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AN INERTIAL MANIFOLD FOR A NON-SELF ADJOINT SYSTEM

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Abstract. In this work, we prove an existence of an inertial manifold for a system of differential equations with a non-self adjoint leading part. The result follows from the existence and uniqueness of negatively bounded solutions. In fact, we show that a sharp spectral condition is sufficient for the proof.

1. Introduction

The concept of an inertial manifold plays a significant role to determine the long-time behavior of solutions of infinite-dimensional nonlinear dynamical systems. By definition, the inertial manifold is a finite-dimensional Lipschitz positively invariant manifold that attracts all trajectory at exponential rates [3]. Furthermore, when an inertial manifold exists, one can reduce the dynamical system from infinite-dimensional to finite-dimensional by a system of ordinary differential equations [7].

The existence of an inertial manifold is proved for many classical partial differential equations but it is still unknown for many others, including 2D Navier-Stokes system. For the proof, a spectral gap condition is necessarily required and it turns out to be a heavy obstacle [1, 8].

Usually, an inertial manifold is constructed as a graph of Lipschitz continuous function. There are several methods to determine that function. The well-known one is the Lyapunov-Perron method, which reduces to a fixed point problem. This method builds on the contraction mapping principle and requires a rather strong spectral gap assumption.

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To avoid this strong gap condition, other approaches must be developed. As an effort, Kwak introduced a transformation for the 2D Navier-Stokes equations [5]. The idea of that goes to convert the original system to a new form of a reaction diffusion system by using a nonlinear change of variables so that the transformed system possesses the same asymptotic dynamics. The transformed system has no gradient terms in nonlinearity and thus has a remarkable gain in the gap condition. However, a non-self adjoint differential operator is involved and requires a further investigation. For this purpose, a negatively bounded solution is introduced and it is revealed that the existence and uniqueness of negatively bounded solutions are crucial for the existence theory of an inertial manifold [2, 4, 6, 8]. In particular, the uniqueness of such a negatively bounded solution allows the Lipschitz continuity of the function which defines the inertial manifold [6].

In this note, we consider a system of differential equations with a non-self adjoint leading part, which is close to the system arising after the Kwak transformation. We mainly demonstrate the existence of an inertial manifold for that system. Section 2 contains basic statements. Section 3 contains a proof of the uniqueness of negatively bounded solutions (Lemma 3.1), which leads to the Lipschitz continuity of the manifold (Lemma 3.2). Finally, the existence of an inertial manifold is derived under the sharp spectral gap condition (Theorem 3.3).

2. Preliminaries

Let H be an infinite-dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. In the space $\mathbb{H} = H \times H$, we consider the following system

$$\begin{cases} \frac{du}{dt} = -Au + AP_0 v, \\ \frac{dv}{dt} = -Av + F(u), \end{cases} \tag{1}$$

where $A : D(A) \to H$ is a linear, positive unbounded self-adjoint operator with compact inverse A^{-1}. There exists a family $\{e_i\}_{i \in N^*}$ that consists of an orthonormal basis of H. Here $\{e_i\}_{i \in N^*}$ are eigenvectors of A corresponding to eigenvalues $\{\lambda_i\}_{i \in N^*}$;

$$Ae_i = \lambda_i e_i, \text{ and } 0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_i \le \cdots, \ \lambda_i \to \infty \text{ as } i \to \infty.$$

The function $F : H \to H$ is nonlinear globally bounded and Lipschitz continuous which satisfies

$$\begin{cases} \|F(u)\| \leq K_0, \\ \|F(u) - F(v)\| \leq K_1\|u - v\|, \end{cases} \tag{2}$$

and the projection operator P_0 is from $P_0H = Span\{e_1, \cdots, e_{N_0}\}$ onto H, where N_0 is a given natural number.

We observe that the system (1) can be rewritten as an abstract form

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = - \begin{pmatrix} A & -AP_{N_0} \\ 0 & A \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ F(u) \end{pmatrix}, \tag{3}$$

which is similar to the system arising after the Kwak transformation. The leading part of differential equations above is a block upper triangular matrix. Thus system (3) has a non-self-adjoint operator term.

Let $P_1 : P_1H \to H$ and $Q : QH \to H$ denote the projection operators from $P_1H = Span\{e_{N_0+1}, \cdots, e_N\}$ and $QH = Span\{e_{N+1}, \cdots\}$ onto H respectively. In fact, $P = P_0 + P_1$ and $Q = I - P$ are orthogonal complemented projection operators which split the Hilbert space H into a finite-dimensional space PH and an infinite-dimensional space QH. Then the space H is the direct sum of PH and QH, that is, $H = PH \oplus QH$.

Let (u, v) be a solution of initial system (1). (u, v) can be decomposed uniquely as a direct sum

$$u = P_0u + P_1u + Qu = r_1 + p_1 + q_1,$$

$$v = P_0v + P_1v + Qv = r_2 + p_2 + q_2,$$

and system (1) is equivalent to the following system

$$\begin{cases} \frac{dr_1}{dt} = -AP_0r_1 + AP_0r_2, \\ \frac{dr_2}{dt} = -AP_0r_2 + P_0F(r_1 + p_1 + q_1), \end{cases} \tag{4}$$

$$\begin{cases} \frac{dp_1}{dt} = -AP_1p_1, \\ \frac{dp_2}{dt} = -AP_1p_2 + P_1F(r_1 + p_1 + q_1), \end{cases} \tag{5}$$

$$\begin{cases} \frac{dq_1}{dt} = -AQq_1, \\ \frac{dq_2}{dt} = -AQq_2 + QF(r_1 + p_1 + q_1). \end{cases} \tag{6}$$

As we know, the existence and uniqueness of negatively bounded solutions of (4), (5) and (6) is essential in the existence theory of inertial manifolds. By the Arzela-Ascoli theorem, there exists negatively bounded solutions of systems (4), (5) and (6) that satisfy $\sup_{t\le 0}\|q_1(t)\| < \infty$ and $\sup_{t\le 0}\|q_2(t)\| < \infty$ regardless of the spectral gap condition.

We only prove that these negatively bounded solutions are unique.

3. The existence of an inertial manifold

Lemma 3.1. *The negatively bounded solutions of systems* (4), (5) *and* (6) *are unique.*

Proof. Let (u_1, v_1) and (u_2, v_2) are two negatively bounded solutions of original system (1) and set

$$u_1 - u_2 = P_0(u_1 - u_2) + P_1(u_1 - u_2) + Q(u_1 - u_2) = \phi_1 + \rho_1 + \sigma_1,$$

$$v_1 - v_2 = P_0(v_1 - v_2) + P_1(v_1 - v_2) + Q(v_1 - v_2) = \phi_2 + \rho_2 + \sigma_2,$$

then ϕ_1, ρ_1, σ_1 and ϕ_2, ρ_2, σ_2 are solutions of the following differential systems

$$\begin{cases} \frac{d\phi_1}{dt} = -AP_0\phi_1 + AP_0\phi_2, \\ \frac{d\phi_2}{dt} = -AP_0\phi_2 + P_0(F(u_1) - F(u_2)), \end{cases} \tag{7}$$

$$\begin{cases} \frac{d\rho_1}{dt} = -AP_1\rho_1, \\ \frac{d\rho_2}{dt} = -AP_1\rho_2 + P_1(F(u_1) - F(u_2)), \end{cases} \tag{8}$$

$$\begin{cases} \frac{d\sigma_1}{dt} = -AQ\sigma_1, \\ \frac{d\sigma_2}{dt} = -AQ\sigma_2 + Q(F(u_1) - F(u_2)), \end{cases} \tag{9}$$

where $\sup_{t\le 0}\|\sigma_1(t)\| < \infty$ and $\sup_{t\le 0}\|\sigma_2(t)\| < \infty$.

Integrating the first equation of (8) and (9) respectively from t to 0 and $-\infty$ to t, we have

$$e^{AP_1 t}\rho_1(t) = \rho_1(0) \text{ and } e^{AQt}\sigma_1(t) = 0.$$

Then taking the norm of both sides of the first equation, we have the estimates

$$e^{\lambda_N t}\|\rho_1(t)\| \le \|\rho_1(0)\|, \tag{10}$$

and

$$\sigma_1(t) = 0. \tag{11}$$

Now, by using the variation of constants formula, the first equation of (7) has the following integral form

$$\phi_1(t) = e^{-AP_0 t}\phi_1(0) - \int_t^0 e^{-AP_0(t-\tau)} AP_0 \phi_2(\tau) d\tau. \tag{12}$$

For notational simplicity, let us denote $\Lambda = (\lambda_N + \lambda_{N+1})/2$, then multiply both sides of (12) by $e^{\Lambda t}$ and estimates to obtain

$$\begin{aligned} \|e^{\Lambda t}\phi_1(t)\| &\le e^{\left(\Lambda - \lambda_{N_0}\right)t}\|\phi_1(0)\| \\ &\quad + \int_t^0 \|(AP_0)^{\frac{1}{2}} e^{-(AP_0-\Lambda)(t-\tau)}\|_{op} \|e^{\Lambda\tau}(AP_0)^{\frac{1}{2}}\phi_2(\tau)\| d\tau. \end{aligned} \tag{13}$$

On the other hand, the second equation of (7) yields that

$$\phi_2(t) = e^{-AP_0 t}\phi_2(0) - \int_t^0 e^{-AP_0(t-\tau)} P_0(F(u_1) - F(u_2)) d\tau. \tag{14}$$

We multiply $e^{\Lambda t}(AP_0)^{\frac{1}{2}}$ to both sides of (14) and take the norm to find

$$\begin{aligned} &\|e^{\Lambda t}(AP_0)^{\frac{1}{2}}\phi_2(t)\| \\ &\le \sqrt{\lambda_{N_0}}\|\phi_2(0)\| \\ &\qquad + \int_t^0 \|(AP_0)^{\frac{1}{2}} e^{-(AP_0-\Lambda)(t-\tau)}\|_{op} \|e^{\Lambda\tau} P_0(F(u_1) - F(u_2))\| d\tau \\ &\le \sqrt{\lambda_{N_0}}\|\phi_2(0)\| + K_1 \gamma_{N_0,N}\left(\|e^{\Lambda\tau}\phi_1(\tau)\|_\infty + \|\rho_1(0)\|\right), \end{aligned}$$

where

$$\begin{aligned} \gamma_{N_0,N} &= \int_t^0 \|(AP_0)^{\frac{1}{2}} e^{-(AP_0-\Lambda)(t-\tau)}\|_{op} d\tau \\ &\le \frac{2\sqrt{\lambda_{N_0}}}{\lambda_N + \lambda_{N+1} - 2\lambda_{N_0}} \\ &\le \frac{\sqrt{\lambda_{N_0}}}{\lambda_N - \lambda_{N_0}}. \end{aligned}$$

Applying this estimate to (13), one has

$$\begin{aligned} \|e^{\Lambda t}\phi_1(t)\| &\le \|\phi_1(0)\| + \sqrt{\lambda_{N_0}}\gamma_{N_0,N}\|\phi_2(0)\| \\ &\quad + K_1\gamma_{N_0,N}^2\left(\|e^{\Lambda\tau}\phi_1(\tau)\|_\infty + \|\rho_1(0)\|\right). \end{aligned} \tag{15}$$

Assume $K_1\gamma^2_{N_0,N} < 1$, we thereby obtain the following estimate

$$\|e^{\Lambda t}\phi_1(t)\|_\infty \leq \frac{1}{1-K_1\gamma^2_{N_0,N}}\|\phi_1(0)\| + \frac{\sqrt{\lambda_{N_0}}\gamma_{N_0,N}}{1-K_1\gamma^2_{N_0,N}}\|\phi_2(0)\| + \frac{K_1\gamma^2_{N_0,N}}{1-K_1\gamma^2_{N_0,N}}\|\rho_1(0)\|. \tag{16}$$

Moreover, using the variation of constants formula, we obtain from the second equation of (8) and (9)

$$\rho_2(t) = e^{-AP_1 t}\rho_2(0) - \int_t^0 e^{-AP_1(t-\tau)}P_1(F(u_1)-F(u_2))d\tau, \tag{17}$$

$$\sigma_2(t) = \int_{-\infty}^t e^{-AQ(t-\tau)}Q(F(u_1)-F(u_2))d\tau. \tag{18}$$

Analogously, we multiply $e^{\Lambda t}$ to both sides of (14), (17) and (18) to get the estimates

$$\begin{aligned}\|e^{\Lambda t}\phi_2(t)\| &\leq \|\phi_2(0)\| + K_1\left(\|e^{\Lambda t}\phi_1(t)\|_\infty + \|\rho_1(0)\|\right)\int_t^0 \|e^{-(AP_0-\Lambda)s}\|_{op}ds \\ &\leq \|\phi_2(0)\| + \frac{2K_1}{\lambda_N+\lambda_{N+1}-2\lambda_{N_0}}\left(\|e^{\Lambda t}\phi_1(t)\|_\infty + \|\rho_1(0)\|\right),\end{aligned} \tag{19}$$

$$\begin{aligned}\|e^{\Lambda t}\rho_2(t)\| &\leq \|\rho_2(0)\| \\ &\quad + K_1\left(\|e^{\Lambda t}\phi_1(t)\|_\infty + \|\rho_1(0)\|\right)\int_t^0 \|e^{-(AP_1-\Lambda)(t-\tau)}\|_{op}d\tau \\ &\leq \|\rho_2(0)\| + \frac{2K_1}{\lambda_{N+1}-\lambda_N}\left(\|e^{\Lambda t}\phi_1(t)\|_\infty + \|\rho_1(0)\|\right),\end{aligned} \tag{20}$$

and

$$\begin{aligned}\|e^{\Lambda t}\sigma_2(t)\| &\leq K_1\left(\|e^{\Lambda t}\phi_1(t)\|_\infty + \|\rho_1(0)\|\right)\int_{-\infty}^t e^{-(AQ-\Lambda)(t-\tau)}\|_{op}d\tau \\ &= \frac{2K_1}{\lambda_{N+1}-\lambda_N}\left(\|e^{\Lambda t}\phi_1(t)\|_\infty + \|\rho_1(0)\|\right).\end{aligned} \tag{21}$$

Suppose $Pu_1(0) = Pu_2(0)$ and $Pv_1(0) = Pv_2(0)$, that is $\phi_1(0) = \phi_2(0) = \rho_1(0) = 0$. The inequality (10) implies $\rho_1(t) = 0$. Now we apply (16) to (15), (19), (20) and (21). It follows that $\phi_1(t) = 0$, $\phi_2(t) = 0$, $\rho_2(t) = 0$ and $\sigma_2(t) = 0$.

□

This asserts that the negatively bounded solutions of (4), (5) and (6) are unique. The Lipschitz continuity of the function defining an inertial manifold follows immediately.

Lemma 3.2. *The map $\Phi : P\mathbb{H} \to Q\mathbb{H}$ is Lipschitz continuous, where Φ defined as*

$$\Phi : (P_0u(0) + P_1u(0), P_0v(0) + P_1v(0)) \mapsto (Qu(0), Qv(0)) .$$

Proof. From estimate (21) in the proof of Lemma 3.1, we have

$$\|e^{\Lambda t}\sigma_2(t)\|_\infty \le \frac{2K_1}{\lambda_{N+1} - \lambda_N} \left(\|e^{\Lambda t}\phi_1(t)\|_\infty + \|\rho_1(0)\|\right) .$$

Take $t = 0$ and combine with (16), we deduce the estimate

$$\begin{aligned} \|\sigma_2(0)\| &\le \|e^{\Lambda t}\sigma_2(t)\|_\infty \\ &\le \frac{2K_1}{\lambda_{N+1} - \lambda_N} \cdot \frac{1}{1 - K_1\gamma^2_{N_0,N}} (\|\phi_1(0)\| + \|\rho_1(0)\|) \\ &\quad + \frac{2K_1}{\lambda_{N+1} - \lambda_N} \cdot \frac{\sqrt{\lambda_{N_0}}\gamma_{N_0,N}}{1 - K_1\gamma^2_{N_0,N}} \|\phi_2(0)\|. \end{aligned} \tag{22}$$

This implies that Φ is a Lipschitz continuous map. □

Addition to the Lipschitz continuity of the map which defines inertial manifolds, we also need to prove that the inertial manifold attracts all trajectory of system (1) exponentially. In fact, we have

Theorem 3.3. *Assume there exists $N \in \mathbf{N}$ such that $\lambda_{N+1} - \lambda_N > 2K_1$, $\lambda_{N_0} \le \frac{1}{2}\lambda_N$ and $\lambda_{N_0} < \left(\sqrt{\frac{K_1}{3} + \lambda_N} - \sqrt{\frac{K_1}{3}}\right)^2$, then there exists an inertial manifold for system* (1)*.*

Proof. We know from Lemma 3.1, negatively bounded solutions of (4), (5) and (6) are unique. On this basis, we can construct an inertial manifold $\mathcal{M}$ as a graph of the Lipschitz continuous function Φ and the positively invariant property holds by the construction of the inertial manifold. In order to complete the proof, we need to prove the exponential tracking property.

Assume that the Lipschitz constant less than 1, that is,

$$\frac{2K_1}{\lambda_{N+1} - \lambda_N} \cdot \frac{1}{1 - K_1\gamma^2_{N_0,N}} < 1 \text{ and } \frac{2K_1}{\lambda_{N+1} - \lambda_N} \cdot \frac{\sqrt{\lambda_{N_0}}\gamma_{N_0,N}}{1 - K_1\gamma^2_{N_0,N}} < 1,$$

which holds when $\lambda_{N+1} - \lambda_N > 2K_1$, $\lambda_{N_0} < \left(\sqrt{\frac{K_1}{3} + \lambda_N} - \sqrt{\frac{K_1}{3}}\right)^2$ and $\lambda_{N_0} \leq \frac{1}{2}\lambda_N$. Then, all the solutions of (4), (5) and (6) tends exponentially to $\mathcal{M}$, see [6]. □

References

[1] Z. Fang and M. Kwak, A finite dimensional property of a parabolic partial differential equation, J. Dynam. Diff. Eqns 17 (2005), 845-855.

[2] Z. Fang and M. Kwak, Negatively bounded solutions for a parabolic partial differential equation, Bull. Korean Math. Soc. 42 (2005), 829-836.

[3] C. Foias, G. Sell and R. Temam, Inertial manifolds for nonlinear evolutionary equations, J. Diff. Eqns 73 (1988), 309-353.

[4] A. Kostianko and S. Zelik, Kwak transform and inertial manifolds revisited, arXiv preprint arXiv:1911.00698 (2019).

[5] M. Kwak, Finite dimensional inertial forms for 2D Navier-Stokes equations, Indiana Univ. Math. J. 41 (1992), 927-982.

[6] M. Kwak and X. Sun, Remarks on the existence of an inertial manifold, (2020), accepted.

[7] R. Temam, Infinite-dimensional dynamical systems in mechnics and physics, Applied Math. Science 68 (1997), Springer.

[8] S. Zelik, Inertial manifolds and finite-dimensional reduction for dissipative PDEs, Proc R Soc Lond Ser A 144A (2014), 1245-1327.

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