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Teaching the Intermediate Value Theorem with Non-Existing Examples[1]

Jihyun Hwang (Researcher)[1*], Dae S. Hong (Professor)[2]
[1] University of Iowa, jihyun-hwang@uiowa.edu
[2] University of Iowa, dae-hong@uiowa.edu

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In this case study, a professor was observed to investigate use of instructional examples when teaching the Intermediate Value Theorem in a calculus course. Video-recorded lessons were analyzed with constant comparison to video-stimulated recall interviews and field notes. The professor employed multiple instructional examples, which was initiated by students and modified by the professor. The professor asked students to build non-existing examples as an informal proof of the Intermediate Value Theorem and assessment of students' previous knowledge. Use of incorrect examples on instructional purpose can be an appropriate way for formative assessment as well as a bridge between informal and formal proofs in college mathematics.

Keywords: instructional example, college mathematics, the intermediate value theorem
MESC Classification: D50
MSC2010 Classification: 97D50

I. INTRODUCTION

Mathematics examples have been studied as fundamental elements of teaching and learning mathematics. Giving examples is a widespread way to teach mathematics concepts, but not a straightforward instructional process (Bardelle & Ferrari, 2011). An example in mathematics instruction could provide students an opportunity to reason and generalize their observations within the given example as a special case of a larger class (Watson & Mason, 2002). Furthermore, teachers can examine what a learner has in one's mind through a process of generating examples (Antonini, Presmeg, Mariotti, & Zaslavsky, 2011).

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[*] Corresponding Author: jihyun-hwang@uiowa.edu

In terms of teaching, examining a collection of instructional examples rather than a single example can considerably contribute to understanding teaching and learning mathematics. A set of examples that teachers use not only mirror a structure of knowledge that a teacher has, but also influence how students build their knowledge through given examples. Rissland (1991) argued, “A central component of knowledge is a collection of examples” (p. 188) and natural relations exist in the collection of examples. A set of multiple examples can be structured by the relations reflecting instructional purposes, for example, one example is used and modified to construct the other example (Rissland, 1978). According to Atkinson, Derry, Renkl, and Wortham (2000), multiple worked examples in proximity could be amalgam of related mathematical concepts in learning mathematics and allow students to transfer their learning across various settings.

Considering the importance of instructional examples to students’ learning of mathematics, prior studies have examined how and why teachers select specific examples for their own instructional purposes. Particularly, teachers’ knowledge for teaching can have significant influence on their choices of examples in their instruction (Zodik & Zaslavsky, 2008). Teachers’ use of examples are possibly related to several pedagogical aspects which emphasize significance of this essential element of mathematics instruction (Zaslavsky, 2010). Mathematics teachers consider transparency of mathematics concepts for their choices of examples. This means that teachers are concerned about whether or not students can clearly realize key concepts within examples. Simultaneously, teachers think about unnecessary noises which distract students’ attention from main mathematics concepts. Also, students’ familiarity and possible mistakes are also essential for teacher to select examples in their instructions (Zodik & Zaslavsky, 2008).

In this article, we will describe a professor’s use of examples to teach the Intermediate Value Theorem (IVT) in a calculus class at a Midwest university in the United States. Her calculus 1 course had been originally observed for a semester in a research project to examine various aspects of instructional practices. As a faculty member of that university, she had taught calculus 1 four times. In the lesson to teach IVT, the professor employed several examples in an interesting way because she asked students to generate non-existing cases.

By examining this episode, we will discuss potential of instructional use of non-existing cases, included in incorrect examples, to check students’ prior knowledge as well as to give an opportunity of informal reasoning. Findings in this study can show how to introduce mathematical proof to undergraduate students because instructors’ use of instructional examples can provide opportunities for students to build their own examples and think logical arguments of mathematical ideas in calculus courses, for example, the IVT, the mean value theorem, and the squeeze theorem. Furthermore, teachers, even at the elementary and secondary levels, can construct these non-existing cases simultaneously from students' initial incorrect example, which might help students to engage in classroom discussion. The research questions guiding our study were: what are the appearance and relations of instructional examples to teach the IVT in calculus 1 courses? What characteristics are found in relations of those instructional examples?

II. TYEPS OF INSTRUCTIONAL EXAMPLES

Instructional examples are defined as examples produced by teachers or students within the context of learning (Zodik & Zaslavsky, 2008). According to Zaslavsky (2010), examples are mathematical objects which can be instruction tools to facilitate student's deep understanding of mathematical ideas, concepts and algorithms as well as communication tools for explanation, argument, and proof with a variety of representations. This argument is exactly aligned with the definitions of an example provided by a couple of previous studies in mathematics education; an example is referred as mathematical object serving a cultural mediating tool between a person and mathematical concepts or theorems (Goldenberg & Mason, 2008). An instructional example should allow learners to mentally interact to mathematical objects with abstract mathematical ideas in educational settings (Zodik & Zaslavsky, 2008).

It should be noticed that the roles of instructional examples should be recognized within learning contexts. Instructional examples have different roles as a special case of a large class "cover[ing] a broad ranges of mathematics genres" (Watson & Mason, 2002, p. 378); (a) illustrating concepts; (b) of problems with strategies; and (c) proving theorems. For instance, 72÷3 can be a concept example to illustrate the concept of division or a procedure example for the long division algorithm depending on learning contexts.

To identifying instructional examples in learning contexts we need to examine two factors in the contexts: mathematical authority (Amit & Fried, 2005) and correctness of examples (Zodik & Zaslavsky, 2008). Most examples exposed to students come from authorities like textbooks and teachers (Watson & Mason, 2002). Instructional examples presented by only a teacher could be distinguished straightforwardly. However, few examples generated by students are identified as instructional examples if teachers use students' examples to facilitate students' learning further. In other words, student-generated examples need to be at least relevant to what is taught in a classroom although they are incorrect examples for lesson topics. For example, students can suggest "$y = |x|$" in the lesson to illustrate an everywhere differentiable function (Zodik & Zaslavsky, 2008). Although, it is incorrect in this learning context, teachers can use it as an instructional example helping students to learn differentiable functions based on their understanding of continuous functions.

1. INCORRECT EXAMPLE

Previous studies have examined that student and teachers can have opportunities to interact with mathematical concepts through construction and use of correct examples in a mathematics classroom (e.g., Antonini et al., 2011). However, incorrect examples can also be pedagogically useful, and it is necessary to examine appearance of incorrect examples in mathematics classrooms as well. Although we often witnessed teachers' use of incorrect examples, uncertain are how and why incorrect examples emerge in teaching and learning mathematics.

To address the above questions, categorizing incorrect examples is helpful because incorrectness of examples potentially has significant roles in teaching and learning when students are asked why they are incorrect. This suggests that instructional purposes have close relationships to characteristics of those incorrect examples. For example, examples in a *general case* can be appropriate to check students' understanding of specific mathematical concepts and those in a *counter example* can be applied to proof conjectures. We utilize the three types of incorrect examples as seen in Table 1, which Zodik and Zaslavsky (2008) discussed.

Instructional examples do not necessitate mathematical correctness within context of learning. Examples in general cases and counter examples can be incorrect depending on learning contexts. Those examples are only inappropriate to mathematics concepts and theorems taught in a classroom. However, examples in non-existing cases should be mathematically incorrect because those do not exist as seen in Figure 1. Although it is mathematically incorrect, with non-existing case in Figure 1, teachers can ask students to construct triangles, which can lead to the conclusion that the sum of the lengths of two sides of a triangle should be greater than the length of the third side

Table 1. Types of "incorrectness" examples (Zodik & Zaslavsky, 2008)

Category		Description	
Type I	General Case (GC)	An example of more general class satisfies the necessary conditions to qualify as such sample	$y = \|x\|$ as an example of an everywhere differentiable function
Type II	Counter Example (CE)	A counterexample for particular conjectures or claims.	$x^2 - 3x + 1 = 0$ for the conjecture that every quadratic equation has at least one real solution
Type III	Non-Existing Case (NC)	An example of mathematically incorrectness manifested in treating a non-existing case as if it is possible.	See Figure 1

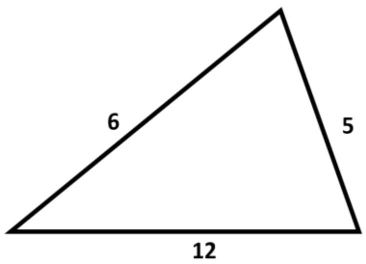

Figure 1. An example of "non-existing case" of isosceles triangles. Modified from Zodik and Zaslavsky (2008, p. 170)

2. PRE–PLANNED VS. SPONTANEOUS EXAMPLES

In addition to incorrectness of examples, Zodik and Zaslavsky (2008) introduced two interesting ways for teachers to create examples. In the lesson planning stage, teachers can prepare instructional examples using their knowledge, textbooks, and other resources. These examples are referred to pre-planned examples. On the other hand, teachers can create spontaneous examples as in-the-moment decisions during the lessons. A remarkable characteristic of spontaneous examples distinguished from pre-planned examples is that spontaneous examples are often generated in response to students' questions or claims (Zodik & Zaslavsky, 2008).

For example, when a student is asked to bisect an acute angle ∠BAC, the student can unexpectedly claim that the line bisecting the angle ∠BAC can be created by finding the midpoint of $\overline{BC}$ and connect the midpoint and the vertex A. This claim is not always true, which the teacher needs to know. Then, how does the teacher convince the students that the claim is not always true? The teacher can create other examples like Figure 2 on this spot. We can consider teacher's examples derived from student's initial answer as spontaneous examples. A teaching moment like this can also be a good learning opportunity for teachers, which result in expanding their example space (Zodik & Zaslavsky, 2008).

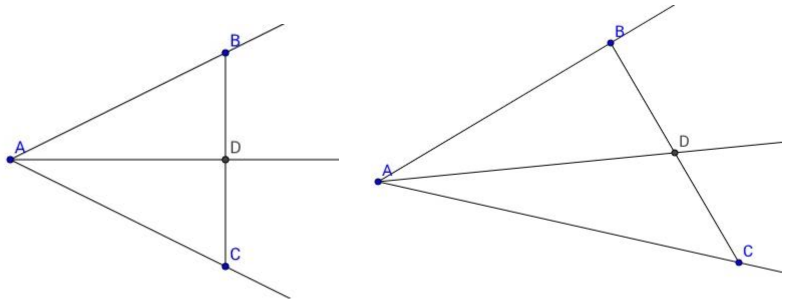

Figure 2. Teacher's possible spontaneous examples from student's claim

III. DATA COLLECTION AND ANALYSIS

The faculty member in the mathematics department at a large university in the United States, Jessica, which is a pseudonym, was observed. We examined Jessica's calculus 1 course collecting three types of qualitative data: video-taped lessons, semi-structured interviews, and field notes. Classroom observation provided direct photos of how Jessica used instructional examples in teaching the concept of limit. Field notes were taken during observation to include environmental factors that could affect teaching but possibly not be captured by videotape, like temperature, atmosphere, and scent (Creswell, 2013).

Two types of semi-structured interviews were conducted: an initial interview before the course started and multiple reflective video-stimulated interviews after initial data analysis. Each interview lasted around 50 minutes to one hour. In the initial interview, Jessica was asked to describe students' role and her role in calculus 1 class, teaching experiences (e.g., how long have you been teaching Calculus 1 in this university?), and philosophy and background (e.g., what are the students' role and your role in Calculus 1 class? And do you have any idea about students who will take your calculus class in the next semester?).

Without discussion right after lessons, we arranged a semi-structured and video-stimulated recall interview after completing preliminary analysis of data in order to ask why Jessica behaved in a certain way. Pieces of the videotaped lessons were played in the video-stimulated interviews so that Jessica was able to recall interesting episodes. The main goal of the video-stimulated interviews was to examine Jessica's perspective on what happen in the classroom episodes, what Jessica was trying to accomplish, and what evidence Jessica's choices are based on. Questions in the reflective video-stimulated interviews varied by episode.

The total number of the lessons that we observed during semester was 38 and each lesson was 50 minutes long. For this study, we selected and transcribed a lesson about the IVT. A rationale to select this lesson was that multiple instructional examples from both students and Jessica were emerged while instructional examples in other lessons were built by Jessica. Another rationale was that use of the instructional examples in this lesson was more strongly related to students' understanding of mathematical ideas. She described that the lesson about the IVT was unique because students should understand and *prove* this theorem beyond memorizing definitions and formulas although what students did in the lesson was not a proof mathematically.

After transcribing all interviews and the lesson that we collected, we used constant comparative technique and coding method (Strauss, 1987) to analyze the videotaped lesson, interviews, and field notes. The open coding methods were applied to the interviews in order to find supportive evidence for use of the instructional examples. In analysis of videotaped lesson, instructional examples were identified with checking mathematical correctness. After identification of a series of instructional examples, we documented those instructional examples with details about how they are changed.

IV. RESULTS

First of all, according to Jessica, the IVT says; Suppose f is defined and continuous on $[a, b]$ and let N be a number between $f(a)$ and $f(b)$ where $f(a) \neq f(b)$. Then there exists a number c between a and b such that $f(c) = N$. This is Jessica's exact statement written on a blackboard, which is probably related to instructional examples in her lesson.

During the lesson about the IVT, Jessica used three examples identified as incorrect examples. Remarkably, these examples were discussed before her introduction of the IVT. The initial example was constructed by a student when the class asked to draw a graph of a continuous function satisfying following conditions; (a) $f(1) = -3$, (b) $f(2) = 4$, and (c) the function never crosses the x-axis. This question was to construct impossible examples, namely the *non-existing* examples. When students were asked to share their answers with the class, no one volunteered. According to observation of students' discussion and Jessica's comments in the interviews, students in her classroom hesitated to show their graphs because students was able to realize that it is an impossible case even intuitively. After she suggested to give extra credits to a person sharing graphs, a female student drew one certainly incorrect graph on a blackboard.

Table 2. First incorrect example of the intermediate value theorem

Code	Constructor	Representation	Type	Math concept for discussion	Math concept to teach
Ex. 1	Student 1		Spontaneous & Non-Existing Case	Vertical line test & Asymptote	Intermediate Value Theorem

Classroom discussion

Teacher: All right. Um, how do you, how does everybody else feel about this picture? It will be a class about feelings. How do you feel about?

Student1: I don't like it

Teacher: Excellent. What do you not like about it?

Student1: It looks like an asymptote at a two but then the bottom part of the graph crosses at two. May be the part of the function is, it looks to me like an asymptote that is just going on. I can't tell.

Teacher: I think this [pointing out the upper part of the graph] is just going on.

Student1: Well, I was talking about it going on like it looks like an asymptote there. Yeah.

Teacher: Um. Does anything bother you about the line? This particular thing does not pass the vertical line test because this intersects this vertical line twice that's what you were trying to say and now, I'm going to fix that. Now this does pass the vertical line test and then you can give me your objection. What's wrong with?

Jessica asked a whole class to share their opinions about the initial graph. We highlight that teacher's pre-planned question employed algebraic expressions while classroom discussion used only graphical representations of functions. Furthermore, the teacher modified student's answer (Ex. 1 in Table 2), which led to teacher's spontaneous but still incorrect examples (Ex. 2 and 3 in Table 3) constructed by the teacher. While non-existing examples were discussed, students' prior knowledge such as vertical line test (Ex. 1 in Table 2) and horizontal asymptote (Ex. 2 in Table 3) were also discussed. And this classroom discussion could be meaningful learning opportunities to students using prior knowledge. Although Jessica reconstructed an example by mistake which violated the original conditions, the non-existing example also helped her to check whether or not students concentrate on solving the question by correcting her mistake.

Table 3. Second and third incorrect examples of the intermediate value theorem

Code	Constructor	Representation	Type	Math concept for discussion	Math concept to teach
Ex. 2	Teacher		Spontaneous & Non-Existing Case	Cross the x-axis	Intermediate Value Theorem
Ex. 3				Asymptote	

Classroom discussion

Student1: The graph passes through *x*-axis.

Teacher: Oh yeah. When it crosses *x*-axis, let me try again. This is actually very bad for my self-esteem. Um. All right now it does not cross the *x*-axis. Does this function do everything it's supposed to do?

Student2: That is a possible asymptote

Teacher: Yeah, there is an asymptote some place over here, which means it's not defined everywhere. So now, you have a definition problem. All right. Did anybody have a graph that does not do that now? All right. So, the real issue is that I'm supposed to have this continuous everywhere defined function and it means to connect these two points right so I'm supposed to connect these two points. And now I'm not allowed to lift the chalk because that’s sort of good working definition for continuous at the moment and to get from there to there. I at some point have to cross the x-axis right. I just don't have any choices, so I give you an impossible task, which is why only people with high self-esteem were allowed to play. And this is what intermediate value theorem says and this together with the squeeze theorem are the two theorems people always forget to use.

Teacher’s use of incorrect examples on purpose can play a role to review what students already learned as well as prove the theorem informally. When we asked Jessica what she had expected students to answer, she said that she did not have a specific expectation. Instead, she recognized that constructing non-existing cases can be a good chance of review previous concepts whatever students answered. Particularly, Jessica indicated that asymptote was the most possible answer based on her experiences.

It should be noted again that these examples were used before presenting the IVT. Thus, we questioned Jessica why she asked students to create non-existing cases before presenting and explaining the IVT in interviews. At that time, the teacher answered as following; “And sort of in the heads, it’s kind of proof because they tried to do it and couldn’t do it, therefore, cannot be done and obviously mathematically that is not a proof, […] this is a sort of beginning of any mathematical proof of something, you try to do it, and figure out why cannot be done or why can be done.”

V. DISCUSSION AND CONCLUSION

The episode, which we discussed here, showed that teachers’ use of incorrect examples on purpose can provide meaningful learning opportunities to students. Jessica’s episode about the IVT showed that use of incorrect examples needs to be well planned and organized. Within her facilitation, these examples made the path for students to understand the IVT. Students engaged in an opportunity to prove the IVT informally. Thus, students’ opportunity to learn depends on how teachers organize and use multiple instructional examples well for one learning goal.

We recognized that her preference on graphs and specific objects might contribute to using a sequence of instructional examples with some modification in introducing the IVT. Relying on algebraic expressions, proving theorems and connecting concepts might be very challenging to both teacher and students. We examined that, in other lessons, Jessica mainly used graphical representations and specific mathematical objects (e.g., $f(x) = x^2$ for a quadratic function instead of $f(x) = ax^2 + b$ and $a \neq 0$) in her examples consistently. In the episode we discussed, she also used graphs and specific objects ($f(1) = -3$, $f(2) = 4$, and the function never crosses the x-axis), which helped students to prove theorems intuitively.

Jessica showed a possibility of non-existing cases for use to check understanding of mathematics concepts. Incorrect examples can be used to indicate whether or not students are prepared for the next lesson by reviewing what they already learned. In addition, creating non-existing cases based on the IVT was an intuitive way to begin proving the theorem. This use of examples could contribute to students’ preliminary opportunity for mathematical formal reasoning. This episode also described how spontaneous examples can contribute to students’ learning combined with incorrectness of examples.

Interestingly, one of Jessica's incorrect examples was spontaneously constructed by mistake violating the given conditions of the continuous function. This showed another possible use of spontaneous and incorrect examples in student-centered learning environment. This is because this example could show whether or not students engaged in the classroom discussion of constructing non-existing cases. In addition, spontaneous examples are more meaningful to students because these are constructed based on students' responses. Thus, proper use of spontaneous and incorrect examples could be significant in mathematics instructions although spontaneous examples could be created incorrectly by mistake.

In this article, we focused on how to use non-existing-case examples for instructional purposes. However, it does not mean that other types of incorrect examples are less important than examples of non-existing cases. For example, Jessica used incorrect examples in a *general case* when she introduced the continuity of functions. By providing discontinuous functions right after discussion of continuity, these examples allowed the teacher to check whether students can apply the definition of continuity to a specific function as well as how students understand the concept of continuity.

More observations and studies are certainly required to have better ideas about teacher's use of incorrect examples for students' better opportunities to learn. This is because this study discussed only one case at a tertiary level. It is possible to consider use of non-existing examples at the elementary or secondary level. For instance, Figure 1 might be used to teach the triangle inequality; the sum of the lengths of any two sides of a triangle must be greater than the third side. It is possible to present this non-existing example to students in a similar way with our episode. However, we need to think more about the best way to use Figure 1 to teach the theorem; for example, how can teachers immediately create follow-up examples to facilitate students' thinking from their initial responses to Figure 1? With better understanding to use incorrect examples, we could help students to prove the triangle inequality informally instead of memorizing and applying this inequality to solve problems.

Lastly, mathematics teachers need to reflect their use of incorrect examples as well as educators need to study how to use those examples as professional development. We talked about only one episode in this article. However, if educators pay more attentions to use of incorrect examples in mathematics instructions, students could have opportunities to understand mathematically correctness, prove theorems informally, and discuss about mathematical concepts with a variety of representations. Moreover, mores studies can link teachers' use of instructional examples to their knowledge for teaching (Zodik & Zaslavsky, 2008). Additionally, replication studies in secondary and elementary education might be required because Jessica is a professional mathematician.

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