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UNIT TANGENT SPHERE BUNDLES OF LOCALLY SYMMETRIC SPACES

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Abstract. We give characterizations of locally symmetric spaces M via the structural operator $h = \frac{1}{2}\mathcal{L}_\xi\phi$ or the characteristic Jacobi operator $\ell = R(\cdot, \xi)\xi$ on the unit tangent sphere bundles T_1M.

1. Introduction

When we study a Riemannian manifold (M, g), it is interesting to investigate the interplay between the manifold and its unit tangent sphere bundle T_1M equipped with the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$.

On the other hand, in contact metric geometry, other than the defining structure tensors η, $\bar{g}$, ϕ and ξ, two self-adjoint operators play a fundamental role, namely, the structural operator $h = \frac{1}{2}\mathcal{L}_\xi\phi$ and the characteristic Jacobi operator $\ell = R(\cdot, \xi)\xi$, where $\mathcal{L}_\xi$ denotes Lie differentiation in the characteristic direction ξ.

The main purpose of this paper is to prove

Theorem 1.1. *Let (M, g) be a Riemannian manifold and T_1M be the unit tangent sphere bundle with the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$. Then T_1M satisfies $\bar{\nabla}_\xi\ell = \mu(h\phi\ell - \ell\phi h)$ for real number $\mu \neq 1$ if and only if (M, g) is a space of constant curvature c ($\neq \frac{2}{3}, 2$) or locally isometric to a symmetric space of rank one where the eigenvalues of R_u are 1 and 4 (or $\frac{1}{4}$).*

The unit tangent sphere bundles of complex space forms were studied in an earlier work ([8]).

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2. Preliminaries

All manifolds in the present paper are assumed to be connected and of class C^∞. We start by collecting some fundamental material about contact metric geometry. We refer to [2] for further details. A $(2n-1)$-dimensional differentiable manifold $\bar{M}^{2n-1}$ is said to be a it contact manifold if it admits a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form η, we have a unique vector field ξ, the *characteristic vector field,* satisfying $\eta(\xi) = 1$ and $d\eta(\xi, \bar{X}) = 0$ for any vector field $\bar{X}$ on $\bar{M}$. It is well-known that there exists a Riemannian metric $\bar{g}$ and a $(1,1)$-tensor field ϕ such that

$$(1)\quad \eta(\bar{X}) = \bar{g}(\bar{X}, \xi), \quad d\eta(\bar{X}, \bar{Y}) = \bar{g}(\bar{X}, \phi\bar{Y}), \quad \phi^2\bar{X} = -\bar{X} + \eta(\bar{X})\xi,$$

where $\bar{X}$ and $\bar{Y}$ are vector fields on $\bar{M}$. From (1) it follows that

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \bar{g}(\phi\bar{X}, \phi\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}) - \eta(\bar{X})\eta(\bar{Y}).$$

A Riemannian manifold $\bar{M}$ equipped with structure tensors $(\eta, \bar{g}, \phi, \xi)$ satisfying (1) is said to be a *contact metric manifold* and is denoted by $\bar{M} = (\bar{M}; \eta, \bar{g}, \phi, \xi)$. Given a contact metric manifold $\bar{M}$, we define the *structural operator* h by $h = \frac{1}{2}\mathcal{L}_\xi\phi$, where $\mathcal{L}$ denotes Lie differentiation. Then we can show that h is symmetric and satisfies

$$(2)\qquad h\xi = 0 \quad \text{and} \quad h\phi = -\phi h,$$

$$(3)\qquad \bar{\nabla}_{\bar{X}}\xi = -\phi\bar{X} - \phi h\bar{X},$$

where $\bar{\nabla}$ is the Levi-Civita connection. From (2) and (3) we see that ξ generates a geodesic flow. Furthermore, we know that $\bar{\nabla}_\xi\phi = 0$ in general (cf. p.67 in [2]). From the second equation of (2) it follows also that

$$(4)\qquad (\bar{\nabla}_\xi h)\phi = -\phi(\bar{\nabla}_\xi h).$$

We denote by $\bar{R}$ the Riemannian curvature tensor defined by

$$\bar{R}(\bar{X}, \bar{Y})\bar{Z} = \bar{\nabla}_{\bar{X}}\bar{\nabla}_{\bar{Y}}\bar{Z} - \bar{\nabla}_{\bar{Y}}\bar{\nabla}_{\bar{X}}\bar{Z} - \bar{\nabla}_{[\bar{X}, \bar{Y}]}\bar{Z}$$

for all vector fields $\bar{X}, \bar{Y}$ and $\bar{Z}$. Along a geodesic flow ξ, the Jacobi operator $\ell = \bar{R}(\cdot, \xi)\xi$ is a symmetric $(1,1)$-tensor field. We call it the *characteristic Jacobi operator*. From the definition of $\bar{R}$ by using (3) we have

$$(5)\qquad \ell = \phi\bar{\nabla}_\xi h - (h^2 + \phi^2).$$

From (5), using the second equation of (2) and (4), we have

$$(6)\qquad \bar{\nabla}_\xi h = \frac{1}{2}(\ell\phi - \phi\ell).$$

3. The contact metric structure of the unit tangent sphere bundle

We briefly review some notations and definitions. Let $M = (M, g)$ be an n-dimensional Riemannian manifold and ∇ the associated Levi-Civita connection. The tangent bundle of (M, g), denoted by TM, consists of pairs (p, u) where p is a point in M and u a tangent vector to M at p. The natural projection mapping is given by $\pi : TM \to M,\ \pi(p, u) = p$. The tangent space to TM at (p, u) splits into the direct sum of the vertical subspace $\mathcal{V}$ =Ker π_* and the the horizontal subspace $\mathcal{H}$ with respect to ∇:

$$T_{(p,u)}TM = \mathcal{V} \oplus \mathcal{H}.$$

The vertival (resp. horizontal) subspace consists of tangent vectors at (p, u) to a curve $\tilde{\gamma}(t) = (\gamma(t), V(t))$ in TM satisfying $\gamma(t) = p$ for each t (resp. $\nabla_{\dot{\gamma}(t)} V(t) = 0$).

For a vector $X \in T_pM$, the *horizontal lift* $X^h \in \mathcal{H}$ is (uniquely) determined by $\pi_* X^h = X$ and the *vertical lift* $X^v \in \mathcal{V}$ is (uniquely) detemined by $X^v(df) = X(f)$ for all smooth fuction f on M. Then we can define a Riemannian metric $\tilde{g}$, the *Sasaki metric* on TM ([14]), in a natural way by

$$\tilde{g}(X^h, Y^h) = \tilde{g}(X^v, Y^v) = g(X, Y) \circ \pi, \quad \tilde{g}(X^h, Y^v) = 0$$

for all vector fields X and Y on M. Also, a natural almost complex structure tensor J of TM is defined by $JX^h = X^v$ and $JX^v = -X^h$. Then we easily see that $(TM; \tilde{g}, J)$ is an almost Hermitian manifold, actually $(\tilde{g}, J)$ is an almost Kähler structure. Note that J is integrable if and only if (M, g) is locally flat ([15]). Now we consider the unit tangent sphere bundle (T_1M, g'), which is an isometrically embedded hypersurface in $(TM, \tilde{g})$ with unit normal vector field $N = u^v$. For $X \in T_pM$, we define the *tangential lift* of X to $(p, u) \in T_1M$ by

$$X^t_{(p,u)} = X^v_{(p,u)} - g(X, u)N_{(p,u)}.$$

Clearly, the tangent space $T_{(p,u)}T_1M$ is spanned by vectors of the form X^h and X^t where $X \in T_pM$. We put

$$\xi' = -JN, \quad \phi' = J - \eta' \otimes N.$$

Then we find $g'(\bar{X}, \phi'\bar{Y}) = 2d\eta'(\bar{X}, \bar{Y})$. By taking $\xi = 2\xi'$, $\eta = \frac{1}{2}\eta'$, $\phi = \phi'$, and $\bar{g} = \frac{1}{4}g'$, we get the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$. Indeed, we easily check that these tensors satisfy (1). Here we notice that ξ determines the geodesic flow. The tensors ξ and ϕ are explicitly given by

$$\xi = 2u^h, \quad \phi X^t = -X^h + \frac{1}{2}\, g(X,u)\xi, \quad \phi X^h = X^t \tag{7}$$

where X and Y are vector fields on M ([16]). From now on, we consider $T_1M = (T_1M; \eta, \bar{g})$ with the standard contact metric structure. We list the fundamental formulae which we need for the proof of our theorems. They are derived in [4, 5, 10]. The Levi-Civita connection $\bar{\nabla}$ of $(T_1M, \bar{g})$ is given by

$$\begin{aligned}
\bar{\nabla}_{X^t} Y^t &= -g(Y,u)X^t, \\
\bar{\nabla}_{X^t} Y^h &= \frac{1}{2}\,(R(u,X)Y)^h, \\
\bar{\nabla}_{X^h} Y^t &= (\nabla_X Y)^t + \frac{1}{2}\,(R(u,Y)X)^h, \\
\bar{\nabla}_{X^h} Y^h &= (\nabla_X Y)^h - \frac{1}{2}\,(R(X,Y)u)^t.
\end{aligned} \tag{8}$$

For the Riemann curvature tensor $\bar{R}$, we give only the two expressions, which we need for the characteristic Jacobi operator ℓ:

$$\begin{aligned}
\bar{R}(X^t,Y^h)Z^h =& -\frac{1}{2}\,\{R(Y,Z)(X - g(X,u)u)\}^t \\
&+ \frac{1}{4}\,\{R(Y,R(u,X)Z)u\}^t \\
&- \frac{1}{2}\,\{(\nabla_Y R)(u,X)Z\}^h, \\
\bar{R}(X^h,Y^h)Z^h =& (R(X,Y)Z)^h + \frac{1}{2}\,\{R(u,R(X,Y)u)Z\}^h \\
&- \frac{1}{4}\,\{R(u,R(Y,Z)u)X - R(u,R(X,Z)u)Y\}^h \\
&+ \frac{1}{2}\,\{(\nabla_Z R)(X,Y)u\}^t
\end{aligned} \tag{9}$$

for all vector fields X, Y and Z on M. In the above, we denote by ∇ the Levi-Civita connection and by R the Riemannian curvature tensor associated with g. From (7) and (8), it follows that

$$\bar{\nabla}_{X^t}\xi = -2\phi X^t - (R_u X)^h, \quad \bar{\nabla}_{X^h}\xi = -(R_u X)^t, \tag{10}$$

where $R_u = R(\cdot, u)u$ is the Jacobi operator associated with the unit vector u. From (3) and (10), it follows that

$$\begin{aligned} hX^t &= X^t - (R_uX)^t, \\ hX^h &= -X^h + \frac{1}{2}\, g(X,u)\xi + (R_uX)^h. \end{aligned} \tag{11}$$

Using the formulae (9), we get

$$\begin{aligned} \ell X^t &= (R_u^2X)^t + 2(R'_uX)^h, \\ \ell X^h &= 4(R_uX)^h - 3(R_u^2X)^h + 2(R'_uX)^t, \end{aligned} \tag{12}$$

where $R'_u = (\nabla_u R)(\cdot, u)u$ and $R_u^2 = R(R(\cdot, u)u, u)u$. By using (6), (7) and (12) we obtain

$$\begin{aligned} h'X^t &= -2(R_uX)^h + 2(R_u^2X)^h - 2(R'_uX)^t, \\ h'X^h &= -2(R_uX)^t + 2(R_u^2X)^t + 2(R'_uX)^h, \end{aligned} \tag{13}$$

where we put $h' = \bar{\nabla}_\xi h$.

The above formulae (10)–(13) are also found in [3, 4]. Finally, from (8) and (12) we compute

(14)

$$\begin{aligned} \ell' X^t &= 4(R'_uR_uX + R_uR'_uX)^t + 4(R''_uX + R_u^2X - R_u^3X)^h, \\ \ell' X^h &= 8(R'_uX - R'_uR_uX - R_uR'_uX)^h + 4(R''_uX + R_u^2X - R_u^3X)^t, \end{aligned}$$

where $\ell' = (\bar{\nabla}_\xi \bar{R})(\cdot, \xi)\xi$. We also refer the formula (14) to [7, 9].

4. Unit tangent sphere bundles of locally symmetric spaces

From (12) and (13), we can easily have

Theorem 4.1. ([4]) *A Riemannian manifold (M, g) is locally symmetric if and only if one of the following statements holds:*

(*a*) *the horizontal* (*or equivalently, the vertical*) *distribution of T_1M is invariant by ℓ;*

(*b*) *the horizontal* (*or equivalently, the vertical*) *distribution of T_1M is anti-invariant by h'.*

Now, we prove

Theorem 4.2. *A Riemannian manifold (M, g) is locally symmetric if and only if one of the following statements holds:*

(*a*) *both the horizontal and the vertical distributions of T_1M are anti-invariant respectively by ℓ';*

(*b*) *the self-adjoint operators* h *and* ℓ *on* T_1M *commute, and the horizontal distribution of* T_1M *is anti-invariant by* ℓ';

(*c*) *the self-adjoint operators* h *and* ℓ *on* T_1M *commute, and the vertical distribution of* T_1M *is anti-invariant by* ℓ'.

Proof. From (14) we find that

$$\ell'(\mathcal{V}) \subset \mathcal{H} \Longleftrightarrow R'_u R_u X + R_u R'_u X = 0, \tag{15}$$

$$\ell'(\mathcal{H}) \subset \mathcal{V} \Longleftrightarrow R'_u X - R'_u R_u X - R_u R'_u X = 0. \tag{16}$$

So, we have that both the horizontal and the vertical distributions of T_1M are anti-invariant respectively by ℓ' if and only if $R'_u X = 0$. Due to Cartan's result ([6]), this condition is satisfied if and only if (M, g) is locally symmetric, which completes to prove (a). In order to prove (b) and (c), we compute

$$\begin{aligned}(h\ell - \ell h)(X^t) &= 2(R'_u R_u X + R_u R'_u X - 2R'_u X)^h,\\ (h\ell - \ell h)(X^h) &= -2(R'_u R_u X + R_u R'_u X - 2R'_u X)^t.\end{aligned} \tag{17}$$

From (17) we get $2R'_u X = R'_u R_u X + R_u R'_u X$. So, combining with (15), (16) respectively we have $R'_u X = 0$ for the both cases. Therefore, we have completed Theorem 4.2.

□

Theorem 4.3. *Let* (M, g) *be a real analytic Riemannian manifold and* T_1M *be the unit tangent sphere bundle with the standard contact metric structure* $(\eta, \bar{g}, \phi, \xi)$. *Then* T_1M *satisfies* $\ell\phi h = \phi h \ell$ *if and only if* (M, g) *is a space of constant curvature* 0 *or* 1.

Proof. We first compute

$$\begin{aligned}(\ell\phi h)(X^h) &= (R_u^3 X - R_u^2 X)^t - 2(R'_u X - R'_u R_u X)^h,\\ (\phi h\ell)(X^h) &= (-4R_u X + 7R_u^2 X - 3R_u^3 X)^t - 2(R'_u X - R_u R'_u X)^h,\end{aligned} \tag{18}$$

$$\begin{aligned}(\ell\phi h)(X^t) &= (-4R_u X + 7R_u^2 X - 3R_u^3 X)^h - 2(R'_u X - R'_u R_u X)^t,\\ (\phi h\ell)(X^t) &= (R_u^3 X - R_u^2 X)^h - 2(R'_u X - R_u R'_u X)^t,\end{aligned} \tag{19}$$

where we have used (7), (11), (12). Thus, from (18) and (19) we have that T_1M satisfies $\ell\phi h = \phi h\ell$ if and only if M satisfies $4R_u^3 X - 8R_u^2 X + 4R_u X = 0$ and $R'_u R_u X = R_u R'_u X$. The first one implies that the eigenvalues of R_u are constants. Moreover, the second condition $R'_u \circ R_u = R_u \circ R'_u$ on real analytic manifold M implies that R_u is diagonalizable by a parallel orthonormal frame field. From those two properties, we have that M is locally symmetric, that is, $\nabla R = 0$ (cf.

[1]), and moreover $4R_u^3X - 8R_u^2X + 4R_uX = 0$ yields that M is of constant curvature 0 or 1. Also, we easily check that such specific spaces satisfy the commutation condition $\ell\phi h = \phi h\ell$. □

5. Proof of Main theorem

In this section, we prove the main therem (Theorem 1.1). Suppose that T_1M satisfies $\bar{\nabla}_\xi\ell = \mu(h\phi\ell - \ell\phi h)$ for real number $\mu \neq 1$. From (2), (18) and (19), we compute

$$(20)\quad \begin{aligned} (h\phi\ell - \ell\phi h)(X^t) =& 2(2R_uX - 3R_u^2X + R_u^3X)^h \\ &+ 2(2R'_uX - R_uR'_uX - R'_uR_uX)^t, \\ (h\phi\ell - \ell\phi h)(X^h) =& 2(2R_uX - 3R_u^2X + R_u^3X)^t \\ &+ 2(2R'_uX - R_uR'_uX - R'_uR_uX)^h \end{aligned}$$

for all $X \perp u$. Then from (14) and (20), we have

$$(21)\quad 2\mu R'_uX = (\mu + 2)(R'_uR_uX + R_uR'_uX),$$

$$(22)\quad 2R''_uX - 2\mu R_uX + (3\mu + 2)R_u^2X - (\mu + 2)R_u^3X = 0$$

and

$$(23)\quad 2(\mu - 2)R'_uX = (\mu - 4)(R'_uR_uX + R_uR'_uX)$$

for all $X \perp u$. Combining (21) and (23), then we get $(\mu - 1)R'_uX = 0$. From the assumption $\mu \neq 1$, we have that M is locally symmetric, and from (22) we have $2\mu R_uX - (3\mu + 2)R_u^2X + (\mu + 2)R_u^3X = 0$. Assuming $R_uX = \lambda X$, then we get $\lambda(\lambda - 1)\big((\mu + 2)\lambda - 2\mu\big) = 0$. If $\mu = -2$, then we have that M is of constant curvature 0 or 1. In case that $\mu \neq -2$, we have $\lambda = 0, 1$, or $\lambda = \frac{2\mu}{\mu+2}$, that is, (M, g) is a globally Osserman space (i.e., the eigenvalues of R_u neither depend on the point p nor on the choice of unit vector u at p). However, a locally symmetric globally Osserman space is locally flat or locally isometric to a symmetric space of rank one ([12]). Then we have that M is either a space of constant curvature $c(\neq \frac{2}{3}, 2)$ or a locally symmetric space of rank one with $\lambda = 1, 4$ (or $\frac{1}{4}$). In order to show the converse, we treat first a Riemannian space (M, g) of constant curvature c. Then we have $R_uX = cX$ and $R'_uX = 0$ for all $X \perp u$. From (14) and (20), we get

$$\begin{aligned} \ell' X^t &= 4c^2(1 - c)X^h, \\ \ell' X^h &= 4c^2(1 - c)X^t \end{aligned}$$

and

$$(h\phi\ell - \ell\phi h)X^t = 2c(1-c)(2-c)X^h,$$
$$(h\phi\ell - \ell\phi h)X^h = 2c(1-c)(2-c)X^t.$$

From the above equations, we first see that T_1M satisfies the condition $\nabla_\xi \ell = \mu(h\phi\ell - \ell\phi h)$ for any real number μ, when $c = 0, 1$. For the other cases, we take $\mu = 2c/(2-c)$, where $c \neq 2$. Next, we consider a rank one symmetric space of non-constant sectional curvature. Those are the compact symmetric spaces of rank one: $\mathbb{C}P^{n/2}$, $\mathbb{H}P^{n/4}$, the Cayley plane $\mathrm{Ca}P^2$ and their non-compact duals. In such spaces the Jacobi operator R_u has only two eigenvalues $k(\neq 0)$ and $\frac{k}{4}$. Taking $\mu = -4$ (or $\mu = \frac{2}{7}$), then they satisfy $\nabla_\xi \ell = \mu(h\phi\ell - \ell\phi h)$. For example, assuming $R_uX = X$ and $R_uY = 4Y$, then we have from (14)

$$\ell' X^t = 0, \quad \ell' Y^t = -4(48Y)^h$$
$$\ell' X^h = 0, \quad \ell' Y^h = -4(48Y)^t.$$

Moreover, we have from (12)

$$(h\phi\ell - \ell\phi h)X^t = 0, \quad (h\phi\ell - \ell\phi h)Y^t = 48Y^h$$
$$(h\phi\ell - \ell\phi h)X^h = 0, \quad (h\phi\ell - \ell\phi h)Y^h = 48Y^t.$$

After all, we have that M is either a space of constant curvature $c(\neq \frac{2}{3}, 2)$ or a locally symmetric space of rank one with $\lambda = 1, 4$(or $\frac{1}{4}$). This completes the proof of Theorem 1.1. ■

From Theorems 1.1 and 4.3, we have

Corollary 5.1. *Let M be an n-dimensional Riemannian manifold and T_1M be the unit tangent sphere bundle with the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$. Then the followings are equivalent:*
(i) $\bar{\nabla}_\xi \ell = 0$,
(ii) $\bar{\nabla}_\xi \ell = \pm 2(h\phi\ell - \ell\phi h)$,
(iii) (M, g) *is a real analytic manifold and* T_1M *satisfies* $\ell\phi h = \phi h \ell$,
(iv) (M, g) *is either a space of constant curvature* 0 *or* 1.

The equivalency of (i) and (iv) was also proved by Perrone ([13]).

Remark 1. In a previous work [11], we have proved that the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$ of T_1M satisfies $\bar{\nabla}_\xi \ell = k(\phi\ell - \ell\phi)$ $(k \neq 2, k \in \mathbb{R})$ if and only if either (M, g) is a space of constant curvature c $(c \neq 2)$ or a locally rank one symmetric space where the eigenvalues of R_u are 1 and 4 (or $\frac{1}{4}$).

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