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INVARIANT SUBMANIFOLDS OF $(LCS)_n$-MANIFOLDS ADMITTING CERTAIN CONDITIONS

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Abstract. The object of the present paper is to study the invariant submanifolds of $(LCS)_n$-manifolds. We study generalized quasi-conformally semi-parallel and 2-semiparallel invariant submanifolds of $(LCS)_n$-manifolds and showed their existence by a non-trivial example. Among other it is shown that an invariant submanifold of a $(LCS)_n$-manifold is totally geodesic if the second fundamental form is any one of (i) symmetric, (ii) recurrent, (iii) pseudo symmetric, (iv) almost pseudo symmetric and (v) weakly pseudo symmetric.

1. Introduction

The notion of Lorentzian concircular structure manifold (briefly $(LCS)_n$ -manifold) was introduced by Shaikh [24] and proved its existence by several examples (see [25]) and later studied by many authors. For details, we refer [3], [4], [11], [12], [13], [14], [15], [16], [17], [18], [19], [22], [23], [27], [28], [29], [30], [31], [32], [33] and also the references therein. Recently, Mantica and Molinari [20] showed that a $(LCS)_n$ -manifold ($n > 3$) is equivalent to the generalized Robertson-Walker spacetime. Also the present authors [6] investigated the applications of $(LCS)_n$-manifolds in general theory of relativity and cosmology.

In the context of $N(k, \mu)$-contact metric manifolds, the notion of generalized quasi-conformal curvature tensor was introduced in [5] and

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defined on an n-dimensional manifold M as

$$\begin{aligned} \omega(X,Y)Z &= \frac{n-2}{n}[(1+(n-1)a-b)-\{1+(n-1)(a+b)\}c] \\ &\quad \times C(X,Y)Z \\ &\quad +[1-b+(n-1)a]E(X,Y)Z+(n-1)(b-a)P(X,Y)Z \\ &\quad +\frac{n-2}{n}(c-1)\{1+2n(a+b)\}L(X,Y)Z \end{aligned} \tag{1}$$

for all vector fields X, Y, Z on M and $a, b, c \in \mathbb{R}$, where E, P, L, C are concircular curvature tensor, projective curvature tensor, conharmonic curvature tensor, conformal curvature tensor respectively. This tensor can also be taken as a special case of generalized tensor defined by Shailkh and Kundu [34].

Throughout the paper such tensor field will be named as ω-tensor and by M (resp. $\bar{M}$) as the $(LCS)_n$-manifold ($n > 2$) (resp. submanifold of M). In particular, if

(i) $a = 0$, $b = 0$, $c = 0$, then ω turns into Riemann curvaure tensor R,

(ii) $a = -\frac{1}{n-2}$, $b = -\frac{1}{n-2}$, $c = 1$, then ω turns into conformal curvaure tensor C,

(iii) $a = -\frac{1}{n-2}$, $b = -\frac{1}{n-2}$, $c = 0$ then ω turns into conharmonic curvaure tensor L,

(iv) $a = 0$, $b = 0$, $c = 1$, then ω turns into concircular curvaure tensor E,

(v) $a - \frac{1}{n-1}$, $b = 0$, $c = 0$, then ω turns into projective curvaure tensor P,

(vi) $a = -\frac{1}{2n-2} = b$, $c = 0$ then ω turns into m-projective curvaure tensor H.

Simplifying (1) can be written as

$$\begin{aligned} \omega(X,Y)Z &= R(X,Y)Z + a[S(Y,Z)X - S(X,Z)Y] \\ &\quad +b[g(Y,Z)QX - g(X,Z)QY] \\ &\quad -\frac{cr}{n}\left(\frac{1}{n-1}+a+b\right)[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{2}$$

r being the scalar curvature of the manifold.

The present paper is outlined as follows: Section 2 is concerned with rudiments of $(\mathrm{LCS})_n$-manifolds. Section 3 deals with the study of some basic properties of invariant submanifolds of $(\mathrm{LCS})_n$-manifolds. In analogous to the work of [8], in section 4, we have studied the notion of generalized quasi-conformally semiparallel submanifolds which is defined as follows:

Definition 1.1. *An immersed submanifold $\bar{M}$ of M is said to be generalized quasi-conformally semiparallel (briefly ω-semiparallel) if*

$$\bar{\omega}(X, Y) \cdot \sigma = 0 \tag{3}$$

holds for all vector fields X, Y tangent to $\bar{M}$, where $\bar{\omega}$ denotes the generalized quasi-conformal curvature tensor with respect to Vander-Waerden-Bortolotti connection $\bar{\nabla}$ of $\bar{M}$ and σ is the second fundamental form.

Semiparallel immersed submanifolds have also been studied in [9], [10], [21], [35]. It is found that for each of (i) $\bar{C}(X, Y) \cdot \sigma = 0$, (ii) $\bar{L}(X, Y) \cdot \sigma = 0$ and (iii) $\bar{H}(X, Y) \cdot \sigma = 0$, invariant submanifold of a $(LCS)_n$-manifold is totally geodesic if and only if $\sigma(Z, QY) = 0$ whereas for each of (i) $\bar{R}(X, Y) \cdot \sigma = 0$, (ii) $\bar{E}(X, Y) \cdot \sigma = 0$ and (iii) $\bar{P}(X, Y) \cdot \sigma = 0$, invariant submanifold of a $(LCS)_n$-manifold is totally geodesic.

Further, keeping the spirit of Arslan et al. [1], we have also studied generalized quasi-conformally 2-semiparallel invariant submanifolds which is defined as follows:

Definition 1.2. *An immersed submanifold $\bar{M}$ of M is said to be generalized quasi-conformally 2-semiparallel (briefly ω 2-semiparallel) if*

$$\bar{\omega}(X, Y) \cdot \bar{\nabla}\sigma = 0 \tag{4}$$

holds for all vector fields X, Y tangent to $\bar{M}$.

It is observed that for each of (i) $\bar{C}(X, Y) \cdot \bar{\nabla}\sigma = 0$, (ii) $\bar{L}(X, Y) \cdot \bar{\nabla}\sigma = 0$ and (iii) $\bar{H}(X, Y) \cdot \bar{\nabla}\sigma = 0$, invariant submanifold of a $(LCS)_n$-manifold with non-vanishing ξ-sectional curvature is totally geodesic if and only if $\sigma(Z, QY) = 0$ whereas for each of (i) $\bar{R}(X, Y) \cdot \bar{\nabla}\sigma = 0$, (ii) $\bar{E}(X, Y) \cdot \bar{\nabla}\sigma = 0$ and (iii) $\bar{P}(X, Y) \cdot \bar{\nabla}\sigma = 0$, invariant submanifold of a $(LCS)_n$-manifold with non-vanishing ξ-sectional curvature is totally geodesic.

In section 5 of this paper we have investigated invariant submanifold of a $(LCS)_n$-manifold whose second fundamental form σ satisfies [36]

$$(\bar{\nabla}_X \sigma)(Y, Z) = B_1(X)\sigma\ (Y, Z) + C_1(Y)\sigma\ (X, Z) + D_1(Z)\sigma\ (Y, X) \tag{5}$$

where B_1, C_1 and D_1 are non-zero 1-forms defined by $B_1(X) = g(X, \delta)$, $C_1(X) = g(X, \theta)$ and $D_1(X) = g(X, \psi)$. Finally, we have cited an example of an invariant submanifold of a $(\text{LCS})_n$-manifold to support our claims.

2. Preliminaries

A $(LCS)_n$-manifold is a Lorentzian manifold M of dimension n endowed with the unit timelike concircular vector field ξ, its associated 1-form η and an $(1,1)$ tensor field ϕ such that

$$\nabla_X \xi = \alpha \phi X, \tag{6}$$

α being a non-zero scalar function such that

$$\nabla_X \alpha = (X\alpha) = \alpha(X) = \rho\eta(X), \tag{7}$$

ρ being a certain scalar function and ∇ is the Levi-Civita connection of the Lorentzian metric g.

In a $(LCS)_n$-manifold, the following relations hold ([24], [25], [26]):

$$\eta(\xi) = -1, \quad \phi \circ \xi = 0, \tag{8}$$

$$\eta(\phi X) = 0, \quad g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{9}$$

$$(\nabla_X \eta)(Y) = \alpha\{g(X,Y) + \eta(X)\eta(Y)\}, \tag{10}$$

$$\eta(R(X,Y)Z) = (\alpha^2 - \rho)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{11}$$

$$R(X,Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{12}$$

$$S(X,\xi) = (n-1)(\alpha^2 - \rho)\eta(X) \tag{13}$$

$$\begin{aligned} R(X,Y)Z &= \phi R(X,Y)Z + (\alpha^2 - \rho)\{g(Y,Z)\eta(X) \\ &\quad - g(X,Z)\eta(Y)\}\xi \end{aligned} \tag{14}$$

for all vector fields $X, Y, Z \in \chi(M)$ and $\beta = -(\xi\rho)$ is a scalar function. In consequence of (12), (13) and (2) we have

$$\begin{aligned} \omega(\xi,Y)Z &= a[S(Y,Z)\xi - (n-1)(\alpha^2-\rho)\eta(Z)Y] \\ &\quad + b[(n-1)(\alpha^2-\rho)g(Y,Z)\xi - \eta(Z)QY] + [(\alpha^2-\rho) \\ &\quad - \frac{cr}{n}\left(\frac{1}{n-1} + a + b\right)][g(Y,Z)\xi - \eta(Z)Y]. \end{aligned} \tag{15}$$

$$\begin{aligned} \omega(\xi,Y)\xi &= [(\alpha^2-\rho)\{(n-1)(a+b)+1\} \\ &\quad - \frac{cr}{n}\left(\frac{1}{n-1} + a + b\right)][\eta(Y)\xi + Y]. \end{aligned} \tag{16}$$

3. Some basic properties of invariant submanifolds of $(\mathbf{LCS})_n$-manifolds

Let $\bar{\nabla}$ and $\bar{\nabla}^{\perp}$ be the induced connection on $T\bar{M}$ and $T^{\perp}\bar{M}$ respectively. Then the Gauss and Weingarten formulae are given by

$$\nabla_X Y = \bar{\nabla}_X Y + \sigma(X,Y) \tag{17}$$

and

$$\nabla_X V = \bar{\nabla}^{\perp}_X V - A_V X \tag{18}$$

for all $X, Y \in \Gamma(T\bar{M})$ and $V \in \Gamma(T^{\perp}\bar{M})$, where σ and A_V are second fundamental form and the shape operator (corresponding to the normal vector field V) respectively for the immersion of $\bar{M}$ into M they are related by [37]

$$g(\sigma(X,Y),V) = g(A_V X, Y). \tag{19}$$

Note that $\sigma(X,Y)$ is bilinear and since $\nabla_{\pi X} Y = \pi \nabla_X Y$ for any smooth function π on a manifold, we have

$$\sigma(\pi X, Y) = \pi\sigma(X,Y). \tag{20}$$

Now, $\bar{M}$ is said to be invariant [7] if the structure vector field ξ is tangent to $\bar{M}$ and $\phi(T\bar{M}) \subset T\bar{M}$ at every point of $\bar{M}$. Also $\bar{M}$ is called totally geodesic if $\sigma(X,Y) = 0$ for any $X, Y \in \Gamma(T\bar{M})$. The covariant derivative of σ is

$$(\bar{\nabla}_X\sigma)(Y,Z) = \nabla^{\perp}_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z) \tag{21}$$

for any vector field X, Y, Z tangent to $\bar{M}$. Then $\bar{\nabla}\sigma$ is a normal bundle valued tensor of type (0,3) and is said to be third fundamental form of $\bar{M}$, $\bar{\nabla}$ is called the Vander–Waerden–Bortolotti connection of M, i.e., $\bar{\nabla}$ is the connection in $T\bar{M} \oplus T^{\perp}\bar{M}$ built with ∇ and $\nabla^{\perp}$. If $\bar{\nabla}\sigma = 0$, then $\bar{M}$ is said to have parallel second fundamental form [37]. From the Gauss and Weingarten formulae, we obtain

$$\bar{R}(X,Y)Z = R(X,Y)Z + A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}X, \tag{22}$$

where $\bar{R}(X,Y)Z$ denotes the tangential part of the curvature tensor of $\bar{M}$. We know that

$$\begin{aligned}(\bar{R}(X,Y)\cdot\sigma)(Z,U) &= R^{\perp}(X,Y)\sigma(Z,U) \\ &\quad -\sigma(R(X,Y)Z,U) - \sigma(Z,R(X,Y)U)\end{aligned} \tag{23}$$

for all vector fields X, Y, Z and U, where

$$R^{\perp}(X,Y) = [\nabla^{\perp}_X, \nabla^{\perp}_Y] - \nabla^{\perp}_{[X,Y]} \tag{24}$$

and $\bar{R}$ denotes the curvature tensor of $\bar{\nabla}$. In the similar manner one can write

$$(\bar{R}(X,Y)\cdot\bar{\nabla}\sigma)(Z,U,W) = R^{\perp}(X,Y)(\bar{\nabla}\sigma)(Z,U,W) -(\bar{\nabla}\sigma)(R(X,Y)Z,U,W) -(\bar{\nabla}\sigma)(Z,R(X,Y)U,W) -(\bar{\nabla}\sigma(Z,U,R(X,Y)W)) \tag{25}$$

for all vector fields X, Y, Z, U and W tangent to $\bar{M}$ and $(\bar{\nabla}\sigma)(Z,U,W) = (\bar{\nabla}_Z\sigma)(U,W)$. Again for the ω-tensor we have [21]

$$(\bar{\omega}(X,Y)\cdot\sigma)(Z,U) = R^{\perp}(X,Y)\sigma(Z,U) - \sigma(\omega(X,Y)Z,U) -\sigma(Z,\omega(X,Y)U) \tag{26}$$

and

$$(\bar{\omega}(X,Y)\cdot\bar{\nabla}\sigma)(Z,U,W) = R^{\perp}(X,Y)(\bar{\nabla}\sigma)(Z,U,W) -(\bar{\nabla}\sigma)(\omega(X,Y)Z,U,W) -(\bar{\nabla}\sigma)(Z,\omega(X,Y)U,W) -(\bar{\nabla}\sigma(Z,U,\omega(X,Y)W)). \tag{27}$$

In an invariant submanifold $\bar{M}$ of a $(LCS)_n$-manifold, we have [2]

$$\sigma(X,\xi) = 0, \tag{28}$$

$$\bar{\nabla}_X\xi = \alpha\phi X, \tag{29}$$

$$\bar{R}(X,Y)\xi = (\alpha^2-\rho)[\eta(Y)X-\eta(X)Y], \tag{30}$$

$$\bar{S}(X,\xi) = (n-1)(\alpha^2-\rho)\eta(X), \tag{31}$$

$$\sigma(X,\phi Y) = \phi\sigma(X,Y). \tag{32}$$

4. Invariant submanifolds of $(LCS)_n$-manifolds admitting $\bar{\omega}(X,Y)\cdot\sigma = 0$ and $\bar{\omega}(X,Y)\cdot\bar{\nabla}\sigma = 0$

Let $\bar{M}$ be an invariant submanifold of a $(LCS)_n$-manifold M satisfying $\bar{\omega}(X,Y)\cdot\sigma = 0$ such that $r = n(n-1)(\alpha^2-\rho)$. Then we have from (26) that

$$R^{\perp}(X,Y)\sigma(Z,U) - \sigma(\omega(X,Y)Z,U) - \sigma(Z,\omega(X,Y)U) = 0. \tag{33}$$

Plugging $X = U = \xi$ in (33) and using (15) and (28), we get

$$\sigma(Z,\omega(\xi,Y)\xi) = 0, \tag{34}$$

which yields

$$b\sigma(Z,QY) = -\left[(\alpha^2-\rho)\{(n-1)a+1\} - \frac{cr}{n}\left(\frac{1}{n-1}+a+b\right)\right]\sigma(Z,Y). \tag{35}$$

This leads to the followings:

Theorem 4.1. *Let $\bar{M}$ be an invariant submanifold of a $(LCS)_n$-manifold M. Then for each of (i) $\bar{C}(X,Y)\cdot\sigma = 0$, (ii) $\bar{L}(X,Y)\cdot\sigma = 0$ and (iii) $\bar{H}(X,Y)\cdot\sigma = 0$, $\bar{M}$ is totally geodesic if and only if $\sigma(Z,QY) = 0$.*

Theorem 4.2. *Let $\bar{M}$ be an invariant submanifold of a $(LCS)_n$-manifold M. Then for each of (i) $\bar{R}(X,Y)\cdot\sigma = 0$, (ii) $\bar{E}(X,Y)\cdot\sigma = 0$ and (iii) $\bar{P}(X,Y)\cdot\sigma = 0$, $\bar{M}$ is totally geodesic.*

Taking account of ([35], Theorem 4.2) and the above theorems, one can state that the following:

Theorem 4.3. *Let $\bar{M}$ be an invariant submanifold of a $(LCS)_n$-manifold M. Then for each of (i) $\bar{C}(X,Y)\cdot\sigma = 0$, (ii) $\bar{L}(X,Y)\cdot\sigma = 0$ and (iii) $\bar{H}(X,Y)\cdot\sigma = 0$, the second fundamental form of the submanifold $\bar{M}$ is parallel if and only if $\sigma(Z,QY) = 0$.*

Theorem 4.4. *Let $\bar{M}$ be an invariant submanifold of a $(LCS)_n$-manifold M. Then for each of (i) $\bar{R}(X,Y)\cdot\sigma = 0$, (ii) $\bar{E}(X,Y)\cdot\sigma = 0$ and (iii) $\bar{P}(X,Y)\cdot\sigma = 0$, the second fundamental form of the submanifold $\bar{M}$ is parallel if and only if $\bar{M}$ is totally geodesic.*

Let $\bar{M}$ be an invariant submanifold of a $(LCS)_n$-manifold M, whose ω-tensor is 2-semi parallel. Then from (27), we get

$$\begin{aligned} 0 \;=\; & R^{\perp}(X,Y)(\bar{\nabla}\sigma)(Z,U,W) - (\bar{\nabla}\sigma)(\omega(X,Y)Z,U,W) \\ & -(\bar{\nabla}\sigma)(Z,\omega(X,Y)U,W) - (\bar{\nabla}\sigma(Z,U,\omega(X,Y)W)) \end{aligned} \tag{36}$$

which yields for $X = U = \xi$

$$\begin{aligned} 0 \;=\; & R^{\perp}(\xi,Y)(\bar{\nabla}\sigma)(Z,\xi,W) - (\bar{\nabla}\sigma)(\omega(\xi,Y)Z,\xi,W) \\ & -(\bar{\nabla}\sigma)(Z,\omega(X,Y)\xi,W) - (\bar{\nabla}\sigma(Z,\xi,\omega(\xi,Y)W)) . \end{aligned} \tag{37}$$

In consequence of (6), (12), (21) and (28), one can easily bring out the following:

$$(\bar{\nabla}\sigma)(Z,\xi,W) = (\bar{\nabla}\sigma_Z)(\xi,W) = -\alpha\sigma(\phi Z,W), \tag{38}$$

$$
\begin{aligned}
(\bar{\nabla}\sigma)(\omega(\xi,Y)Z,\xi,W) &= (\bar{\nabla}_{\omega(\xi,Y)Z}\sigma)(\xi,W)\\
&= \alpha b[\eta(Z)\sigma(\phi QY,W)]\\
&\quad +\alpha[(\alpha^2-\rho)\{a(n-1)+1\}\\
&\quad -\frac{cr}{n}\left(\frac{1}{n-1}+a+b\right)]\eta(Z)\sigma(\phi Y,W), \qquad (39)
\end{aligned}
$$

$$
\begin{aligned}
(\bar{\nabla}\sigma)(Z,\omega(\xi,Y)\xi,W) &= (\bar{\nabla}_Z\sigma)(\omega(\xi,Y)\xi,W)\\
&= [(\alpha^2-\rho)\{(n-1)(a+b)+1\}\\
&\quad -\frac{cr}{n}\left(\frac{1}{n-1}+a+b\right)][\nabla^{\perp}_Z(\sigma(Y,W))\\
&\quad -\sigma(\nabla_Z\{Y+\eta(Y)\xi\},W)-\sigma((Y,\nabla_Z W)] \qquad (40)
\end{aligned}
$$

and

$$
\begin{aligned}
(\bar{\nabla}\sigma)(Z,\xi,\omega(\xi,Y)W) &= (\bar{\nabla}_Z\sigma)(\xi,\omega(\xi,Y)W)\\
&= b\alpha\eta(W)\sigma(\phi Z,QY)\\
&\quad +\alpha[(\alpha^2-\rho)\{(n-1)a+1\}\\
&\quad -\frac{cr}{n}\left(\frac{1}{n-1}+a+b\right)]\eta(W)\sigma(\phi Z,Y). \qquad (41)
\end{aligned}
$$

Using (6), (38)–(41) in (37), we obtain

$$
\begin{aligned}
0 &= -\alpha R^{\perp}(\xi,Y)\sigma(\phi Z,W)-\alpha b\,[\eta(Z)\sigma(\phi QY,W)]\\
&- \alpha\left[(\alpha^2-\rho)\{a(n-1)+1\}-\frac{cr}{n}\left(\frac{1}{n-1}+a+b\right)\right][\eta(Z)\sigma(\phi Y,W)]\\
&- \left[(\alpha^2-\rho)\{(n-1)(a+b)+1\}-\frac{cr}{n}\left(\frac{1}{n-1}+a+b\right)\right]\\
&\quad \times[\bar{\nabla}^{\perp}_Z(\sigma(Y,W))-\sigma(\bar{\nabla}_Z\{Y+\eta(Y)\xi\},W)-\sigma((Y,\bar{\nabla}_Z W)]\\
&- b\alpha\eta(W)\sigma(\phi Z,QY)\\
&- \alpha\left[(\alpha^2-\rho)\{(n-1)a+1\}-\frac{cr}{n}\left(\frac{1}{n-1}+a+b\right)\right]\\
&\quad \times[\eta(W)\sigma(\phi Z,Y)]. \qquad (42)
\end{aligned}
$$

Putting $W=\xi$ in (42) and using (28) and (29), we get

$$
\begin{aligned}
0 &= \left[(\alpha^2-\rho)\{(n-1)(2a+b)+2\}-\frac{cr}{n}\left(\frac{1}{n-1}+a+b\right)\right]\sigma(Y,\phi Z)\\
&\quad +b\sigma(\phi Z,QY). \qquad (43)
\end{aligned}
$$

In consequence of ([35], Theorem 4.3), one can state that

Theorem 4.5. *Let $\bar{M}$ be an invariant submanifold of a $(LCS)_n$-manifold M with non-vanishing ξ-sectional curvature. Then for each of (i) $\bar{C}(X,Y)\cdot\bar{\nabla}\sigma = 0$, (ii) $\bar{L}(X,Y)\cdot\bar{\nabla}\sigma = 0$ and (iii) $\bar{H}(X,Y)\cdot\bar{\nabla}\sigma = 0$, $\bar{M}$ is totally geodesic if and only if $\sigma(Z,QY) = 0$.*

Theorem 4.6. *Let $\bar{M}$ be an invariant submanifold of a $(LCS)_n$-manifold M with non-vanishing ξ-sectional curvature. Then $\bar{M}$ is totally geodesic if and only if*
(i) $\bar{R}(X,Y)\cdot\bar{\nabla}\sigma = 0$,
(ii) $\bar{E}(X,Y)\cdot\bar{\nabla}\sigma = 0$ and
(iv) $\bar{P}(X,Y)\cdot\bar{\nabla}\sigma = 0$.

5. Invariant submanifolds of a $(LCS)_n$-manifold whose second fundamental form σ is weakly symmetric type

In this section, since M has parallel second fundamental form, it follows from (21) that

$$(\bar{\nabla}_X\sigma)(Y,Z) = \bar{\nabla}^{\perp}_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z) \tag{44}$$

Putting $Z = \xi$ in (44) and making use of (28) and (29), we have

$$(\bar{\nabla}_X\sigma)(Y,\xi) = \alpha\sigma(Y,X). \tag{45}$$

In consequence of (5) and (45) one can easily bring out

$$[\alpha - D_1(\xi)]\sigma(Y,X) = 0. \tag{46}$$

This leads to the following:

Theorem 5.1. *Let $\bar{M}$ be an invariant submanifold of a $(LCS)_n$-manifold M with $\alpha \neq D_1(\xi)$. Then $\bar{M}$ is totally geodesic if second fundamental form σ is of the following types*
(i) symmetric,
(ii) recurrent,
(iii) pseudo symmetric,
(iv) almost pseudo symmetric and
(v) weakly pseudo symmetric.

6. Example

Example 6.1. *Let us consider a 4-dimensional connected manifold $M = \{(x^1,x^2,x^3,x^4) \in \mathbb{R}^4 : x^4 \neq 0\}$, where (x^1,x^2,x^3,x^4) being standard coordinates in $\mathbb{R}^4$. Let $\{e_1,e_2,e_3,e_4\}$ be a linearly independent* *global frame on M given by*

$$e_1 = x^1x^4\left(\frac{\partial}{\partial x^1}+\frac{\partial}{\partial x^3}\right),\quad e_2 = x^4\frac{\partial}{\partial x^2},\quad e_3 = x^4\frac{\partial}{\partial x^3},\quad e_4 = (x^4)^3\frac{\partial}{\partial x^4}.$$

Let g be the Lorentzian metric defined by $g(\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^1}) = \left(\frac{1}{x^1x^4}\right)^2$, $g(\frac{\partial}{\partial x^2},\frac{\partial}{\partial x^2}) = g(\frac{\partial}{\partial x^3},\frac{\partial}{\partial x^3}) = \left(\frac{1}{x^4}\right)^2$, $g(\frac{\partial}{\partial x^4},\frac{\partial}{\partial x^4}) = -\left(\frac{1}{x^4}\right)^6$ and $g(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}) = 0$ for $i \neq j = 1,2,3,4$. Let η be the 1-form defined by $\eta(U) = g(U, e_4)$ for any $U \in \chi(M)$. Let ϕ be the (1,1) tensor field defined by $\phi e_1 = e_1$, $\phi e_2 = e_2$, $\phi e_3 = e_3$ and $\phi e_4 = 0$. Then using the linearity of ϕ and g we have $\eta(e_4) = -1$, $\phi^2 U = U + \eta(U)\, e_4$ and $g(\phi U, \phi W) = g(U,W) + \eta(U)\eta(W)$ for any $U, W \in \chi(M)$. Thus for $e_4 = \xi$, (ϕ,ξ,η,g) defines a Lorentzian paracontact structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. Then we have
$[e_1,e_2] = -x^4e_2$, $[e_1,e_4] = -\left(x^4\right)^2 e_1$, $[e_2,e_4] = -\left(x^4\right)^2 e_2$, $[e_3,e_4] = -\left(x^4\right)^2 e_3$.

Taking $e_4 = \xi$ and using Koszul formula for the Lorentzian metric g, we can easily calculate

$$\nabla_{e_1}e_4 = -\left(x^4\right)^2 e_1,\qquad \nabla_{e_2}e_4 = -\left(x^4\right)^2 e_2,\qquad \nabla_{e_3}e_4 = -\left(x^4\right)^2 e_3,$$
$$\nabla_{e_1}e_1 = -\left(x^4\right)^2 e_4,\qquad \nabla_{e_2}e_1 = x^4e_2,\qquad \nabla_{e_3}e_3 = -\left(x^4\right)^2 e_4,$$
$$\nabla_{e_2}e_2 = -\left(x^4\right)^2 e_4 - x^4e_1,\qquad \nabla_{e_4}e_1 = 0,\qquad \nabla_{e_3}e_2 = 0,$$
$$\nabla_{e_1}e_3 = 0,\qquad \nabla_{e_1}e_2 = 0,\qquad \nabla_{e_3}e_1 = 0,\qquad \nabla_{e_4}e_4 = 0,$$
$$\nabla_{e_4}e_2 = 0,\qquad \nabla_{e_4}e_3 = 0.$$

From the above it can be easily seen that (ϕ,ξ,η,g) is a $(LCS)_4$ - structure on M. Consequently $M(\phi,\xi,\eta,g)$ is a $(LCS)_4$-manifold with $\alpha = -\left(x^4\right)^2 \neq 0$ and $\rho = 2\left(x^4\right)^2$.

Let $\bar{M}$ be a subset of M and consider the isometric immersion $\pi : \bar{M} \longrightarrow M$ defined by $\pi(x^1,x^2,x^4) = (x^1,x^2,0,x^4)$. It can be easily proved that $\bar{M} = \{(x^1,x^2,x^4) \in \mathbb{R}^3\ (x^1,x^2,x^4) \neq 0\}$ is a 3-dimensional submanifold of M, where (x^1,x^2,x^4) are standard coordinates in $\mathbb{R}^3$. We choose the vector fields

$$e_1 = x^1x^4\left(\frac{\partial}{\partial x^1}+\frac{\partial}{\partial x^3}\right),\qquad e_2 = x^4\frac{\partial}{\partial x^2},\qquad e_4 = (x^4)^3\frac{\partial}{\partial x^4}.$$

Let g be the Lorentzian metric defined by $g(\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^1}) = \left(\frac{1}{x^1x^4}\right)^2$, $g(\frac{\partial}{\partial x^2},\frac{\partial}{\partial x^2}) = \left(\frac{1}{x^4}\right)^2$, $g(\frac{\partial}{\partial x^4},\frac{\partial}{\partial x^4}) = -\left(\frac{1}{x^4}\right)^6$ and $g(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}) = 0$ for

$i \neq j = 1, 2, 4$. Let ϕ be the $(1,1)$ tensor field defined by $\phi e_1 = e_1$, $\phi e_2 = e_2$, $\phi e_4 = 0$. Then using the linearity of ϕ and g we have

$$\begin{aligned} \eta(e_4) &= -1, \ \phi^2 U = U + \eta(U) e_4 \quad \text{and} \\ g(\phi U, \phi W) &= g(U, W) + \eta(U)\eta(W). \end{aligned}$$

Thus for $e_4 = \xi$, (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on $\bar{M}$.

Let $\bar{\nabla}$ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. Then we have

$$[e_1, e_2] = -x^4 e_2, \quad [e_1, e_4] = -\left(x^4\right)^2 e_1, \quad [e_2, e_4] = -\left(x^4\right)^2 e_2.$$

Taking $e_4 = \xi$ and using Koszul formula for the Lorentzian metric g, we can easily calculate

$$\begin{gathered} \bar{\nabla}_{e_1} e_4 = -\left(x^4\right)^2 e_1, \qquad \bar{\nabla}_{e_2} e_4 = -\left(x^4\right)^2 e_2, \\ \bar{\nabla}_{e_1} e_1 = -\left(x^4\right)^2 e_4, \qquad \bar{\nabla}_{e_2} e_1 = x^4 e_2, \\ \bar{\nabla}_{e_2} e_2 = -\left(x^4\right)^2 e_4 - \ x^4 e_1, \qquad \bar{\nabla}_{e_4} e_1 = 0, \\ \bar{\nabla}_{e_4} e_2 = 0, \qquad \bar{\nabla}_{e_1} e_2 = 0, \qquad \bar{\nabla}_{e_4} e_4 = 0. \end{gathered}$$

From the above it can be easily seen that (ϕ, ξ, η, g) is a $(LCS)_3$ - structure on $\bar{M}$. Consequently $M^3(\phi, \xi, \eta, g)$ is a $(LCS)_3$-manifold with $\alpha = -\left(x^4\right)^2 \neq 0$ and $\rho = 2\left(x^4\right)^2$. Finally, from the values of $\nabla_{e_i} e_j$ and $\bar{\nabla}_{e_l} e_k$, where i, $j \in \{1, 2, 3, 4\}$ and l, $k \in \{1, 2, 4\}$, one can easily obtain $\sigma = 0$ and hence $\bar{M}$ is totally geodesic. Thus, Theorems 4.1- 4.6 and 5.1 are verified.

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