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QUASI HEMI-SLANT SUBMANIFOLDS OF KAEHLER MANIFOLDS

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Abstract. In the present paper, we introduce the notion of quasi hemi-slant submanifolds of almost Hermitian manifolds and give some of its examples. We obtain the necessary and sufficient conditions for the distributions to be integrable. We also investigate the necessary and sufficient conditions for these submanifolds to be totally geodesic and study the geometry of foliations determined by the distributions. Finally, we obtain the necessary and sufficient condition for a quasi hemi-slant submanifold to be local product of Riemannian manifold.

1. Introduction

The theory of submanifolds is a vast and rich research field. It is playing an important role in the development of modern differential geometry. It has the origin in the study of the geometry of plane curves initiated by Fermat. Nowadays this theory plays a key role in computer design, image processing, economic modelling as well as in mathematical physics and mechanics.

In 1990, B. Y. Chen [5] introduced the notion of slant submanifolds of almost Hermitian manifolds. It was a natural generalization of both holomorphic and totally real submanifolds. The theory of submanifolds has been studied by several geometers such as ([1]-[4],[6],[7],[9]-[11],[13]-[15] and [16]). Further the notion of slant submanifold is generalized as semi-slant submanifold, pseudo-slant submanifold, bi-slant submanifold

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etc. The purpose of the present paper is to introduce the notion of quasi-hemi-slant submanifolds of almost Hermitian manifolds which includes the classes of anti-invariant, semi-invariant, slant, semi-slant and hemi-slant submanifolds as its particular cases.

The paper is organized as follows : In section 2, we mention the basic definitions and properties of almost complex manifolds. In section 3, we define quasi-hemi-slant submanifolds of an almost Hermitian manifold and studied some of its basic properties. Section 4 deals with necessary and sufficient conditions for the distributions to be integrable. We also find the necessary and sufficient conditions for the submanifolds to be totally geodesic. In the last section, we construct some examples of such submanifolds.

2. Preliminaries

Let N be a Riemannian manifold with an almost complex structure J and a Hermitian metric g satisfying

$$J^2 = -I, \tag{1}$$

$$g(JX, JY) = g(X, Y), \tag{2}$$

for any $X, Y \in \Gamma(TN)$, where $\Gamma(TN)$ is the Lie algebra of vector fields in N, then (N, g) is called an almost Hermitian manifold. If an almost complex structure J satisfies

$$(\bar{\nabla}_X J)Y = 0 \tag{3}$$

for any $X, Y \in \Gamma(TN)$, where $\bar{\nabla}$ is the Levi-Civita connection on N, then N is called a Kaehler manifold [8].

Let N be a Kaehler manifold with an almost complex structure J and let M be a Riemannian manifold isometrically immersed in N. Then M is called holomorphic or complex if $J(T_xM) \subset T_xM$ for any $x \in M$, where T_xM denotes the tangent space of M at the point $x \in M$; and is called totally real if $J(T_xM) \subset T_x^{\perp}M$, for every $x \in M$, where $T_x^{\perp}M$ denotes the normal space of M at the point $x \in M$. There are three other important classes of submanifolds of a Kaehler manifold determined by the behaviour of the tangent bundle of the submanifold under the action of an almost complex structure of the ambient manifold. A distribution D on a manifold N is called autoparallel if $\bar{\nabla}_X Y \in D$ for any $X, Y \in D$ and is called parallel if $\bar{\nabla}_U X \in D$ for any $X \in D$ and $U \in TN$. If a distribution D on N is autoparallel, then it is called integrable, and thus by the Gauss formula the distribution D is totally geodesic in N. If D

is parallel, then the orthogonal complementary distribution $D^{\perp}$ is also parallel which implies that D is parallel if and only if $D^{\perp}$ is parallel. In this case N is locally product of the leaves of the distributions D and $D^{\perp}$. Let M be a submanifold (integral submanifold) of N. For the distributions D_1 and D_2 on M, we say that M is (D_1, D_2)-mixed totally geodesic if all $X \in D_1$ and $Y \in D_2$ and thus we have $h(X, Y) = 0$, where h is the second fundamental form of M.

Throughout this paper, we denote A and h the shape operator and the second fundamental form of submanifold M into manifold N, respectively. If ∇ is the induced Riemannian connection on M, then the Gauss and Weingarten formulae are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{4}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{5}$$

for all vector fields $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where $\nabla^{\perp}$ denotes the connection on the normal bundle $(T^{\perp}M)$ of M. The shape operator A and the second fundamental form h are related by

$$g(A_V X, Y) = g(h(X, Y), V). \tag{6}$$

The mean curvature vector is defined by

$$H = \frac{1}{n} trace(h) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i). \tag{7}$$

For any $X \in \Gamma(TM)$, we can write

$$JX = \phi X + \omega X, \tag{8}$$

where ϕX and ωX are the tangential and normal components of JX on M, respectively. Similarly for any $V \in \Gamma(T^{\perp}M)$, we have

$$JV = BV + CV, \tag{9}$$

where BV and CV are the tangential and normal components of JV on M, respectively.

The covariant derivative of projection morphisms given in (8) and (9) are defined by

$$(\bar{\nabla}_X \phi) Y = \nabla_X \phi Y - \phi \nabla_X Y, \tag{10}$$

$$(\bar{\nabla}_X \omega) Y = \nabla^{\perp}_X \omega Y - \omega \nabla_X Y, \tag{11}$$

$$(\bar{\nabla}_X B) V = \nabla_X BV - B \nabla^{\perp}_X V,$$

and

$$(\bar{\nabla}_X C) V = \nabla^{\perp}_X CV - C \nabla^{\perp}_X V,$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

A submanifold M of a Kaehler manifold N is said to be totally umbilical if

$$h(X,Y) = g(X,Y)H, \tag{12}$$

where H is the mean curvature vector defined in (7) . If $h(X,Y) = 0$ for all $X, Y \in \Gamma(TM)$, then M is said to be totally geodesic; and if $H = 0$, then M is said to be a minimal submanifold.

Now we introduce the notion of quasi hemi-slant submanifolds of almost Hermitian manifolds:

3. Quasi hemi-slant submanifolds of almost Hermitian manifolds

In this section, we introduce quasi hemi-slant submanifolds of almost Hermitian manifolds and obtain the necessary and sufficient conditions for the distributions to be integrable.

Definition 3.1. *A submanifold M of an almost Hermitian manifold N is called a quasi hemi-slant submanifold if there exist the distributions D, D_θ and $D^{\perp}$ such that*
(i) TM admits the orthogonal direct decomposition as

$$TM = D \oplus D_\theta \oplus D^{\perp}, \tag{13}$$

(ii) the distribution D is invariant, i.e., $JD = D$,
(iii) for any non-zero vector field $X \in (D_\theta)_p$, $p \in M$, the angle θ between JX and $(D_\theta)_p$ is constant and is independent of the choice of the point p and X in $(D_\theta)_p$,
(iv) the distribution $D^{\perp}$ is anti-invariant, i.e., $JD^{\perp} \subseteq T^{\perp}M$.

We call the angle θ a quasi hemi-slant angle of M. Suppose that the dimensions of the distributions D, D_θ and $D^{\perp}$ are n_1, n_2 and n_3, respectively. Then we can easily see the following particular cases:
(i) If $n_1 = 0$, then M is a hemi-slant submanifold,
(ii) If $n_2 = 0$, then M is a semi invariant submanifold,
(iii) If $n_3 = 0$, then M is a semi-slant submanifold.
We say that a quasi hemi-slant submanifold M is proper if $D \neq \{0\}$, $D^{\perp} \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$.
This means the notion of quasi hemi-slant submanifold is a generalization of invariant, anti-invariant, semi-invariant, slant, semi-slant and hemi-slant submanifolds.

Let M be a quasi hemi-slant submanifold of an almost Hermitian manifold N. We denote the projections of $X \in \Gamma(TM)$ on the distributions D, D_θ and $D^\perp$ by P, Q and R, respectively. Then we can write

$$X = PX + QX + RX \tag{14}$$

for any $X \in \Gamma(TM)$. Now, we put

$$JX = \phi X + \omega X, \tag{15}$$

where ϕX and ωX are the tangential and normal component on M. From (14) and (15), we obtain

$$JX = \phi PX + \omega PX + \phi QX + \omega QX + \phi RX + \omega RX. \tag{16}$$

Since $JD = D$ and $JD^\perp \subset (T^\perp M)$, therefore we have $\omega PX = 0$ and $\phi RX = 0$. Thus (16) reduces to

$$JX = \phi PX + \phi QX + \omega QX + \omega RX. \tag{17}$$

This means that

$$J(TM) = D \oplus \phi D_\theta \oplus \omega D_\theta \oplus JD^\perp.$$

Since $\omega D_\theta \subset \Gamma(T^\perp M)$ and $JD^\perp \subset (T^\perp M)$, so we have

$$T^\perp M = \omega D_\theta \oplus JD^\perp \oplus \mu, \tag{18}$$

where μ is the orthogonal complement of $\omega D_\theta \oplus JD^\perp$ in $(T^\perp M)$ and it is invariant with respect to J.
For any $Z \in \Gamma(T^\perp M)$, we put

$$JZ = BZ + CZ, \tag{19}$$

where $BZ \in \Gamma(TM)$ and $CZ \in \Gamma(T^\perp M)$.

Lemma 3.2. *Let M be a quasi hemi-slant submanifold of an almost Hermitian manifold N. Then we obtain*

$$\phi D = D,\ \phi D_\theta = D_\theta,\ \phi D^\perp = \{0\},\ B\omega D_\theta = D_\theta,\ B\omega D^\perp = D^\perp. \tag{20}$$

Now, by comparing the tangential and normal components in (14) and (19) and using (1), we have the following:

Lemma 3.3. *Let M be a quasi hemi-slant submanifold of an almost Hermitian manifold N. Then the endomorphisms ϕ and the projection morphisms ω, B and C on the tangent bundle of M satisfy the following identities:*
(i) $\phi^2 + B\omega = -I$ *on* TM,
(ii) $\omega\phi + C\omega = 0$ *on* TM,

(iii) $\omega B + C^2 = -I$ *on* $(T^{\perp}M)$,
(iv) $\phi B + BC = 0$ *on* $(T^{\perp}M)$,
where I *is the identity operator.*

Lemma 3.4. *Let* M *be a quasi hemi-slant submanifold of an almost Hermitian manifold* N*. Then we have*
(i) $\phi^2 X = -(\cos^2\theta)X$,
(ii) $g(\phi X, \phi Y) = (\cos^2\theta)g(X,Y)$,
(iii) $g(\omega X, \omega Y) = (\sin^2\theta)g(X,Y)$
for any $X, Y \in \Gamma(D_\theta)$.

Proof. The proof follows using similar steps as in (Proposition (2.8) in [12]). □

Using the equations (3), (4), (5), (8) and (9) and comparing the tangential and normal components, we have the following:

Lemma 3.5. *Let* M *be a quasi hemi-slant submanifold of a Kaehler manifold* N*. Then, we have*

$$\nabla_X \phi Y - A_{\omega Y} X - \phi \nabla_X Y - Bh(X,Y) = 0$$

and

$$h(X, \phi Y) + \nabla^{\perp}_X \omega Y - \omega(\nabla_X Y) - Ch(X,Y) = 0$$

for any $X, Y \in \Gamma(TM)$.

Lemma 3.6. *Let* M *be a quasi hemi-slant submanifold of a Kaehler manifold* N*. Then, we have*

$$(\bar{\nabla}_X \phi)Y = A_{\omega Y} X + Bh(X,Y),$$

$$(\bar{\nabla}_X \omega)Y = Ch(X,Y) - h(X, \phi Y)$$

for any $X, Y \in \Gamma(TM)$.

Proof. Using the equations (10) and (11) in the Lemma 3.5, Lemma 3.6 follows. □

4. Integrability of the distributions and decomposition theorems

In this section we investigate the conditions for the distributions to be integrable.

Theorem 4.1. *Let M be a proper quasi hemi-slant submanifold of a Kaehler manifold N. Then the slant distribution D_θ is integrable if and only if*

$$g(A_{\omega W}Z - A_{\omega Z}W, JPX) = g(A_{\omega\phi W}Z - A_{\omega\phi Z}W, X) + g(\nabla^\perp_Z \omega W - \nabla^\perp_W \omega Z, JRX)$$

for any $W, Z \in \Gamma(D_\theta)$ and $X \in \Gamma(D \oplus D^\perp)$.

Proof. Using (2), (6) and (17), we find

$$\begin{aligned} g([Z,W],X) &= g(\bar\nabla_Z \omega W, JX) - g(\bar\nabla_Z J\phi W, X) \\ &\quad - g(\bar\nabla_W \omega Z, JX) + g(\bar\nabla_W J\phi Z, X) \end{aligned}$$

for any $Z, W \in \Gamma(D_\theta)$ and $X = PX + RX \in \Gamma(D \oplus D^\perp)$. Then from (4), (5) and (17), we have

$$\begin{aligned} g([Z,W],X) &= -g(A_{\omega W}Z - A_{\omega Z}W, JX) + \cos^2\theta g([Z,W],X) \\ &\quad + g(A_{\omega\phi W}Z - A_{\omega\phi Z}W, X) + g(\nabla^\perp_Z \omega W - \nabla^\perp_W \omega Z, JX) \end{aligned}$$

which leads to

$$\begin{aligned} \sin^2\theta g([Z,W],X) = g(A_{\omega\phi W}Z - A_{\omega\phi Z}W, X) \\ + g(\nabla^\perp_Z \omega W - \nabla^\perp_W \omega Z, JRX) \\ - g(A_{\omega W}Z - A_{\omega Z}W, JPX). \end{aligned}$$

This completes the proof. □

From above theorem we have the following sufficient conditions for the slant distribution D_θ to be integrable:

Theorem 4.2. *Let M be a proper quasi hemi-slant submanifold of a Kaehler manifold N. If*

$$\begin{aligned} \nabla^\perp_Z \omega W - \nabla^\perp_W \omega Z &\in \omega D_\theta \oplus \mu, \\ A_{\omega\phi W}Z - A_{\omega\phi Z}W &\in D_\theta, \quad \text{and} \\ A_{\omega W}Z - A_{\omega Z}W &\in D^\perp \oplus D_\theta \end{aligned}$$

for any $Z, W \in \Gamma(D_\theta)$, then the slant distribution D_θ is integrable.

Proof. Theorem 4.2 follows from the equations (18) and (20). □

Theorem 4.3. *Let M be a quasi hemi-slant submanifold of a Kaehler manifold N. Then the anti-invariant distribution $D^\perp$ is integrable if and only if*

$$g(A_{JZ}W - A_{JW}Z, \phi X) = -g(\nabla^\perp_Z JW - \nabla^\perp_W JZ, \omega X)$$

for any $W, Z \in \Gamma(D^{\perp})$ *and* $X \in \Gamma(D \oplus D_{\theta})$.

Proof. For any $W, Z \in \Gamma(D^{\perp})$ and $X = PX + QX \in \Gamma(D \oplus D_{\theta})$, by making use of (2), (4), (5) and (17), we obtain

$$\begin{aligned} g([Z,W],X) &= g(\bar{\nabla}_Z\, JW, JX) - g(\bar{\nabla}_W\, JZ, JX) \\ &= g(A_{JZ}W - A_{JW}Z, \phi X) + g(\nabla^{\perp}_Z JW - \nabla^{\perp}_W JZ, \omega X). \end{aligned}$$

Hence the proof follows. □

Now for any Z, $W \in \Gamma(D)$ and $X = QX + RX \in \Gamma(D_{\theta} \oplus D^{\perp})$, by using (4) and (15), we have

$$\begin{aligned} g([Z,W],X) &= g(\bar{\nabla}_Z\, \phi W, JX) - g(\bar{\nabla}_W\, \phi Z, JX), \\ &= g(\nabla_Z\, \phi W - \nabla_W\, \phi Z, \phi QX) \\ &+ g(h(Z,\phi W) - h(W,\phi Z), \omega X). \end{aligned}$$

Hence, we have the following:

Theorem 4.4. *Let* M *be a proper quasi hemi-slant submanifold of a Kaehler manifold* N*. Then the invariant distribution* D *is integrable if and only if*

$$g(\nabla_Z \phi W - \nabla_W \phi Z, \phi QX) = g(h(W,\phi Z) - h(Z,\phi W), \omega X)$$

for any Z, $W \in \Gamma(D)$ *and* $X \in \Gamma(D_{\theta} \oplus D^{\perp})$.

Now, we obtain the necessary and sufficient condition for a quasi hemi-slant submanifold M to be totally geodesic.

Theorem 4.5. *Let* M *be a proper quasi-hemi-slant submanifold of a Kaehler manifold* N*. Then* M *is totally geodesic if and only if*

$$\begin{aligned} g(h(X,PY) + \cos^2\theta h(X,QY), U) =& g(\nabla^{\perp}_X \omega\phi QY, U) - g(\nabla^{\perp}_X \omega Y, CU) \\ &+ g(A_{\omega QY}X + A_{\omega RY}X, BU) \end{aligned}$$

for any $U \in \Gamma(TM)^{\perp}$ *and* $X, Y \in \Gamma(TM)$.

Proof. For any $X, Y \in \Gamma(TM)$, $U \in \Gamma(TM)^{\perp}$, we have

$$\begin{aligned} g(\bar{\nabla}_X Y, U) &= g(\bar{\nabla}_X PY, U) + g(\bar{\nabla}_X QY, U) + g(\bar{\nabla}_X RY, U) \\ &= g(\bar{\nabla}_X JPY, JU) + g(\bar{\nabla}_X \phi QY, JU) + g(\bar{\nabla}_X \omega QY, JU) \\ &\quad + g(\bar{\nabla}_X JRY, JU). \end{aligned}$$

By using (4), (5) and (13) in the above equation, we have

$$g(\bar{\nabla}_X Y, U) = g(\bar{\nabla}_X PY, U) - g(\bar{\nabla}_X \phi^2 QY, U) - g(\bar{\nabla}_X \omega\phi QY, U) + g(\bar{\nabla}_X \omega QY, JU) + g(\bar{\nabla}_X \omega RY, JU)$$
$$= g(h(X, PY), U) + \cos^2\theta g(h(X, QY), U) - g(\nabla^{\perp}_X \omega\phi QY, U) + g(-A_{\omega QY}X + \nabla^{\perp}_X \omega QY, JU) + g(-A_{\omega RY}X + \nabla^{\perp}_X \omega RY, JU)$$

as $\omega V = \omega PV + \omega QV + JRV$, $\omega PV = 0$, so we have

$$g(\bar{\nabla}_X Y, U) = g(h(X, PY) + \cos^2\theta h(X, QY), U) - g(\nabla^{\perp}_X \omega\phi QY, U) - g(A_{\omega QY}X + A_{\omega RY}X, BU) + g(\nabla^{\perp}_X \omega Y, CU). \tag{21}$$

Hence the proof follows from (21). □

Now we investigate the geometry of leaves of foliations determine by the above distributions.

Theorem 4.6. *Let M be a proper quasi hemi-slant submanifold of a Kaehler manifold N. Then the anti-invariant distribution $D^{\perp}$ defines a totally geodesic foliation on M if and only if*

$$g(h(X, Y), \omega\phi QZ) = g(\nabla^{\perp}_X \omega Y, \omega QZ) \tag{22}$$

and

$$g(A_{\omega Y}X, B\xi) = g(\nabla^{\perp}_X \omega Y, C\xi) \tag{23}$$

for any X, $Y \in \Gamma(D^{\perp})$, $Z \in \Gamma(D \oplus D_\theta)$ and $\xi \in \Gamma(TM)^{\perp}$.

Proof. For any X, $Y \in \Gamma(D^{\perp})$, $Z = PZ + QZ \in \Gamma(D \oplus D_\theta)$ and using (14), we have

$$\begin{aligned} g(\bar{\nabla}_X Y, Z) &= g(\bar{\nabla}_X JY, JZ) \\ &= g(\bar{\nabla}_X JY, JPZ) + g(\bar{\nabla}_X JY, \phi QZ) + g(\bar{\nabla}_X JY, \omega QZ) \\ &= g(\bar{\nabla}_X Y, PZ) - g(\bar{\nabla}_X Y, \phi^2 QZ) - g(\bar{\nabla}_X Y, \omega\phi QZ) \\ &\quad + g(\bar{\nabla}_X \omega Y, \omega QZ). \end{aligned} \tag{24}$$

As $g(\bar{\nabla}_X Y, Z) = g(\bar{\nabla}_X Y, PZ) + g(\bar{\nabla}_X Y, QZ)$, in view of (4) and (5) and (12) the equation (24) reduces to

$$g(\bar{\nabla}_X Y, \sin^2\theta QZ) = -g(h(X, Y), \omega\phi QZ) + g(\nabla^{\perp}_X \omega Y, \omega QZ)$$

which gives (22). Now for any X, $Y \in \Gamma(D^{\perp})$, $\xi \in \Gamma(TM)^{\perp}$, and using (5) and (9) we have

$$\begin{aligned} g(\bar{\nabla}_X Y, \xi) &= g(\bar{\nabla}_X JY, J\xi) = g(\bar{\nabla}_X \omega Y, B\xi) + g(\bar{\nabla}_X \omega Y, C\xi) \\ &= -g(A_{\omega Y}X, B\xi) + g(\nabla^{\perp}_X \omega Y, C\xi) \end{aligned}$$

which gives (23). □

Theorem 4.7. *Let M be a proper quasi hemi-slant submanifold of a Kaehler manifold N. Then the slant distribution D_θ defines a totally geodesic foliation on M if and only if*

$$g(\nabla^\perp_X \omega Y, \omega RZ) = g(A_{\omega Y}X, \phi PZ) - g(A_{\omega\phi Y}X, Z), \quad \text{and} \tag{25}$$

$$g(A_{\omega Y}X, B\xi) = g(\nabla^\perp_X \omega Y, C\xi) - g(\nabla^\perp_X \omega\phi Y, \xi) \tag{26}$$

for any $X, Y \in \Gamma(D_\theta)$, $Z \in \Gamma(D \oplus D^\perp)$ and $\xi \in \Gamma(TM)^\perp$.

Proof. For any $X, Y \in \Gamma(D_\theta)$, $Z = PZ + RZ \in \Gamma(D \oplus D^\perp)$ by using (14), we have

$$\begin{aligned} g(\bar\nabla_X Y, Z) &= g(\bar\nabla_X JY, JZ) = g(\bar\nabla_X \phi Y, JZ) + g(\bar\nabla_X \omega Y, JZ) \\ &= -g(\bar\nabla_X \phi^2 Y, Z) - g(\bar\nabla_X \omega\phi Y, Z) + g(\bar\nabla_X \omega Y, \phi PZ + \omega RZ). \end{aligned}$$

Then using (14), (5) and the fact that $\omega PZ = 0$, we have

$$\begin{aligned} g(\bar\nabla_X Y, Z) &= \cos^2\theta g(\bar\nabla_X Y, Z) + g(A_{\omega\phi Y}X, Z) \\ &\quad - g(A_{\omega Y}X, \phi PZ) + g(\nabla^\perp_X \omega Y, \omega RZ). \end{aligned}$$

This implies

$$\begin{aligned} \sin^2\theta g(\bar\nabla_X Y, Z) &= g(A_{\omega\phi Y}X, Z) \\ &\quad - g(A_{\omega Y}X, \phi PZ) + g(\nabla^\perp_X \omega Y, \omega RZ). \end{aligned} \tag{27}$$

Similarly, we get

$$\sin^2\theta g(\bar\nabla_X Y, \xi) = -g(\nabla^\perp_X \omega\phi Y, \xi) - g(A_{\omega Y}X, B\xi) + g(\nabla^\perp_X \omega Y, C\xi). \tag{28}$$

Thus from (27) and (28), we have the assertions. □

Theorem 4.8. *Let M be a proper quasi hemi-slant submanifold of a Kaehler manifold N. Then the invariant distribution D defines a totally geodesic foliation on M if and only if*

$$g(\nabla_X \phi Y, \phi QZ) = -g(h(X, \phi Y), \omega Z), \tag{29}$$

and

$$g(\nabla_X \phi Y, B\xi) = -g(h(X, \phi Y), C\xi) \tag{30}$$

for any $X, Y \in \Gamma(D), Z \in \Gamma(D_\theta \oplus D^\perp)$ and $\xi \in \Gamma(TM)^\perp$.

Proof. For any $X, Y \in \Gamma(D), Z = QZ + RZ \in \Gamma(D_\theta \oplus D^\perp)$ and using $\omega Y = 0$, we have

$$\begin{aligned} g(\bar\nabla_X Y, Z) &= g(\bar\nabla_X \phi Y, JZ), \\ &= g(\nabla_X \phi Y, \phi QZ) + g(h(X, \phi Y), \omega Z). \end{aligned}$$

Now for any $\xi \in \Gamma(TM)^{\perp}$ and $X, Y \in \Gamma(D)$, we have

$$\begin{aligned} g(\bar{\nabla}_X Y, \xi) &= g(\bar{\nabla}_X \phi Y, J\xi) \\ &= g(\nabla_X \phi Y, B\xi) + g(h(X, \phi Y), C\xi). \end{aligned}$$

Hence the proof. □

In view of last three theorems, we have the following decomposition theorem:

Theorem 4.9. *Let M be a proper quasi hemi-slant submanifold of a Kaehler manifold N. Then M is a local product Riemannian manifold of the form $M_D \times M_{D_\theta} \times M_{D^\perp}$, where M_D, M_{D_θ} and $M_{D^\perp}$ are the leaves of D, D_θ and $D^\perp$ respectively, if and only if the conditions* (22), (23) (25) , (26), (29) *and* (30) *hold.*

5. Example

Example 5.1. *Consider $\mathbb{R}^{2n}$ with standard coordinates $(x_1, x_2, x_3, x_4, ..., x_{2n-1}, x_{2n})$. We can canonically choose an almost complex structure J on $\mathbb{R}^{2n}$ as follows :*

$$\begin{aligned} & J(a_1\frac{\partial}{\partial x_1}+a_2\frac{\partial}{\partial x_2}+a_3\frac{\partial}{\partial x_3}+a_4\frac{\partial}{\partial x_4}+\cdots+a_{2n-1}\frac{\partial}{\partial x_{2n-1}}+a_{2n}\frac{\partial}{\partial x_{2n}}) \\ = & (a_1\frac{\partial}{\partial x_2}-a_2\frac{\partial}{\partial x_1}+a_3\frac{\partial}{\partial x_4}-a_4\frac{\partial}{\partial x_3}+\cdots+a_{2n-1}\frac{\partial}{\partial x_{2n}}-a_{2n}\frac{\partial}{\partial x_{2n-1}}), \end{aligned}$$

where $a_1, a_2, a_3, ...a_{2n}$ are C^∞ functions defined on $\mathbb{R}^{2n}$. Consider a submanifold M of $\mathbb{R}^{14}$ defined by

$$\begin{aligned} & f(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \\ = & (\frac{x_1+\sqrt{3}x_2}{2}, \frac{x_1-\sqrt{3}x_2}{2}, x_3\cos\theta, x_4, x_3\sin\theta, 0, x_5\cos\theta, \\ & 0, -x_5\sin\theta, 0, \frac{\sqrt{5}x_7+2x_8}{2}, \frac{\sqrt{5}x_7-2x_8}{2}, x_6, 0), \end{aligned}$$

where θ is a constant.

By the direct computation, it is easy to check that the tangent bundle of M is spanned by the set $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\}$, where

$$Z_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, \qquad Z_2 = \sqrt{3}(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}),$$
$$Z_3 = \cos\theta\frac{\partial}{\partial x_3} + \sin\theta\frac{\partial}{\partial x_5}, \qquad Z_4 = \frac{\partial}{\partial x_4},$$
$$Z_5 = \cos\theta\frac{\partial}{\partial x_7} - \sin\theta\frac{\partial}{\partial x_9}, \qquad Z_6 = \frac{\partial}{\partial x_{13}},$$
$$Z_7 = \frac{\sqrt{5}}{2}(\frac{\partial}{\partial x_{11}} + \frac{\partial}{\partial x_{12}}), \qquad Z_8 = \frac{\partial}{\partial x_{11}} - \frac{\partial}{\partial x_{12}}.$$

Then, by using the canonical complex structure of $\mathbb{R}^{14}$, we have

$$JZ_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}, \qquad JZ_2 = -\sqrt{3}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}),$$
$$JZ_3 = -\cos\theta\frac{\partial}{\partial x_4} - \sin\theta\frac{\partial}{\partial x_6}, \qquad JZ_4 = \frac{\partial}{\partial x_3},$$
$$JZ_5 = -\cos\theta\frac{\partial}{\partial x_8} + \sin\theta\frac{\partial}{\partial x_{10}}, \qquad JZ_6 = -\frac{\partial}{\partial x_{14}},$$
$$JZ_7 = \frac{\sqrt{5}}{2}(\frac{\partial}{\partial x_{11}} - \frac{\partial}{\partial x_{12}}), \qquad JZ_8 = -(\frac{\partial}{\partial x_{11}} + \frac{\partial}{\partial x_{12}}).$$

Now, let the distributions defined by $D = Span\{Z_1, Z_2, Z_7, Z_8\}, D_\theta = Span\{Z_3, Z_4\}$ and $D^\perp = Span\{Z_5, Z_6\}$. Then one can easily see that D is invariant, D_θ is slant with slant angle θ and $D^\perp$ is anti-invariant distributions.

Example 5.2. *Consider a $12-$dimensional differentiable manifold* $\overline{M} = R^{12}$

$$\overline{M} = \{(x_i, y_i) = (x_1, x_2, ..., x_6, y_1, y_2, ..., y_6) \in \mathbb{R}^{12}; \quad i = 1, 2, ..., 6\}.$$

We choose the vector fields

$$E_i = \frac{\partial}{\partial y_i}, \quad E_{6+i} = \frac{\partial}{\partial x_i}, \quad \textit{for} \;\; i = 1, 2, ..., 6.$$

Let g be a Hermitian metric defined by

$$g = (dx_1)^2 + (dx_2)^2 + ... + (dx_6)^2 + (dy_1)^2 + (dy_2)^2 + ... + (dy_6)^2.$$

Here $\{E_1, E_2, ..., E_{12}\}$ forms an orthonormal basis. We define $(1,1)$-tensor field J as

$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \quad J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}, \quad \forall\; i, j = 1, 2, ..., 6.$$

By using linearity of J and g, we have

$$J^2 = -I,$$

$$g(JX, JY) = g(X, Y), \quad \text{for any } X, Y \in \Gamma(T\overline{M}).$$

We can easily show that $(\overline{M}, J, g)$ is a Kaehler manifold of dimension 12. Now, we consider a submanifold M of $\overline{M}$ defined by immersion f as follows:

$$f\,(u, v, w, r, s, t) = (u, w, 0, s, 0, 0, v, r\cos\theta, r\sin\theta, 0, s, t)$$

where θ is a constant.

By direct computation, it is easy to check that the tangent bundle of M is spanned by the set $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}$, where

$$Z_1 = \frac{\partial}{\partial x_1}, \quad Z_2 = \frac{\partial}{\partial y_1}, \quad Z_3 = \frac{\partial}{\partial x_2},$$

$$Z_4 = \cos\theta\frac{\partial}{\partial y_2} + \sin\theta\frac{\partial}{\partial y_3}, \quad Z_5 = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial y_5}, \quad Z_6 = \frac{\partial}{\partial y_6}.$$

Then using almost complex structure of $\overline{M}$, we have

$$JZ_1 = \frac{\partial}{\partial y_1}, \quad JZ_2 = -\frac{\partial}{\partial x_1}, \quad JZ_3 = \frac{\partial}{\partial y_2},$$

$$JZ_4 = -\left(\cos\theta\frac{\partial}{\partial x_2} + \sin\theta\frac{\partial}{\partial x_3}\right), \quad JZ_5 = \frac{\partial}{\partial y_4} - \frac{\partial}{\partial x_5}, \quad JZ_6 = -\frac{\partial}{\partial x_6}.$$

Now, let the distributions $D = Span\{Z_1, Z_2\}$, $D_\theta = Span\{Z_3, Z_4\}$ and $D^\perp = Span\{Z_5, Z_6\}$. Then it is easy to see that D is invariant, D_θ is slant with slant angle θ and $D^\perp$ is anti-invariant distributions. Hence f defines a 6-dimensional quasi hemi-slant submanifold M of Kaehler manifold $\overline{M}$.

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