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Zengcheng Kaifangfa and Zeros of Polynomials

增乘開方法과 多項方程式의 解

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Dedicated to our collaborators, Kim Young Wook (金英郁) and Lee Seung On (李承溫) on their retirements

Extending the method of extractions of square and cube roots in Jiuzhang Suanshu, Jia Xian introduced zengcheng kaifangfa in the 11th century. The process of zengcheng kaifangfa is exactly the same with that in Ruffini-Horner method introduced in the 19th century. The latter is based on the synthetic divisions, but zengcheng kaifangfa uses the binomial expansions. Since zengcheng kaifangfa is based on binomial expansions, traditional mathematicians in East Asia could not relate the fact that solutions of polynomial equation $p(x) = 0$ are determined by the linear factorization of $p(x)$. The purpose of this paper is to reveal the difference between the mathematical structures of zengcheng kaifangfa and Ruffini-Honer method. For this object, we first discuss the reasons for zengcheng kaifangfa having difficulties to connect solutions with linear factors. Furthermore, investigating multiple solutions of equations constructed by tianyuanshu, we show differences between two methods and the structure of word problems in the East Asian mathematics.

Keywords: East Asian mathematics, polynomial equations, zengcheng kaifangfa, Ruffini-Horner method, multiple zeros of polynomials, word problems.

MSC: 01A13, 01A25, 01A27, 12-03

1 Introduction

Theory of equations in China has been established in the period from Song Dynasty (960–1279) to Yuan Dynasty (1271–1368). Introducing tianyuanshu (天元術) to represent polynomials with their basic operations, mathematicians in the dynasties can construct polynomial equations. Extending the extractions of square and cube

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roots in Jiuzhang Suanshu (九章算術), they introduced zengcheng kaifangfa (增乘開方法) for solving polynomial equations. Zhu Shijie (朱世傑) deals with tianyuanshu in his Suanxue Qimeng (算學啓蒙, 1299) and then added the siyuanshu (四元術), the method to represent polynomials of up to four variables in his Siyuan Yujian (四元玉鑑, 1303). Although the books dealing with eryuanshu (二元術) and sanyuanshu (三元術), were mentioned in the preface of Siyuan Yujian, they were not transmitted to the present. Thus, Siyuan Yujian is a unique book dealing with representaions of several variable polynomials [8]. We also note that Suanxue Qimeng and Siyuan Yujian were almost forgotten in China until the 19th century.

The fourth king Sejong (世宗, 1397–1450, r. 1418–1450) of the Joseon Dynasty (1392–1910) imported YangHui Suanfa (楊輝算法, 1274–1275) and Suanxue Qimeng for improving the calendar system in Joseon along with Chinese astromical books. Since then, YangHui Suanfa and Suanxue Qimeng had become the most important references for the development of Joseon mathematics. The two books were transmitted from Joseon to Japan in the last decade of the 16th century and played the fundamental role to establish the Japanese mathematics, wasan (和算). Indeed, Japanese mathematicians in the 17th–18th centuries established the wasan based on tianyuanshu and zengcheng kaifangfa. YangHui Suanfa includes zengcheng kaifangfa for quadratic equations and one quartic equation. Although Zhu Shijie obtained polynomial equations by tianyuanshu, he did not include the process to solve them except the extractions up to quartic roots by zengcheng kaifangfa.

Since Kim Si-jin (金始振, 1621–1668) republished Suanxue Qimeng in 1660, Joseon mathematicians, notably Park Yul (朴繘, 1621–1668) and Hong Jeong-ha (洪正夏, 1684–1727) paid once again their attentions to the theory of equations based on tianyuanshu and zengcheng kaifangfa.

Hong Jeong-ha served mathematical officer in Hojo (戶曹) and completed the most important mathematical book, Gu-il Jib (九一集, 1713–1724) [3] in the history of Joseon mathematics. Hong Jeong-ha showed the mathematical structure of zengcheng kaifangfa in Gu-il Jib. Indeed, he showed that the structure is based on the binomial expansions as the extractions of square and cube roots in Jiuzhang Suanshu (see [7]). Furthermore, he obtained equations by tianyuanshu and hence they are of the form $p(x) = 0$ along Suanxue Qimeng. Furthermore, Hong Jeong-ha applied zengcheng kaifangfa to solve equations $p(x) = 0$. We recall that equations in YangHui Suanfa are of the form $p(x) = a$, where x divides $p(x)$ and a is a positive number [6].

Joseon scholar, Hong Dae-yong (洪大容, 1731–1783) wrote a book Juhae Suyong (籌解需用) [2] which is divided into two parts. The first part deals with the traditional mathematics and the second one with mathematics influenced by Shuli Jingyun (數理精蘊, 1723) and the western astronomy. Hong Dae-yong visited Yanjing (燕京) as an envoy in 1765 and discussed mathematics and astronomy with Chinese scholars and Catholic missonaries. He completed Juhae Suyong around the early part of 1770. We will discuss the detail of Juhae Suyong in a separate paper. He introduced the terminology *Cheonwon Hae* (天元解), literally solving tianyuan, or solutions of equations obtained by tianyuanshu. In the section Cheonwon Hae, Hong explained the details of zengcheng kaifangfa for polynomial equations in the last section, kaifang shisuo men (開方釋鎖門) of Suanxue Qimeng, which were omitted as we mentioned above. The first one is the well known problem to solve a rectangle given with the sum of two sides, and area. Hong Dae-yong included zengcheng kaifangfa for the length and width of the given rectangle unlike the usual method in the other books. Clearly the equations for them are the same because of the symmetry of the given conditions. By Ruffini-Horner method, the other solution is immediate because it gives rise to another linear factor of the equation.

The above fact motivates our paper. Zengcheng kaifangfa is a suitable method to solve polynomial equations in the field of rational numbers. We also compare zengcheng kaifangfa with Ruffini-Horner method. Discussing the mathematical structure of zengcheng kaifangfa, we show that zengcheng kaifangfa contains numerous impediments to reveal the structure of solutions of polynomial equations.

In Section 2, we deal with the reasons why zengcheng kaifangfa misses to relate its processes to a linear factor of the equation given by solutions. In Section 3, we discuss multiple solutions of polynomial equations and the structure of word problems in Yigu Yanduan (益古演段, 1259) and Suanxue Qimeng.

The reader may find all the Chinese sources of this paper in [1] and hence they are not numbered as an individual reference.

2 Zengcheng Kaifangfa and Factorizations

In this section, we discuss the structure of zengcheng kaifangfa which give rise to essential impediments for East Asian mathematicians to relate solutions of polynomial equations to their factorizations.

As established in Jiuzhang Suanshu, mathematical structures in East Asian mathematics are revealed by solving word problems. They took the word problems related to daily ones so that answers should be positive numbers. We should mention a note Kaihō Honhen no Hō (開方飜變之法, 1685) [10] written by Seki Takakazu (關孝和, ?–1708) which dealt with solutions of polynomial equations without referring any word problems. The detail will be discussed later.

In the following, $p(x)$ will denote a polynomial.

For solving an polynomial equation $p(x) = 0$, zengcheng kaifangfa uses the repeating processes to find the next digits of solutions, called cishang (次商) after their first digits, chushang (初商). They used the calculating rods for the basic operations. Assume that the place value of chushang is $\alpha \times 10^k$ (α is a digit and k is an integer) for the equation. Interestingly, they use α instead of the place value in zengcheng kaifangfa since Jiuzhang Suanshu. Thus, they have to change the equation $p(x) = 0$ into $p(10^k y) = 0$ by shifting coefficients of the original equation. Although they obtain the usual result by reversing the shifts, $p(x)$ and $p(10^k y)$ is completely different and hence it would be difficult to relate zengcheng kaifangfa to the division of $p(x)$ by $x - \alpha$.

The extant historical records dealing with zengcheng kaifangfa is Shushu Jiuzhang (數書九章, 1247) written by Qin Jiushao (秦九韶, 1202–1261). In the famous example, zhengfu kaisanchengfang tu (正負 開三乘方圖), Qin Jiushao used the present zero's 0 to indicate the place values in the calculating rods representations but retained the traditional processes of shifting coefficients. We also note that Suanxue Qimeng begins with the section, zongheng yinfa men (縱橫因法門), which deals with multiplication with one multiplier of the form $\alpha \times 10^k$ (α is a digit, k is a natural number) and the final section, kaifang shisuo men (開方釋鎖門) of Suanxue Qimeng deals with the theory of equations as mentioned above. But Zhu Shijie used again the processes of shiftings in his zengcheng kaifangfa.

Now we return to Seki's Kaihō Honhen no Hō mentioned above. He classified equations along numbers of solutions and those with negative solutions. The topic is also included in Taisei Sankei (大成算經, 1711) [11], compiled by Seki, Takebe Katahiro (建部賢弘, 1664–1739) and his brother Takebe Kataa'kira (建部賢明, 1661–1716). The presentaions in Kaihō Honhen no Hō were much more polished in Book 3 of Taisei Sankei.

Except Seki Takakazu, every mathematician in the East Asia did not go further after having a solution, called *qiajin* (恰盡) in the process of zengcheng kaifangfa. Seki *continued* zengcheng kaifangfa to have the representaion, $p(x) = \sum_{k=0}^{n} c_k (x - \alpha)^k$ of $p(x)$ for a solution α of the equation $p(x) = 0$ (also see [4, 5]). The representation is known as the Taylor polynomial of $p(x)$. The final represention is called *henshiki* (變式) in Taisei Sankei. Using the henshiki, they could have classified equations along the number of solutions including negative solutions. We include examples in Taisei Sankei to show the differences between zengcheng kaifangfa and Ruffini-Horner method. But we omit the process of zengcheng kaifangfa.

Example 1. For the equation $x^2 - 4x + 4 = 0$, Seki has the henshiki $(x - 2)^2$ for a solution 2 and claims that the equation has a unique solution. We note that by Ruffini-Horner method, or synthetic division, one can have the quotient $x - 2$ in the division of $x^2 - 4x + 4$ by $x - 2$ in the *first* step and hence the factorization.

Example 2. For the equation $x^2 - 3x + 2 = 0$, the henshiki for a solution 1 is $(x-1)^2 - (x-1)$, i.e., $(x-1)\{(x-1)-1\}$. Thus, they conclude the equation has two solutions 1 and 2. They also include the expansion $(x-2)^2 + (x-2)$ at the solution 2 which gives rise to another solution 1. Clearly one can have another solution by the first step of the synthetic divisions.

Example 3. For the equation $-x^3 + 7x + 6 = 0$, the henshiki for a solution 3 is $-(x-3)^3 - 9(x-3)^2 - 20(x-3)$, i.e., $-x^3 - 7x + 6 = -(x-3)^3 - 9(x-3)^2 - 20(x-3)$. Further, $-y^2 - 9y - 20 = -(y+4)^2 - (y+4)$ by zengcheng kaifangfa at -4. This gives rise to the following factorizations :

$$\begin{aligned}
-x^3 - 7x + 6 &= (x-3)\{-(x-3)^2 - 9(x-3) - 20\} \\
&= (x-3)[-\{(x-3)+4\}^2 - \{(x-3)+4\}] \\
&= (x-3)[-(x+1)\{(x+1)+1\}] \\
&= -(x-3)(x+1)(x+2).
\end{aligned}$$

In all, the equation has three solutions $3, -1, -2$.

Extending zengcheng kaifangfa to have the equation for cishang with a chushang, Seki finds all the possible solutions of the given equation. Once again, one has immediately the factorization $-x^3 + 7x + 6 = (x-3)(-x^2 - 3x - 2)$ by the synthetic division and then one can have easily the remaining solutions.

The more decisive problem to relate solutions of $p(x) = 0$ obtained by the zengcheng kaifangfa to the linear factorizations of $p(x)$ is resulted by repeating zengcheng kaifangfa for each digit of multi-digit solutions. Indeed, the readers could easily lost the connections between the resulting polynomials in the processes for each digit. We note that every example in Taisei Sankei discussed in the above has *single* digit solutions.

3 Zeros of polynomial equations and zengcheng kaifangfa

In this section, we deal with word problems and their solutions based on zengcheng kaifangfa.

In Qin Jiushao's zhengfu kaisanchengfang tu mentioned in the previous section, the equation $x^4 - 763,200x^2 + 40,642,560,000 = 0$ has two solutions, 240 and 840. Indeed, the problem, namely the sum of two isoscels triangles can be easily obtained. To solve the problem, one must have the heights of two triangles which involve the square roots. In the given problem, one can have immediately the heights. But constructing the equation for the problem, Qin has to eliminate the terms of square roots so that he has the quartic equation. This implies the extraneous root.

Except Qin's example, there are numerous cases which have multiple solutions. Discussing problems with multiple solutions in Yigu Yanduan (益古演段, 1259) of Li Ye (李冶, 1192–1279), and Suanxue Qimeng, we reveal the structures of solutions based on zengcheng kaifangfa and word problems in East Asian mathematics.

Since Yigu Yanduan and Suanxue Qimeng include details for constructions of equations by tianyuanshu, we just pay attentions to their solutions (also see [9] for Yigu Yanduan).

Li Ye completed Ceyuan Haijing (測圓海鏡, 1248) where he constructed equations by tianyuanshu. As is well known, every problem in Ceyuan Haijing is to find a length under the strict dimensions and hence it should have a unique solution.

Indeed, the equation for Problem 8 in Book 4, is $0.5x^2 - 820x + 230,400 = 0$. It has two positive solutions 360 and $1,280$. Clearly the latter is extraneous one.

Thus, we take Li's Yigu Yanduan to discuss multiple solutions. Yigu Yanduan contains 64 word problems. They relate to geometrical problems and construction of equations by tianyuanshu to solve them as well in Ceyuan Haijing. Except 6 linear equations, Li has the remaining 58 quadratic equations. Among them, there are 20 problems with two positive solutions. We recall once again that negative solutions relating to geometrical problems are meaningless or extraneous.

Among 20 problems with two positive solutions, 12 problems have extraneous solutions and the remaining 8 problems have another solutions. Since they are all quadratic equations, another solutions are immediate from the first step of the synthetic divisions on the given solutions. We disclose some examples, where the problem's number is one given in Yigu Yanduan. We first discuss problems with extraneous solution among Problem 2, 3, 5, 14, 18, 26, 29, 41, 46, 51, 55, and 61.

Problem 3.

> 今有方田一段 內有圓池 水占之外 計地一萬一千三百二十八步
> 只云從外田角斜至內池楞各五十二步 問內徑外方各多少.

Li Ye has the equation $0.47x^2 - 208x + 11,386.88 = 0$,
for the diameter x of the circular pond (圓池) inside the square (方田). The detail can be found in [9]. The equation has two solutions 64 and $378\dfrac{25}{47}$. Since the additional condition (只云) says that the diagonal a of the given square and the diameter $d = x$ of the circle satisfies $a = 2 \times 52 + x$, i.e., $a = x + 104$. Li Ye takes the convention $\sqrt{2} = 1.4$ and hence the relation between a side of a square, say y and its diagonal a should be $a = 1.4y$ for the sake of *consistency*. Furthermore, $a^2 = 1.96y^2$ instead of the usual $a^2 = 2y^2$. Using them, Li obtains the above equation $0.47x^2 - 208x + 11,386.88 = 0$ and $y = \dfrac{a}{1.4}$ or $y = \dfrac{5a}{7}$. Li has the solution $y = 120$ for the solution $x = 60$. But for $x = 378\dfrac{25}{47}$, we have $y < x$ which contradicts to the circular pond

being inside the square. Thus, the solution $378\frac{25}{47}$ is an extraneous solution for the word problem.

Problem 5

今有方田一段 內有圓池 水占之外 計地一十三畝二分
只云內圓周不及外方周 一百六十八步 問方圓周各多少

For the circumferences x and y of the given circle (圓池) and square (方田), one has $y - 168 = x$ by the given condition. Thus, we have the equation $-x^2 + 1,008x - 67,392 = 0$, which has solutions 72 and 936. For $x = 72$, one has $y = 168 + 72 (= 240)$. For the other solution, $x = 936$, we have $y = 168 + 936 (= 1,104)$. But for $(936, 1, 104)$, the diameter of the circle is $\frac{936}{3} = 312$ ($\pi = 3$) and the side of the square is $\frac{1,104}{4} = 276$. Thus $(936, 1, 104)$ is not a solution for the word problem.

Problem 41

今有直田一段 中心有圓池 水占之外 計地三千九百二十四步
只云從外田角斜通池徑七十一步 外田長闊相和得一百五十八步 問三事各多少

For the diameter x of the circle (圓池) and the diameter y of the rectangle (直田), one has $y = 2 \times 71 - x$. Using this together with the given sum of two sides, say l, m, we have the equation $2.5x^2 - 284x + 3,048 = 0$. It has two solutions 12 and $\frac{254}{2.5} = 101.6$. Using the sum $l + m = 158$ and $y^2 = l^2 + m^2$, we have an equation $(158-m)^2+m^2 = (142-x)^2$. For $x = 12$, one has the equation $m^2-158m+4,032 = 0$ which has two solutions, namely the width (闊) 32 and the length (長) 126. But for the another solution 101.6, we have the equation $2m^2 - 316m + 158^2 = 40.4^2$, i.e., $m^2 - 158m + 11,665.92 = 0$, which has no *real* solution. Thus, the given set $(12, 32, 126)$ is a unique solution. Clearly, the above arguments cannot be understood by the traditional mathematicians. But we include the problem for the equations with multiple positive solutions.

Problem 46

今有方 圓田各一段 共計積一百二十步 只云方面大如圓徑
圓徑穿方斜共得二十步 問面 徑各多少.

For the diameter x of the circle (圓田) and the diagonal y of the square (方田), $x + y = 20$ and $x < y$. One has the equation $-2.47x^2 + 40x - 151.08 = 0$. In Yigu Yanduan, one has solution $x = 6$. By the given condition $x + y = 20$, we have $y = 14$ so that the side of the square is $\frac{14}{1.4} = 10$. But the equation has another solution $\frac{25.18}{2.47}$ which is bigger than 10 and hence $y < 10$. Since $x < y$, $\frac{25.18}{2.47}$ cannot be a solution.

We should mention that $x = 6$ is a single digit solution so that the second solution can be easily obtained by zengcheng kaifangfa. Furthermore, the most famous problem of a rectangle with the area and the sum of two sides, say a, b with $a < b$ and $\alpha = a + b$, the equation for two sides is $x^2 - (a + b)x + ab = 0$. Then one can choose a and b by $a < \frac{\alpha}{2} < b$.

We choose the above 3 problems with the single solution for the word problems but the equations have multiple positive solutions. As shown above, the other solutions of equations become extraneous by the additional conditions given by identities or inequalities. We omit the remaining 7 problems.

Now we discuss 8 problems with multiple solutions for *word problems*. They are Problem 11, 22, 24, 30, 42, 50, 54, and 57.

Problem 11

今有圓田一段 內有方池 水占之外 計地二十五畝 餘二百四步
只云外田楞至四邊各三十二步 問外圓內方各多少

For the side x of the square (方池), the diameter y of the circle (圓田), $y = x + 64$. Thus, one has the equation $x^2 - 384x + 12,528 = 0$. It has solutions $x = 36$ or $x = 348$, and therefore $y = 100$ or 412. In all, the problem has two sets of solutions, $(36, 100)$ and $(348, 412)$.

Problem 22

今有方田一段 其西北隅被斜水占之外 計地一千二百一十二步七分半
只云從田東南隅至水楞四十五步半 問田方面多少

Let x be the height of the triangle (西北隅被斜水占) along the diagonal of the square (方田), then the diagonal of the square is $x + 45.5$. Using the given area $1,212.75$, one has the equation $0.96x^2 - 91x + 306.74 = 0$. It has two solutions, namely 3.5 and $\frac{2,191}{24}$. The first solution 3.5 implies the diagonal $45.5 + 3.5 = 49$ and hence the side of the square is 35 ($\sqrt{2} = 1.4$). For the second solution $\frac{2,191}{24}$, one has another solution $\frac{82,075}{840}$ by the diagonal $\frac{2,191}{24} + 45.5$. Thus Problem 22 has also two sets of solution.

Problem 30

今有圓田二段(一段依圓三徑一率 一段依密率) 共積六百十一步
只云二徑共相和得四十步 問二徑各數

Let x be the diameter of a circle (圓田) measured on the milü ($\pi = \frac{22}{7}$) and y that of circle (圓田) on the gulü ($\pi = 3$). Using the given conditions, one has the equation $43x^2 - 1,680x + 15,092 = 0$ which has solutions 14 and $\frac{1,078}{43}$. Since $x + y = 40$, we have two sets $(14, 26)$ and $(\frac{1,078}{43}, \frac{642}{43})$ for (x, y).

Problem 54

今有方田一段 內有直池結角占之 外計地一千一百五十步
只云田角至水兩頭各一十四步 至水兩邊一十九步 問三事各多少

Let x and y be the width (闊) and the length (長) of the given rectangle (直池) and z the side of the square (方田). Then the diagonal of the square is $x+2\times 19 = x+38$ which equals $y+2\times 14 = y+28$. Thus, $y-x = 10$. Using these, one has the equation $0.96x^2 - 56.4x + 810 = 0$, which has solutions 25 and 33.75. For $x = 25$, the diagonal of the square is $25 + 38 = 63$, which implies $z = \frac{63}{1.4} = 45$ ($\sqrt{2} = 1.4$). Thus, Li has the solution $(x, y, z) = (25, 35, 63)$. Similarly, for $x = 33.75$, we have another solution $(33.75, 43.75, 51.25)$.

Li Ye dealt with quadratic and linear equations in his Yigu Yanduan but using tianyuanshu, Zhu Shijie constructed equations up to quintic equations, altogether 35 problems. Among them, 9 problems are of the type $ax^n - b = 0$. In the remaining 26 problems, we have 9 problems with multiple positive solutions. Except one problem, the remaining 8 problems have extraneous solutions. Zhu missed just one problem with another solution. We recall that Zhu Shijie introduced the construction by tianyuanshu in the section fangcheng zhengfu men (方程正負門). Except this problem, we use the numbers of problems as those in kaifang shisuo men.

We first discuss the problem with two solutions.

Problem 17

今有圓田一段 內有方池 容邊而占之外餘地八畝六十五步七分半
只云四弧矢各闊一十三步 問圓徑池方各幾何

For the diameter x of the circle (圓田) and the side y of the square (方池), $y = x - 26$. Thus, one has the equation $x^2 - 208x + 10,647 = 0$. The equation has two solutions 91 and 117. Thus, by the given condition, we have two solutions $(91, 65)$ and $(117, 91)$ for (x, y).

We will deal with word problems whose equations have two positive solutions but their solutions for the word problems are unique. The problems are problem 9 in the section fangcheng zhengfu men and number 8, 14, 18, 19, 22, 24 and 27 in the section kaifang shisuo men.

Problem 9 in the section fangcheng zhengfu men

> 今有直田句弦和取七分之四 股弦和取七分之六 二數相減餘二十二步
> 又股弦和取三分之一不及句弦和八分之五一十四步 問句股弦各幾何

For the gou (句), gu (股), xian (弦) z, y, x, the system of linear equations on $z + x, y + x$ has the solution $(56, 63)$ and hence $z = 56 - x$ and $y = 63 - x$. Thus one has the equation $x^2 - 239x + 7,105 = 0$ with solutions 35 and 203. Clearly, $x = 35$ implies $z = 21$ and $y = 28$. Since for $x = 203$, z and y are negative, $x = 203$ is extraneous. In all, the word problem has the single solution $(21, 28, 35)$.

Problem 14

> 今有直田九畝八分 只云長取八分之五 平取三分之二相併得六十三步
> 問長平各幾何

For the width (平) x and the length(長) y of the rectangle (直田), we have the equation $-16x^2 + 1,512x - 35,280 = 0$. It then has solutions 42 and $\frac{840}{16} = 52.5$. The first solution 42 gives rise to $y = 56$, but 52.5 to $y = 44.8$. The latter is extraneous for $x < y$. Thus, the word problem has a single solution.

Problem 22

> 今有大小方田二段 只云大方冪內減小方面餘一千二百六十八步
> 又云小方冪內減大方面餘七百四十八步 問大小方面各幾何

For the side x of the smaller square (小方田) and the side a of the larger square (大方田), the given conditions mean $a^2 - x = 1,268$ and $x^2 - a = 748$. Thus, one has the equation $x^4 - 1,496x^2 - x + 558,236 = 0$. It has the solution 28 and the resulting factorization $(x - 28)(x^3 + 28x^2 - 712x - 19,937)$. Let $q(x)$ denote the quotient. Using *curve sketching* of the function $y = q(x)$, the equation $q(x) = 0$ has two negative solutions and one positive solution. Thus, the original quartic equation has two positive solutions. Furthermore, the second positive solution, or that of $q(x) = 0$ is less than 27. Since $x^2 - 748 > 0$, $x > \sqrt{748} > 27$, the second positive solution is extraneous. In all, the word problem has a unique solution.

Problem 24

> 今有直田長平相乘爲實平方開之得數加長平和得一百二十九步
> 只云差三十九步 問長平各幾何

Let a, b be the length and width of the rectangle (直田), then the given conditions are $\sqrt{ab} + (a + b) = 129$ and $a - b = 39$. Zhu took $a + b$ as tianyuan x and obtained the equation $3x^2 - 1,032x + 68,085 = 0$. It has two positive solutions 89 and 255.

The first solution gives rise to the solution $(a, b) = (64, 25)$. Since the first condition $\sqrt{ab} + (a + b) = 129$ implies $129 - x > 0$, the second solution 255 is extraneous. Noting that the first condition contains an irrational term $\sqrt{ab}$, Zhu Shijie dealt with a system of irrational equations.

Problem 27

今有圓田一段 周爲實平方開之得數加入圓積共得一百一十四步 問周徑各幾何

Let x and c be the diameter and circumference of the circle (圓田), then the given conditions are $\sqrt{c} + \frac{3x^2}{4} = 114$ and $c = 3x$ ($\pi = 3$). Thus, one has the equation $9x^4 - 2,736x^2 - 48x + 207,936 = 0$. It has a solution 12 and hence the factorization is given as $(x - 12)(9x^3 + 108x^2 - 1,440x - 17,328)$.

As in Problem 22, we denote the quotient $q(x)$. By *sketching curve* of the function $y = q(x)$, the equation $q(x) = 0$ has two negative solutions and one positive solution. Since $q(12.5) < 0$, its positive solution should be bigger than 12.5. Thus, the original equation has two positive solutions. The condition $\sqrt{c}+\frac{3x^2}{4} = 114$ implies $456 - 3x^2 > 0$, i.e., $152 - x^2 > 0$. Thus, one has the range $-\sqrt{152} < x < \sqrt{152}$. Since $\sqrt{152} < 12.5$, the positive solution of $q(x) = 0$ is extraneous for the word problem.

We point out that the equations for Problem 28 - 34 have a *unique positive* solution. Indeed, the equation for Problem 30 (31. resp.) is quartic (quintic, resp.) and the remaining problems are solved by cubic equations. For those problems with the equation $p(x) = 0$ and its solution α, let $q(x)$ be the factor of $p(x) = (x-\alpha)q(x)$. The quadratic equations $q(x) = 0$ for Problem 28 and 29 do not have *real* roots. Every coefficient of $q(x) = 0$ for the remaining problems is positive and hence they do not have any positive solution.

We will discuss word problems of Hong Jeong-ha's Gu-il Jib in a separate paper.

4 Conclusions

There were many historians who associated mistakenly zengcheng kaifangfa (增乘開方法) to Ruffini-Horner method. Although multiplications of polynomials are introduced as a basic operation in tianyuanshu (天元術), East Asian mathematicians did not consider its inverse operation, namely division except that by x^n. Furthermore, zengcheng kaifangfa was originated by the extractions of square and cube roots in Jiuzhang Suanshu (九章算術) and binomial expansions. Thus, they could not relate solution, say α of a polynomial equation $p(x) = 0$ to the factorization of $p(x)$ by $x - \alpha$. Furthermore, they used zengcheng kaifangfa to find digits of the solution in turn. These processes block the relation between them. As established in

Jiuzhang Suanshu, mathematical structures were described by solving word problems. Except the Japanese mathematician Seki Takakazu (關孝和) and his students, Takebe (建部) brothers, East Asian mathematicians did not give any concern about multiple solutions and negative ones for word problems. We note that Seki and his associates introduced their theory of equations free from word problems but missed the nature of synthetic divisions.

Using multiple solutions of word problems in Li Ye's (李冶) Yigu Yanduan (益古演段) and Zhu Shijie's (朱世傑) Suanxue Qimeng (算學啓蒙), we reveal once again the mathematical structures of zengcheng kaifangfa and word problems in the 13th century Chinese mathematics.

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