\left(\lambda^2w-a^2\mu^2\right)<0.\end{aligned}$$

Thus if $Z\geq 0$, then $E\left(3\right)$ as in (2.6) has no flat extension $E\left(4\right)$. The condition $Z\geq 0$ is equivalent to

$$\frac{b\left(c-ab\right)\left(ac-b^2\right)}{ac^2}\leq\left|\gamma_{02}\right|^2.$$

Finally we obtain the following result.

Theorem 2.6. *If $ac>b^2$ and*

$$\text{(NC)}\qquad\frac{b\left(c-ab\right)\left(ac-b^2\right)}{ac^2}\leq\left|\gamma_{02}\right|^2<b-a^2,$$

then $E\left(3\right)$ as in (2.6) is positive and invertible but has no flat extension $E\left(4\right)$.

We give an illustrating example for Theorem 2.6 below.

Example 2.7. We consider

$$E(3)=\begin{bmatrix} 1 & 0 & \frac{1}{2}+\frac{1}{2}i & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ \frac{1}{2}-\frac{1}{2}i & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 2 & 0 & 0 & 0 & 5 \end{bmatrix}. \tag{2.7}$$

Then $E(3)$ as in (2.7) is positive and invertible. Since the condition (NC) of Theorem 2.6 is satisfied, we know that $E(3)$ as in (2.7) has no flat extension $E(4)$.

The following example shows that there is a moment matrix $E(3)$ that the condition (NC) of Theorem 2.6 is not satisfied but it has a flat extension $E(4)$.

Example 2.8. We consider

$$E(3)=\begin{bmatrix} 1 & 0 & \frac{1}{2}+\frac{1}{2}i & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ \frac{1}{2}-\frac{1}{2}i & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 12 & 0 \\ 0 & 2 & 0 & 0 & 0 & 12 \end{bmatrix}. \tag{2.8}$$

Then $E(3)$ as in (2.8) is positive and invertible. Let $\gamma_{04}=p+iq$, $\gamma_{15}=m+in$, $\gamma_{34}=u+iv$ and $\gamma_{25}=x+iy$. Then $E(3)$ as in (2.8) has a flat extension $E(4)$ if and only if

(1) $\frac{3}{2}m^2+\frac{3}{2}n^2-4mp-4nq+4p^2+4q^2-\frac{1}{8}u^2-\frac{1}{8}v^2+\frac{1}{8}x^2+\frac{1}{8}y^2-144=0$,
(2) $\frac{1}{12}u^2+\frac{1}{12}v^2-\frac{1}{12}x^2-\frac{1}{12}y^2+8=0$,
(3) $6m+6n-12p-12q-\frac{1}{24}u^2+\frac{1}{24}xu+\frac{1}{24}v^2+\frac{1}{24}yv-48=0$,
(4) $6n-6m+12p-12q-\frac{1}{12}uv+\frac{1}{24}uy-\frac{1}{24}vx-48=0$.

We can obtain one of the solutions

$$p=-4,\ q=3\sqrt{2}-4,\ m=-8,\ n=6\sqrt{2},\ u=v=0,\ x=y=4\sqrt{3}.$$

That is, $\gamma_{04}=-4+\left(3\sqrt{2}-4\right)i, \gamma_{15}=-8+6\sqrt{2}i, \gamma_{34}=0$ and $\gamma_{25}=4\sqrt{3}+4\sqrt{3}i$. Thus $E(3)$ as in (2.8) that has a flat extension $E(4)$.

2.3.2. *Sufficient condition for $E(3)$ has a flat extension $E(4)$.* Let $\gamma_{02}=s+it$, $\gamma_{04}=p+iq$, $\gamma_{15}=m+in$, $\gamma_{34}=0$ and $\gamma_{25}=x+iy$. Then

$$c_{11}=a\frac{x^2+y^2}{ac-b^2}-b\frac{p^2+q^2}{w}+2a\frac{mp+nq}{w}+\left(m^2+n^2\right)\frac{s^2+t^2-b}{bw},$$

$$c_{22}=\frac{c^2}{bw}\left(a^2-b\right),$$

$$c_{33}=\frac{1}{c}\left(x^2+y^2\right)-\frac{b^3+c^2-2abc}{w}+\frac{c^2}{b}\cdot\frac{s^2+t^2}{w},$$

$$\begin{aligned}
c_{12} &= -c\,(s+it)\,\frac{am-bp-i\,(an-bq)}{wb},\\
c_{13} &= \frac{cm\left(s^2+t^2\right)-bq\left(b^2-ac\right)+ab^2m-bcm}{bw}\\
&\quad + i\frac{bq\left(b^2-ac\right)-ab^2n+bcn-cn\left(s^2+t^2\right)}{bw},\\
c_{23} &= c\,(s-it)\,\frac{b^2-ac}{bw}.
\end{aligned}$$

Thus $E(3)$ as in (2.6) has a flat extension $E(4)$ if

(1) $b^2\left(b^2-ac\right)(p^2+q^2)+2ab\left(ac-b^2\right)(pm+qn)$
$+\left(|\gamma_{02}|^2-b\right)\left(ac-b^2\right)(m^2+n^2)+abw(x^2+y^2)+c^2\left(b-a^2\right)\left(ac-b^2\right)$
$=0,$

(2) $bw(x^2+y^2)+c^3\,|\gamma_{02}|^2-c\left(ac-b^2\right)^2=0,$

(3) $b^2s-acs+ams+ant-bps-bqt=0,$

(4) $b^2t-act-amt+ans+bpt-bqs=0.$

From (3), (4), we obtain

$$\begin{aligned}
m &= \frac{1}{a\left(s^2+t^2\right)}\left(\left(ac-b^2\right)\left(s^2-t^2\right)+bp\left(s^2+t^2\right)\right),\\
n &= \frac{1}{a\left(s^2+t^2\right)}\left(bq\left(s^2+t^2\right)+2\left(ac-b^2\right)st\right).
\end{aligned}$$

From (2), we have

$$x^2+y^2=\frac{c\left(\left(ac-b^2\right)^2-c^2\,|\gamma_{02}|^2\right)}{bw}.$$

So we need the condition

$$\text{(C1)}\qquad \left(ac-b^2\right)^2-c^2\,|\gamma_{02}|^2\le 0.$$

By (1), we obtain

$$\begin{aligned}
&-w\frac{ac-b^2}{a^2}\left(\left(bp+(s-t)\,(s+t)\,\frac{\left(ac-b^2\right)}{\left(s^2+t^2\right)}\right)^2+\left(bq+2st\frac{\left(ac-b^2\right)}{\left(s^2+t^2\right)}\right)^2\right)\\
&=\left(ac-b^2\right)\left(b^2\left(ac-b^2\right)+c^2\left(b-a^2\right)\right)-\left(ac^3\right)|\gamma_{02}|^2.
\end{aligned}$$

Thus if

$$\text{(C2)}\qquad |\gamma_{02}|^2\le\left(ac-b^2\right)\frac{b^2\left(ac-b^2\right)+c^2\left(b-a^2\right)}{ac^3},$$

then we can solve the equations. Finally, we obtain the following result.

Theorem 2.9. *If $ac>b^2$ and*

$$\text{(2.9)}\quad \frac{\left(ac-b^2\right)^2}{c^2}\le|\gamma_{02}|^2<\min\left\{b-a^2,\left(ac-b^2\right)\frac{b^2\left(ac-b^2\right)+c^2\left(b-a^2\right)}{ac^3}\right\},$$

then $E(3)$ *as in* (2.6) *is positive and invertible, furthermore it has a flat extension* $E(4)$.

We give an illustrating example for Theorem 2.9 below.

Example 2.10. Let $a = 1, b = 2, c = 12$ and $\gamma_{02} = \sqrt{\frac{2}{3}}i$. That is,

$$E(3) = \begin{bmatrix} 1 & 0 & \sqrt{\frac{2}{3}}i & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ -\sqrt{\frac{2}{3}}i & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 12 & 0 \\ 0 & 2 & 0 & 0 & 0 & 12 \end{bmatrix}. \tag{2.10}$$

Then the condition (2.9) of Theorem 2.9 is satisfied. Thus $E(3)$ as in (2.10) is positive and invertible, furthermore it has a flat extension $E(4)$. Indeed, we can find an extension $E(4)$ of $E(3)$ as following:

$$E(4) = \begin{bmatrix} 1 & 0 & \sqrt{\frac{2}{3}}i & 1 & 0 & 0 & 2\sqrt{2}i & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ -\sqrt{\frac{2}{3}}i & 0 & 2 & 0 & 0 & 0 & 0 & 12 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & -8+4\sqrt{2}i & 0 & 12 \\ 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 12\sqrt{2}\,(1-i) \\ 0 & 2 & 0 & 0 & 0 & 12 & 12\sqrt{2}\,(1+i) & 0 & 0 \\ \star & \star & \star & \star & \star & \star & 216 & -48\sqrt{6}i & -144-24\sqrt{2}i \\ \star & \star & \star & \star & \star & \star & 48\sqrt{6}i & 216 & -48\sqrt{6}i \\ \star & \star & \star & \star & \star & \star & -144+24\sqrt{2}i & 48\sqrt{6}i & 216 \end{bmatrix}.$$

Remark. The moment matrix $E\,(3)$ as in (2.8) does not satisfy the condition (2.9) of Theorem 2.9.

2.4. Case of γ_{24}

We consider

$$E(3) = \begin{bmatrix} 1 & 0 & 0 & a & 0 & 0 \\ 0 & a & 0 & 0 & 0 & b \\ 0 & 0 & b & 0 & 0 & 0 \\ a & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & c & \gamma_{42} \\ 0 & b & 0 & 0 & \gamma_{24} & c \end{bmatrix} \quad \text{with } a, b, c \in \mathbb{R}^+ \text{ and } \gamma_{24} = \bar{\gamma}_{42} \neq 0. \tag{2.11}$$

Proposition 2.11. $E(3)$ *as in* (2.11) *is positive and invertible if and only if* $b > a^2$ *and* $w := c\left(b^2 - ac\right) + a\left|\gamma_{24}\right|^2 < 0$.

Let $\gamma_{24} = s + it, \gamma_{25} = x + iy, \gamma_{04} = p + iq$ and $\gamma_{15} = m + in$. Similarly, we know that $E\,(3)$ as in (2.11) has a flat extension $E\,(4)$ if the following conditions hold (here we take $\gamma_{34} = \bar{\gamma}_{43} = 0$):

(1) $abc\left(b-a^2\right)\left(x^2+y^2\right)+wa^2\left(s^2+t^2-c^2\right)-wb^2\left(p^2+q^2\right)$
$+bw\left(c^2-m^2-n^2+2amp+2anq\right)=0,$
(2) $b\left(a^2-b\right)\left(b^2-ac\right)\left(x^2+y^2\right)+wa^2\left(s^2+t^2-c^2\right)+wb^2\left(2ac-b^2\right)=0,$
(3) $cs-ms-nt-abs+aps+aqt=0,$
(4) $ct+mt-ns-abt-apt+aqs=0.$

If $s\neq 0$, then by (3), (4), we have

$$m=\frac{1}{s^2+t^2}\left(\left(c-ab+ap\right)s^2-\left(c-ab-ap\right)t^2\right),$$
$$n=\frac{1}{s^2+t^2}\left(aqs^2+aqt^2+2cst-2abst\right).$$

By (2), we have

$$x^2+y^2=-\frac{1}{b}\frac{w}{\left(ac-b^2\right)\left(b-a^2\right)}\left(a^2\left|\gamma_{24}\right|^2-a^2c^2-b^4+2ab^2c\right).$$

It has a solution if $0\leq a^2\left|\gamma_{24}\right|^2-a^2c^2-b^4+2ab^2c$, that is

$$\frac{\left(ac-b^2\right)^2}{a^2}\leq\left|\gamma_{24}\right|^2.$$

Substituting γ_{04},γ_{15} with m,n and x^2+y^2 to (1), we obtain

$$p^2+q^2=ab\frac{w-ab^3+a^2bc}{\left(a^2-b\right)\left(ac-b^2\right)}.$$

It has a solution if $w-ab^3+a^2bc\leq 0$, that is,

$$\left|\gamma_{24}\right|^2\leq\frac{1}{a}\left(ac-b^2\right)\left(c-ab\right).$$

Since $\left|\gamma_{24}\right|^2<\frac{c}{a}\left(ac-b^2\right)$, and

$$\frac{1}{a}\left(ac-b^2\right)\left(c-ab\right)-\frac{c}{a}\left(ac-b^2\right)=-b\left(ac-b^2\right)<0,$$

we obtain a sufficient condition

$$\frac{\left(ac-b^2\right)^2}{a^2}\leq\left|\gamma_{24}\right|^2\leq\frac{1}{a}\left(ac-b^2\right)\left(c-ab\right).$$

If $s=0$, then similarly we can obtain a sufficient condition as above. Thus we obtain the following result.

Theorem 2.12. *Assume that $b>a^2$. If*

$$\frac{\left(ac-b^2\right)^2}{a^2}\leq\left|\gamma_{24}\right|^2\leq\frac{1}{a}\left(ac-b^2\right)\left(c-ab\right), \tag{2.12}$$

then $E(3)$ as in (2.11) is positive and invertible, furthermore it has a flat extension $E(4)$.

We give an illustrating example for Theorem 2.12 below.

Example 2.13. Let $a = 1, b = 2, c = 5$ and $\gamma_{24} = 1 + i$. That is

$$E(3) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1-i \\ 0 & 2 & 0 & 0 & 1+i & 5 \end{bmatrix}. \tag{2.13}$$

Then the condition (2.12) of Theorem 2.12 is satisfied. Thus $E(3)$ as in (2.13) is positive and invertible, furthermore it has a flat extension $E(4)$. Indeed, we can find an extension $E(4)$ of $E(3)$ as following

$$E(4) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & i\sqrt{2} & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1+i & 5 & 1-i \\ 1 & 0 & 0 & 2 & 0 & 0 & \left(3+\sqrt{2}\right)i & 1+i & 5 \\ 0 & 0 & 0 & 0 & 5 & 1-i & 0 & 0 & \frac{\sqrt{6}}{2} \\ 0 & 2 & 0 & 0 & 1+i & 5 & \frac{\sqrt{6}}{2} & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \frac{29}{2} & \frac{11(1-i)}{2} & -\frac{1+\left(21+4\sqrt{2}\right)i}{2} \\ \star & \star & \star & \star & \star & \star & \frac{11(1+i)}{2} & \frac{29}{2} & \frac{11(1-i)}{2} \\ \star & \star & \star & \star & \star & \star & \frac{-1+\left(21+4\sqrt{2}\right)i}{2} & \frac{11(1+i)}{2} & \frac{29}{2} \end{bmatrix}.$$

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