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# MULTI-PARAMETER TIKHONOV REGULARIZATION PROBLEM WITH MULTIPLE RIGHT HAND SIDES

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Abstract. This study shows that image deblurring problems can be transformed into the multi-parameter Tikhonov type with multiple right hand sides. Also, this paper proposes the extension of the global generalized cross validation to obtain an appropriate choice of the regularization parameters for this problem. The experimental results of using the preconditioned Gl-CGLS algorithm were analyzed.

## 1. Introduction

A linear ill-posed system with $s$ multiple right hand sides occurring in the image deblurring problem can be modeled as

$$B = HX + \mathcal{E}, \tag{1.1}$$

where $B \in R^{M\times s}$ is a matrix that represents the blurred and noisy image, $H \in R^{M\times N}(M \geq N)$, a large ill-conditioned blur matrix, and $\mathcal{E} \in R^{M\times s}$, a matrix that models additive random noise. Our goal is to compute an approximation of the matrix $X \in R^{N\times s}$ which represents the original image.

Since the measurement noises and round-off errors always exists, methods for constructing a stable approximation of (1.1) should be developed to restrain the noise effects. Regularization methods can produce an approximation with parameters to control the ill-posed degree of problem (1.1). The classical Tikhonov regularization method uses a single constraint and hence the general-form Tikhonov regularization

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problem with multiple right hand sides naturally has a single regularization parameter:

$$\min_{X}\{\|HX - B\|_F^2 + \lambda^2 \|LX\|_F^2\}, \tag{1.2}$$

where $\lambda$ is a positive regularization parameter which governs the trade-off between the fit to the observation data and the smoothness of the restoration and the $L$ is a regularization matrix which provides the a priori information ([7, 8]). The effectiveness of regularization methods depends strongly on the reliability of the estimated regularization parameter. As the single-parameter regularization uses one penalty, a regularized solution is too smooth to preserve certain features of the original solution in some case. Thus, concerns of the multi-parameter regularization method using multiple constraints have been increasing. The multi-parameter Tikhonov regularization method adds multiple different penalties to exhibit multi-scale features of the solution.

When $s = 1$(single right hand side), the multi-parameter regularization method has been researched in several papers. One discusses a multi-parameter regularization method as the solution for over-determined and ill-conditioned linear systems [1]. Another study found the optimal regularization parameter by minimizing the average of the errors between the filtered solutions and the true data in the multi-parameter Tikhonov problem ([3]). A damped Morozov discrepancy principle for choosing regularization parameters was presented for the multi-parameter regularization method ([10]). The superiority of the multi-parameter regularization over single-parameter regularization has been shown in ([2, 6, 5]).

This research aims at applying multi-parameter concept to the regularization problem (1.2) with multiple right hand sides and extending the global GCV function for determining optimal regularization parameters. Moreover, we update the preconditioned global conjugate gradient linear least squares(Gl-CGLS) method in [8] to simulate the multi-parameter Tikhonov regularization problem with multiple right hand sides.

This article is organized as follows. A brief review of the multi-parameter regularization method with single right hand side is summarized in Section 2. In Section 3, we introduce an appropriate extension of global generalized cross validation to decide regularization parameters and Gl-CGLS method for solving image deblurring problems which is the multi-parameter regularization method with multiple right hand sides. Lastly section 4 contains numerical experiments and final remarks.

## 2. Review of the multi-parameter Tikhonov problem in $s = 1$

The simplest and the most well known regularization method is Tikhonov's method with single right hand side which solves the following regularized least-squares problem:

$$\min_x (\|Hx - b\|_2^2 + \lambda^2 \|Lx\|_2^2), \tag{2.1}$$

where $\lambda > 0$ is called the regularization parameter. Tikhonov's method can be extended by using multiple constraints

$$\min_x (\|Hx - b\|_2^2 + \sum_{j=1}^{J} \lambda_j^2 \|L_j x\|_2^2), \tag{2.2}$$

where $J$ denotes the number of constraints and $\lambda_1, \ldots, \lambda_J$ are the corresponding regularization parameters.

Using the normal equations for (2.2)

$$(H^T H + \sum_{j=1}^{J} \lambda_j^2 L_j^T L_j)x = H^T b,$$

the regularization solution is written by $x_\lambda = (H^T H + \sum_{j=1}^{J} \lambda_j^2 L_j^T L_j)^{-1} H^T b$.

Under certain assumption on the boundary conditions, $H$ and $L_j$ are simultaneously diagonalizable as

$$H = QCQ^* \quad \text{and} \quad L_j = QS_jQ^* \quad j = 1, \ldots, J,$$

where $Q$ is an orthogonal (or unitary) matrix.

The multi-parameter Tikhonov solution can be written as a filtered solution :

$$\begin{aligned} x_\lambda &= Q|C|^2(|C|^2 + \sum_{j=1}^{J} \lambda_j^2 |S_j|^2)^{-1} C^{-1} Q^* b \\ &= Q\Phi C^{-1} Q^* b \\ &= H_\lambda^\dagger b, \end{aligned} \tag{2.3}$$

where $\Phi$ is a diagonal matrix with the diagonal elements as the filter factors $\phi_i, i = 1, \ldots, N$, which are given by

$$\phi_i = \frac{|c_i|^2}{|c_i|^2 + \sum_{j=1}^{J} \lambda_j^2 |s_{i,j}|^2}$$

with $c_i$ and $s_{i,j}$ being the $i$th diagonal elements of matrices $C$ and $S_j$ respectively.

In this case, the GCV method can provide a vector of regularization parameters $\lambda = [\lambda_1, \ldots, \lambda_J]$ by minimizing the GCV function,

$$G(\lambda) = \frac{\left\| (HH_\lambda^\dagger - I)b \right\|_2^2}{[\mathrm{trace}(I - HH_\lambda^\dagger)]^2}. \tag{2.4}$$

Using the decomposition of (2.3), the GCV function can be written as

$$G(\lambda) = \frac{\sum_{i=1}^{N} ((1-\phi_i)[Q^* b]_i)^2}{\left( \sum_{i=1}^{N} 1 - \phi_i \right)^2}. \tag{2.5}$$

## 3. Multi-parameter Tikhonov problem with multiple right hand sides

Before focusing on the multi-parameter Tikhonov problem with multiple right hand sides, we refer to a single regularization parameter case of Tikhonov problem with multiple right hand sides([9]).

In [9], the image restoration problem with a single regularization parameter is written as

$$\min_X \{ \|HX - B\|_F^2 + \lambda^2 \|X\|_F^2 \}. \tag{3.1}$$

Considering the reflective boundary conditions, $H$ can be diagonalized by the orthogonal two-dimensional discrete cosine transform matrix $\mathcal{C}$,

$$H = \mathcal{C}^T \Lambda_H \mathcal{C}, (\Lambda_H = diag(\eta_1, \eta_2, \ldots, \eta_N)). \tag{3.2}$$

Then the GCV function to decide a single regularization parameter $\lambda$ in (3.1) could be changed to the formula (3.3).

LEMMA 3.1. [9] *If $\{\eta_i\}_{i=1,\ldots,N}$ represents the spectrum of $H$, the global GCV function $\mathcal{G}_{global}(\lambda)$ is*

$$\mathcal{G}_{\mathrm{global}}(\lambda) = \frac{\sum_{j=1}^{s} \sum_{i=1}^{N} \left( \frac{1}{\eta_i^2 + \lambda^2} [\mathcal{C} B_j]_i \right)^2}{\left( \sum_{i=1}^{N} \frac{1}{\eta_i^2 + \lambda^2} \right)^2}, \tag{3.3}$$

*where $B_j$ is the $j$-th column of $B$.*

A regularization parameter $\lambda_{gGCV}$ is a solution the constrained optimization problem

$$\min_{\lambda} \quad \mathcal{G}_{\text{global}}(\lambda) \text{ subject to } \eta_1 \leq \lambda \leq \eta_N, \tag{3.4}$$

where $\eta_1$ is the smallest eigenvalue of $H$ and $\eta_N$ is the largest eigenvalue of $H$.

The multi-parameter Tikhonov regularization finds a regularization solution as the minimizer of the function;

$$\mathcal{J}(X; \lambda_1, \ldots, \lambda_J) = \|HX - B\|_F^2 + \sum_{j=1}^{J} \lambda_j^2 \|L_j X\|_F^2, \tag{3.5}$$

where multiple regularization parameter $\lambda = [\lambda_1, \ldots, \lambda_J]$ and multiple regularization matrices $L_j \in \mathbb{R}^{p_j \times n}$ are incorporated.

The matrix $X_\lambda$ minimizing $\mathcal{J}(X; \lambda_1, \ldots, \lambda_J)$ for given $[\lambda_1, \ldots, \lambda_J]$, is the solution of the system

$$(H^T H + \sum_{j=1}^{J} \lambda_j^2 L_j^T L_j) X = H^T B. \tag{3.6}$$

Definition 3.2. The global GCV function with respect to a vector of regularization parameters $\lambda_1, \ldots, \lambda_J$ is defined by

$$\begin{aligned} &\mathcal{G}_{\text{global}}(\lambda_1, \ldots, \lambda_J) \\ &= \frac{\|HX_\lambda - B\|_F^2}{[\text{trace}(I - H(H^T H + \sum_{j=1}^{J} \lambda_j^2 L_j^T L_j)^{-1} H^T)]^2}. \end{aligned} \tag{3.7}$$

Under the reflective boundary conditions, $L_j$ can be also written as a sum of BTTB(block Toeplitz with Toeplitz blocks), BTHB(block Toeplitz with Hankel blocks), BHTB(block Hankel with Toeplitz block), and BHHB(block Hankel with Hankel blocks) matrices. Thus $L_j$ is diagonalizable

$$L_j = \mathcal{C}^T S_j \mathcal{C} \quad j = 1, \ldots, J, \tag{3.8}$$

where $S_j = diag(s_{1,j}, s_{2,j}, \ldots, s_{N,j})$.

Using (3.2) and (3.8), the regularized solution $X_\lambda$ of the system (3.6) can be written as

$$X_\lambda = \mathcal{C}^T ({\Lambda_H}^2 + \sum_{j=1}^{J} \lambda_j^2 S_j^2)^{-1} {\Lambda_H}^T \mathcal{C} B. \tag{3.9}$$

LEMMA 3.3. *From a unitary spectral decomposition of $H$ and $L_j$, the GCV function $\mathcal{G}_{global}(\lambda_1, \ldots, \lambda_J)$ in (3.7) implies*

$$\mathcal{G}_{global}(\lambda_1, \ldots, \lambda_J) = \frac{\sum_{j=1}^{s} \sum_{i=1}^{N} ((\phi_i - 1)[\mathcal{C}B_j]_i)^2}{\left(\sum_{i=1}^{N} 1 - \phi_i\right)^2}. \tag{3.10}$$

*where* $\phi_i = \frac{|\eta_i|^2}{|\eta_i|^2 + \sum_{j=1}^{J} \lambda_j^2 |s_{i,j}|^2}$ *and $B_k$ is the $k$-th column of $B$.*

*Proof.* The residual matrix $HX_\lambda - B$ from (3.9) yields

$$HX_\lambda - B = \mathcal{C}(\Lambda_H(\Lambda_H^2 + \sum_{j=1}^{J} \lambda^2 S_j^2)^{-1}\Lambda_H - I)\mathcal{C}^T B.$$

Th Frobenius norm of the above squared equals

$$\|HX_\lambda - B\|_F^2 = \sum_{k=1}^{s} \sum_{i=1}^{N} \left( \left( \frac{\eta_i^2}{\eta_i^2 + \sum_{j=1}^{J} \lambda_j^2 |s_{i,j}|^2} - 1 \right) [\mathcal{C}B_k]_i \right)^2.$$

Also the trace part of the denominator of (3.7) can be represented as

$$\begin{aligned} &\mathrm{trace}(I - H(H^T H + \sum_{j=1}^{J} \lambda_j^2 L_j^T L_j)^{-1} H^T) \\ &= \sum_{i=1}^{N} \left( 1 - \frac{\eta_i^2}{\eta_i^2 + \sum_{j=1}^{J} \lambda_j^2 |s_{i,j}|^2} \right). \end{aligned}$$

Thus substitution of two expressions above into the global GCV function (3.7) allows (3.10). □

To find regularization parameters $[\lambda_1, \ldots, \lambda_J]$ by minimizing the global GCV function (3.10), we solve the following constrained optimization problem with the bounded constraints for $[\lambda_1, \ldots, \lambda_J]$ :

$$\begin{aligned} &\min_{\lambda_1, \ldots, \lambda_J} \quad \mathcal{G}_{\mathrm{global}}(\lambda_1, \ldots, \lambda_J) \\ &\text{subject to} \quad \eta_1 \le \lambda_j \le \eta_N, \ \text{ for } \ j = 1, \ldots, J. \end{aligned} \tag{3.11}$$

For the remainder of this section, we are going to develop an algorithm related to the preconditioned GL-CGLS for the multi-parameter Tikhonov problem with the global GCV.

The minimizer of the function (3.5) implies that the best fit in the least squares minimizes the sum of squared residuals;

$$\min_{X} \left\| \begin{pmatrix} H \\ \mathbf{L}_\lambda \end{pmatrix} X - \begin{pmatrix} B \\ O \end{pmatrix} \right\|_F, \tag{3.12}$$

where

$$\mathbf{L}_\lambda = \begin{pmatrix} \lambda_1 L_1 \\ \vdots \\ \lambda_J L_J \end{pmatrix}$$

and its normal equation is

$$(H^T H + \mathbf{L}_\lambda{}^T \mathbf{L}_\lambda) X = H^T B. \tag{3.13}$$

The global conjugate gradient linear least squares(Gl-CGLS) method as an iterative regularization method was designed for solving large sparse systems (3.13) of equations with multiple right hand sides. Let $X_0$ denote the initial, and define $R_0 = \begin{pmatrix} B \\ O \end{pmatrix} - \begin{pmatrix} H \\ \mathbf{L}_\lambda \end{pmatrix} X_0$, $P_0 = S_0 = \begin{pmatrix} H \\ \mathbf{L}_\lambda \end{pmatrix}^T R_0$, and $\gamma_0 = (S_0, S_0)_F$. Then the Gl-CGLS iterations take the following form for $k = 0, 1, ...$

1. $Q_k = \begin{pmatrix} H \\ \mathbf{L}_\lambda \end{pmatrix} P_k$, $\alpha_k = \gamma_k / (Q_k, Q_k)_F$,
2. $X_{k+1} = X_k + \alpha_k P_k$, $R_{k+1} = R_k - \alpha_k Q_k$,
3. $S_{k+1} = \begin{pmatrix} H \\ \mathbf{L}_\lambda \end{pmatrix}^T R_{k+1}$, $\gamma_{k+1} = (S_{k+1}, S_{k+1})_F$,
4. $\beta_k = \gamma_{k+1}/\gamma_k$, $P_{k+1} = S_{k+1} + \beta_k P_k$,

When $\Omega^{-T}$ is a preconditioning matrix and $Y = \Omega X$, the normal equations for the preconditioned problem of (3.13) becomes

$$\Omega^{-T}\left((H^T H + \mathbf{L}_\lambda{}^T \mathbf{L}_\lambda)\Omega^{-1} Y - H^T B\right) = O. \tag{3.14}$$

Here, the matrix $\begin{pmatrix} H \\ \mathbf{L}_\lambda \end{pmatrix} \Omega^{-1}$ is well conditioned. The next algorithm is designed to solve the multi-parameter Tikhonov problem with the extended global GCV.

Algorithm 1. Preconditioned Gl-CGLS for the multi-parameter Tikhonov problem with the global GCV

1. Determine the minimizer $\lambda_1, \dots, \lambda_J$ for the constrained minimization problem:

$$\begin{array}{ll} \min\limits_{\lambda_1,\dots,\lambda_J} & \mathcal{G}_{\text{global}}(\lambda_1, \dots, \lambda_J) \\ \text{subject to} & \eta_1 \le \lambda_j \le \eta_N, \quad \text{for } j = 1, \dots, J \end{array}$$

2. Set $\mathbf{L}_\lambda = \begin{pmatrix} \lambda_1 L_1 & \dots & \lambda_J L_J \end{pmatrix}^T$.
3. Solve $\Omega^{-T}(H^T H + \mathbf{L}_\lambda{}^T \mathbf{L}_\lambda) X = \Omega^{-T} H^T B$ using preconditioned Gl-CGLS:

i. $R_0 = \begin{pmatrix} B \\ O \end{pmatrix} - \begin{pmatrix} H \\ \mathbf{L}_\lambda \end{pmatrix} X_0$, $P_0 = S_0 = \Omega^{-T} \begin{pmatrix} H \\ \mathbf{L}_\lambda \end{pmatrix}^T R_0$, $\gamma_0 = (S_0, S_0)_F$,

ii. For $k = 0, 1, \ldots$ until convergence do

(i) $T_k = \Omega^{-1} P_k$ $Q_k = \begin{pmatrix} H \\ \mathbf{L}_\lambda \end{pmatrix} T_k$, $\alpha_k = \gamma_k / (Q_k, Q_k)_F$,

(ii) $X_{k+1} = X_k + \alpha_k T_k$, $R_{k+1} = R_k - \alpha_k Q_k$,

(iii) $S_{k+1} = \Omega^{-T} \begin{pmatrix} H \\ \mathbf{L}_\lambda \end{pmatrix}^T R_{k+1}$, $\gamma_{k+1} = (S_{k+1}, S_{k+1})_F$,

(iv) $\beta_k = \gamma_{k+1}/\gamma_k$, $P_{k+1} = S_{k+1} + \beta_k P_k$.

It is natural to precondition the Tikhonov problem (3.13) with the following preconditioner

$$\begin{aligned} M &= (H^T H + \mathbf{L}_\lambda{}^T \mathbf{L}_\lambda)^{-1} \\ &= \mathcal{C}^T (\Lambda_H^2 + \sum_{j=1}^{J} \lambda_j^2 S_j{}^2)^{-1} \mathcal{C} \end{aligned} \tag{3.15}$$

From its definition, $M$ is real and symmetric and if we choose $\Omega$ as the square root of M;

$$\Omega = \mathcal{C}^T (\Lambda_H^2 + \sum_{j=1}^{J} \lambda_j^2 S_j{}^2)^{1/2} \mathcal{C},$$

then it follows that $\Omega$ is also real and symmetric.

## 4. Numerical experiments

This section deals with the effectiveness of the multi-parameter regularization. All computations were done by Matlab environment.

Our eight test images are 512-by-512, and these images are divided into the collection of 4 smaller block images of 256-by-256 and 16 block images of 128-by-128 respectively.

For simple simulation, we concentrate on the case of two-parameter regularization method by considering minimizing the function

$$\mathcal{J}(X; \lambda_1, \lambda_2) = \|HX - B\|_F^2 + \lambda_1^2 \|L_1 X\|_F^2 + \lambda_2^2 \|L_2 X\|_F^2, \tag{4.1}$$

where $L_1$ is an identity matrix and $L_2$ ([4]) represents simple expressions in terms of Kronecker products

$$L_2 = I_n \otimes \mathcal{D}_2 + \mathcal{D}_2 \otimes I_m, \quad \mathcal{D}_2 = \begin{pmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \\ 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

Note that $L_2$ could also be chosen as $I_n \otimes \mathcal{D}_2$ or $\mathcal{D}_2 \otimes I_m$ instead of the above.

TABLE 1. Comparisons of the relative accuracy and PSNR between single- and two-parameter regularization.

| s | Test image | $L_1$ | PSNR | $L_1$ and $L_2$ | PSNR |
|---|---|---|---|---|---|
| 4 | I | 0.063950 | 34.466259 | 0.027258 | 41.873253 |
| | II | 0.051149 | 29.838596 | 0.027258 | 39.284432 |
| | III | 0.200348 | 25.285330 | 0.136308 | 28.630619 |
| | IV | 0.137789 | 21.470479 | 0.079226 | 26.277440 |
| | V | 0.152878 | 17.838260 | 0.123324 | 19.704181 |
| | VI | 0.051910 | 37.991887 | 0.020442 | 46.086606 |
| | VII | 0.066862 | 28.874629 | 0.043110 | 32.686622 |
| | X | 0.043107 | 31.849449 | 0.033856 | 33.947774 |
| 16 | I | 0.063956 | 28.444850 | 0.020713 | 38.237612 |
| | II | 0.050994 | 23.844364 | 0.016558 | 33.614668 |
| | III | 0.200891 | 19.241215 | 0.133396 | 22.797584 |
| | IV | 0.137807 | 15.448726 | 0.081275 | 20.035009 |
| | V | 0.152931 | 11.814671 | 0.131362 | 13.135137 |
| | VI | 0.051207 | 32.089699 | 0.016277 | 42.044590 |
| | VII | 0.104142 | 19.005148 | 0.042208 | 26.849728 |
| | X | 0.047328 | 25.017416 | 0.032303 | 28.334994 |

In order to get the local minimizer $[\lambda_1, \lambda_2]$ of $\mathcal{G}_{\text{global}}(\lambda_1, \lambda_2)$, we can use *fmincon* which finds a constrained minimum of a function of several variables. The stopping criteria of the preconditioned Gl-CGLS method is either current residual satisfies the condition $\|R_k\|_F \,/\, \|R_0\|_F \leq tol$,( $tol = 10^{-2}$) or the maximum number of iterations is set to 500.

To measure how well the true image has been restored, we investigate the relative accuracy $\left\|X^* - \widehat{X}\right\|_F / \|X^*\|_F$, when $X^*$ is the original image and $\widehat{X}$ is an approximated solution, and the peak-to-signal ratio(PSNR) defined as $10 \log_{10} \left( \frac{255^2}{\frac{1}{mn} \sum_{i,j} (x^*_{i,j} - \widehat{x}_{i,j})^2} \right)$, where $x^*_{i,j}$ and $\widehat{x}_{i,j}$ denote the pixel value of the original and restored image respectively. Typical values for PSNR in lossy image are between 30 and 50 dB. The higher PSNR generally represents the reconstruction images to be of higher quality.

In our test, the single-parameter regularization problem is setting $L_1 = I$ and $\lambda_2 = 0$ in the problem (4.1). The local minimizer $\lambda_1$ of the single-parameter function $\mathcal{G}_{\text{global}}(\lambda_1)$ can be obtained by using the matlab function *fminbnd*, a method based on the golden section search and the parabolic interpolation([9]).

Table 1 presents the performance results of the preconditioned Gl-LSQR method with single- and two-parameter regularization problem. To look at the data distribution of the relative accuracy, boxplots are in Figure 1. The box represents 50% of the data from 25th to 75th percentile. The horizontal line within the box correspond to the median. Whisker lines correspond to extreme data points.

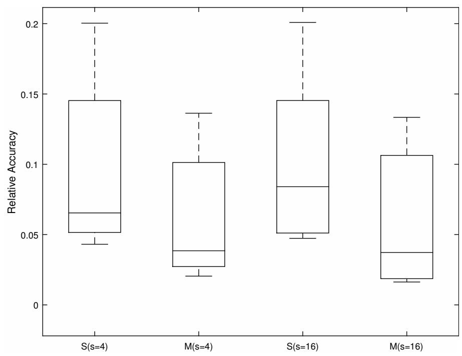


FIGURE 1. Boxplots for the relative accuracy in S(single-) and M(multi-)parameter Tikhonov regularization when $s = 4$ and $s = 16$.

In $s = 4$, the mean value for relative accuracy is 0.0960 for single-parameter and 0.0613 for multi-parameter. In $s = 16$, the mean value for relative accuracy is 0.1012 for single-parameter and 0.0593 for multi-parameter. Mean value of the relative accuracy for two-parameter is lower than that of single-parameter. However the two-parameter brings more meaningful relative accuracy compared to the single-parameter.

For $s = 16$, the degraded and reconstructed image for the moon image(I) by the preconditioned Gl-CGLS with the extended global GCV are given in Figure 2. The reconstruction error is 0.020713 and PSNR is 38.237612. The values of regularization parameters are determined as

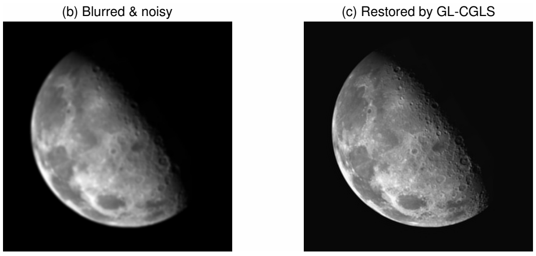


FIGURE 2. (b) Gaussian blurred and noisy image and (c) Reconstructed image using two-parameter Tikhonov regularization method when $s = 16$.

$\lambda_1 \approx 0.0122$ and $\lambda_2 \approx 0.0015$. On the other hand, in the case of single-parameter regularization problem, the relative accuracy is 0.063957 and PSNR 28.44850. Hence, we can make the relative accuracy decrease and PSNR increase by using two-parameter regularization problem. Figure 3 shows the corresponding residual norm decreasing stably.

Note that an extension to more than two-parameter regularization is possible.

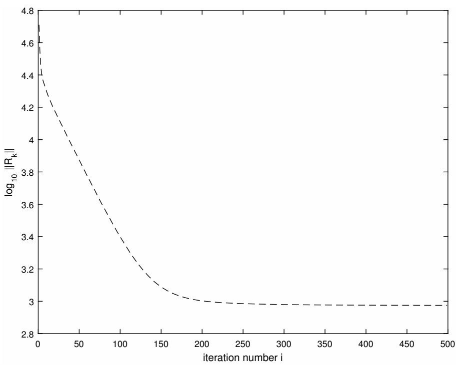


FIGURE 3. The logarithm graph of residual norm $||R_k||$.

This study is about the multi-parameter Tikhonov regularization problem with multiple right hand sides. First, we applied the multi-parameter technique to the $F$-norm based Tikhonov regularization problem with multiple right hand sides. Second, we extended the global GCV function to choose multiple regularization parameters. The results illustrate that multi-parameter performs better than single-parameter in relative accuracy and PSNR. Consequently we can get the best approximation of the true image by means of the two multi-parameter technique.

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