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COEFFICIENT BOUNDS FOR INVERSE OF FUNCTIONS CONVEX IN ONE DIRECTION

Sudhananda Maharana, Jugal Kishore Prajapat, and Deepak Bansal*

Abstract. In this article, we investigate the upper bounds on the coefficients for inverse of functions belongs to certain classes of univalent functions and in particular for the functions convex in one direction. Bounds on the Fekete-Szegö functional and third order Hankel determinant for these classes have also investigated.

1. Introduction and Preliminaries

Let $\mathcal{H}$ denote the family of all analytic functions in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ denote the class of functions $f \in \mathcal{H}$, having the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \quad z \in \mathbb{D}. \tag{1}$$

We denote by $\mathcal{S}$, the class of univalent functions in $\mathcal{A}$.

It is well-known that the function $f \in \mathcal{S}$ of the form (1) has an inverse f^{-1}, which is analytic in $|w| < r_0(f)$ $(r_0(f) \geq 1/4)$. If $f \in \mathcal{S}$ given by (1), then

$$f^{-1}(w) = w + \gamma_2 w^2 + \gamma_3 w^3 + \cdots, \quad |w| < r_0(f). \tag{2}$$

Löwner [21] proved that, if $f \in \mathcal{S}$ and its inverse is given by (2), then the sharp estimate

$$|\gamma_n| \leq \frac{\Gamma(2n+1)}{\Gamma(n+1)\Gamma(n+2)} \tag{3}$$

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*Corresponding author

holds. It has been shown that the inverse of the Koebe function $k(z) = z/(1-z)^2$ provides the best bounds for all $|\gamma_n|$ $(n = 2, 3, \cdots)$ in (3) over all members of $\mathcal{S}$.

Using (1) and (2) with

$$w = f^{-1}(w) + a_2(f^{-1}(w))^2 + a_3(f^{-1}(w))^3 + \cdots,$$

Libera *et al.* [17] (see also [18, 19]) obtained the following relationship between the coefficients of f and f^{-1} for all f of the form (1):

$$(4) \quad \begin{cases} \gamma_2 + a_2 = 0, \quad \gamma_3 + 2a_2\gamma_2 + a_3 = 0, \\ \gamma_4 + a_2(\gamma_2^2 + 2\gamma_3) + 3a_3\gamma_2 + a_4 = 0, \\ \text{and} \\ \gamma_5 + a_2(2\gamma_4 + 2\gamma_2\gamma_3) + a_3(3\gamma_3 + 3\gamma_2^2) + 4a_4\gamma_2 + a_5 = 0. \end{cases}$$

On the other hand, Krzyz *et al.* [15] investigated bounds on initial coefficients of inverse of starlike functions and their results were extended by Kapoor and Mishra [12]. Further, Ali [1] studied sharp bounds on early coefficients of inverse functions and corresponding Fekete-Szegö functional, when function belongs to the class of strongly starlike functions.

Umezawa [30, Theorem 1] studied that, if function f of the form (1) be meromorphic in $\mathbb{D}$ and satisfying the relation

$$(5) \qquad \alpha > \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{\alpha}{2\alpha - 3},$$

where α is an arbitrary number not less than $3/2$, then f is analytic and univalent in $\mathbb{D}$. Moreover, $f(z)$ maps $|z| = r$ for every $r < 1$ into a curve which is convex in one direction, and $|a_n| \leq n$ for all n. We recall that a domain $D \subset \mathbb{C}$ is called convex in the direction φ $(0 \leq \varphi < \pi)$, if every line parallel to the line through 0 and $e^{i\varphi}$ has a connected or empty intersection with D. A function $f \in \mathcal{S}$ is said to be convex in the direction φ, if $f(\mathbb{D})$ is convex in the direction φ.

Several special cases of inequality (5) can be drawn by allowing different values of $\alpha \geq 3/2$. In particular when $\alpha \to 3/2$ in (5), the function $f \in \mathcal{A}$ satisfying the analytic condition

$$(6) \qquad \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{3}{2},$$

is convex in one direction in $\mathbb{D}$. Let the class of all functions $f \in \mathcal{A}$ satisfying (6) be denoted by $\mathcal{G}$ and $\mathcal{G}^{-1}$ be the class of inverse function f^{-1} of functions $f \in \mathcal{G}$. Ozaki [25] studied the class $\mathcal{G}$ and proved that functions in $\mathcal{G}$ are univalent in $\mathbb{D}$. Singh and Singh [29, Theorem 6] proved that the functions in $\mathcal{G}$ are starlike in $\mathbb{D}$.

Furthermore, when $\alpha \to \infty$ in (5), the function $f \in \mathcal{A}$ satisfying the analytic condition

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2}, \tag{7}$$

is convex in one direction in $\mathbb{D}$. Let the class of all functions $f \in \mathcal{A}$ satisfying (7) be denoted by $\mathcal{F}$ and $\mathcal{F}^{-1}$ be the class of inverse function f^{-1} of functions $f \in \mathcal{F}$. Note that the inequality (7) is a consequence of Kaplan characterization [6, p. 48, Theorem 2.18], therefore functions in $\mathcal{F}$ are also close-to-convex (hence univalent) in $\mathbb{D}$. Recently Ponnusamy *et al.* [27] investigated the radius of convexity of partial sums of functions $f \in \mathcal{F}$.

The Hankel determinant of Taylor coefficients of functions $f \in \mathcal{A}$ of the form (1), is denoted by $H_{q,n}(f)$, and is defined by

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}$$

$$(a_1 = 1;\ n, q \in \mathbb{N} = \{1, 2, \cdots\}).$$

Several researchers including Noonan and Thomas [23], Pommerenke [26], Hayman [10], Ehrenborg [7], Noor [24] have studied the Hankel determinant and given some remarkable results, which are useful, for example, in showing that a function of bounded characteristic in $\mathbb{D}$.

Indeed, $H_{2,1}(f) = \Lambda_1(f)$ is the *Fekete-Szegö functional*, which have been studied for various subclasses of $\mathcal{S}$ (see e.g. [2, 8, 13, 14, 20]). Recently many authors have studied the problem of calculating the upper bounds of $|H_{2,2}(f)|$ for various subclasses of $\mathcal{A}$ (see e.g. [3, 11, 16]). The third Hankel determinant $H_{3,1}(f)$ is given by

$$H_{3,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).$$

Recently, the authors have obtained bounds on $|H_{3,1}(f)|$ for certain classes of analytic functions (see e.g. [4, 22]). Also, Raza and Malik [28] have obtained the bounds on $|H_{3,1}(f)|$ for a subclasses of analytic functions associated with right half of the lemniscate of Bernoulli $(x^2 + y^2)^2 - 2(x^2 - y^2) = 0$.

In this paper, we examine the upper bounds on the coefficients $|\gamma_i|$ $(i = 2, 3, 4, 5)$, the upper bounds on $|H_{2,1}(f)|$ and $|H_{3,1}(f)|$ for the inverse functions f^{-1} of the form (2), when f belongs to the function classes $\mathcal{G}$ and $\mathcal{F}$, respectively. In order to obtain our main results, we need the following known results for the class $\mathcal{P}$ of *Carathéodory functions* [6, p.40] that consists of functions $p \in \mathcal{H}$ with $\Re(p(z)) > 0$, having the form

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots, \quad z \in \mathbb{D}. \tag{8}$$

For more details about class $\mathcal{P}$ one can refer the survey article [5].

Lemma 1.1. [6, 13, 18] *Let the function $p \in \mathcal{P}$ be given by (8). Then*

- **(a)** $|c_n| \leq 2,\ n \in \mathbb{N} := \{1, 2, \cdots\}$. *This inequality is sharp and equality holds for every function* $p_\epsilon(z) = \dfrac{1 + \epsilon z}{1 - \epsilon z} \quad (z \in \mathbb{D},\ |\epsilon| = 1)$.
- **(b)** $\max |c_2 - \lambda c_1^2| = 2 \max\{1, |2\lambda - 1|\}$, *for any complex number* λ.

Lemma 1.2. [9] *Let the function $p \in \mathcal{P}$ be given by (8). Then*

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{9}$$

and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z, \tag{10}$$

for some x and z, such that $|x| \leq 1$ and $|z| \leq 1$.

Lemma 1.3. *If $p(z) \in \mathcal{P}$ be given by (8) and* $1/p(z) = 1 + \sum_{n=1}^{\infty} c_n^* z^n$, *then*

$$c_1^* = -c_1, \qquad c_2^* = c_1^2 - c_2,$$
$$c_3^* = 2c_1c_2 - c_3 - c_1^3, \qquad c_4^* = c_1^4 + c_2^2 + 2c_1c_3 - 3c_1^2c_2 - c_4$$

and $|c_n^*| \leq 2$, for all $n \in \mathbb{N}$.

Note that Lemma 1.3 is a revised form of a known result [18, Lemma 1] and its last statement follows from the observation that both the p and its reciprocal are in $\mathcal{P}$.

2. Main Results

Theorem 2.1. *Let $f \in \mathcal{G}$ be of the form* (1) *and its inverse f^{-1} be given by* (2)*. Then*

$$|\gamma_2| \leq \frac{1}{2}, \quad |\gamma_3| \leq \frac{1}{2}, \quad |\gamma_4| \leq \frac{5}{8}, \quad \textit{and} \quad |\gamma_5| \leq \frac{7}{8}. \tag{1}$$

The equalities in (1) *hold for the inverse of function $f_0(z) = z - z^2/2$.*

Proof. Let $g(z) = zf'(z)$ be given by

$$g(z) = z + b_2 z^2 + b_3 z^3 + \cdots + b_n z^n + \cdots . \tag{2}$$

Then $b_n = na_n$, for $n \geq 2$. If $f \in \mathcal{G}$, then clearly it follows that

$$\frac{3}{2} - \frac{zg'(z)}{g(z)} = \frac{1}{2}p(z), \tag{3}$$

where $p \in \mathcal{P}$. Substituting (8) and (2) in (3), we obtain

$$\begin{aligned} &a_2 = -\tfrac{1}{4}c_1,\ a_3 = \tfrac{1}{24}(c_1^2 - 2c_2),\ a_4 = \tfrac{1}{192}(6c_1c_2 - c_1^3 - 8c_3) \\ &\text{and} \\ &a_5 = \tfrac{1}{1920}(32c_1c_3 + c_1^4 + 12c_2^2 - 48c_4 - 12c_1^2c_2). \end{aligned} \tag{4}$$

By using (4) in (4), we estimate

$$\begin{aligned} &\gamma_2 = \tfrac{1}{4}c_1,\ \gamma_3 = \tfrac{1}{12}(c_1^2 + c_2),\ \gamma_4 = \tfrac{1}{96}(4c_3 + 3c_1^3 + 7c_1c_2) \\ &\text{and} \\ &\gamma_5 = \tfrac{1}{960}(12c_1^4 + 46c_1^2c_2 + 44c_1c_3 + 14c_2^2 + 24c_4). \end{aligned} \tag{5}$$

Now by using Lemma 1.1, and triangle inequality in (5), we get the desired result (1). Finally for equality in (1), the function $f_0(z) = z - z^2/2$ is given by

$$\begin{aligned} w &= f_0(z) = f_0(f_0^{-1}(w)) \\ &= f_0^{-1}(w) - \frac{1}{2}(f_0^{-1}(w))^2 \\ &= w + (\gamma_2 - \frac{1}{2})w^2 + (\gamma_3 - \gamma_2)w^3 \\ &\qquad + (\gamma_4 - \gamma_3 - \frac{1}{2}\gamma_2^2)w^4 + (\gamma_5 - \gamma_4 - \gamma_2\gamma_3)\,w^5 + \cdots . \end{aligned}$$

By equating the coefficients, it gives the equalities in (1), and this completes the proof of the theorem. □

Theorem 2.2. *Let $f \in \mathcal{G}$ be of the form* (1) *and its inverse f^{-1} be given by* (2). *Then for any complex number μ, we have*

$$|\gamma_3 - \mu\gamma_2^2| \leq \frac{1}{6}\, max\left\{1, \frac{3|\mu-2|}{2}\right\}. \tag{6}$$

The equality in (6) *is attained for the inverse of function $f_0(z) = z - z^2/2$.*

Proof. By using (5), we get

$$|\gamma_3 - \mu\gamma_2^2| = \frac{1}{12}\left|c_2 - \frac{3\mu-4}{4}c_1^2\right|.$$

The result now follows from the application of Lemma 1.1. □

If we take $\mu = 1$ in Theorem 2.2, we obtain the following result.

Corollary 2.3. *Let $f \in \mathcal{G}$ be of the form* (1) *and its inverse f^{-1} be given by* (2). *Then*

$$|\gamma_3 - \gamma_2^2| \leq \frac{1}{4},$$

and the equality is attained for the inverse of function $f_0(z) = z - z^2/2$.

Theorem 2.4. *Let $f \in \mathcal{G}$ be of the form* (1) *and its inverse f^{-1} be given by* (2). *Then*

$$|\gamma_2\gamma_3 - \gamma_4| \leq \frac{3}{8} \quad \text{and} \quad |\gamma_2\gamma_4 - \gamma_3^2| \leq \frac{1}{16}. \tag{7}$$

The equalities in (7) *is attained by the inverse of function $f_0(z) = z - z^2/2$.*

Proof. If $f \in \mathcal{G}$ be of the form (1) and its inverse f^{-1} be given by (2), then the coefficients of f^{-1} are given by (5). Using these coefficients, we estimate

$$\begin{aligned} &|\gamma_2\gamma_3 - \gamma_4| = \frac{1}{96}\left|c_1^3 + 5c_1c_2 + 4c_3\right| \\ &\text{and} \\ &|\gamma_2\gamma_4 - \gamma_3^2| = \frac{1}{1152}\left|c_1^4 + 5c_1^2c_2 - 8c_2^2 + 12c_1c_3\right|. \end{aligned} \tag{8}$$

By using Lemma 1.2 and (8), we obtain

$$\begin{aligned} &|\gamma_2\gamma_3 - \gamma_4| = \frac{1}{192}\left|9c_1^3 + (4-c_1^2)\{9c_1x - 2c_1x^2 + 4(1-|x|^2)z\}\right| \\ &\text{and} \\ &|\gamma_2\gamma_4 - \gamma_3^2| = \frac{1}{2304}\left|9c_1^4 + (4-c_1^2)\{9c_1^2x - 4x^2(4-c_1^2) \right. \\ &\qquad\qquad\qquad\qquad \left. -6c_1^2x^2 + 12c_1(1-|x|^2)z\}\right|. \end{aligned} \tag{9}$$

As per Lemma 1.1, it is clear that $|c_1| \leq 2$. Therefore, letting $c_1 = c$, we may assume without restriction that $c \in [0,\, 2]$. Hence, applying triangle inequality with $\mu = |x|$, we obtain

$$|\gamma_2\gamma_3 - \gamma_4| \leq \frac{1}{192}\left[9c^3 + (4-c^2)\{9c\mu + 2c\mu^2 + 4(1-\mu^2)\}\right] := C(c,\mu)$$

and

$$|\gamma_2\gamma_4 - \gamma_3^2| \leq \frac{1}{2304}\left[9c^4 + (4-c^2)\{9c^2\mu + 16\mu^2 + 2c^2\mu^2 + 12c(1-\mu^2)\}\right] := D(c,\mu).$$

Now to prove our results, we need to maximize the values of C and D over the region $\Omega = \{(c,\mu) : 0 \leq c \leq 2, 0 \leq \mu \leq 1\}$. For this, first differentiating C with respect to μ and c, we obtain

$$\frac{\partial C}{\partial \mu} = \frac{1}{192}\left[(4-c^2)(9c + 4c\mu - 8\mu)\right]$$

and

$$\frac{\partial C}{\partial c} = \frac{1}{192}\left[(27 - 27\mu - 6\mu^2)c^2 + 8(\mu^2 - 1)c + 36\mu + 8\mu^2\right].$$

A critical point of $C(c,\mu)$ must satisfy $\frac{\partial C}{\partial \mu} = 0$ and $\frac{\partial C}{\partial c} = 0$. The condition $\frac{\partial C}{\partial \mu} = 0$ gives $c = \pm 2$ or $\mu = \frac{9c}{4(2-c)}$. Points (c,μ) satisfying such conditions are not interior point of Ω. So the maximum cannot be attained in the interior of Ω. Now to see on the boundary, first taking the boundary line $L_1 = \{(0,\mu) : 0 \leq \mu \leq 1\}$, we have $C(0,\mu) = (1-\mu^2)/12$, and its maximum on this line is equal to $1/12$, which is attained at the point $(0,0)$. On the boundary line $L_2 = \{(2,\mu) : 0 \leq \mu \leq 1\}$, we have $C(2,\mu) = 3/8$, which is a constant. On the boundary line $L_3 = \{(c,0) : 0 \leq c \leq 2\}$, we have $C(c,0) = (9c^3 - 4c^2 + 16)/192$, and its maximum on this line is equal to $3/8$, which is attained at the point $(2,0)$. On the line $L_4 = \{(c,1) : 0 \leq c \leq 2\}$, we have $C(c,1) = (22c - c^3)/96$, and the maximum on this line is $3/8$, which is attained at the point $(2,1)$. Comparing these results, we get

$$\max_{\Omega} C(c,\mu) = C(2,\mu) = 3/8.$$

Further, differentiating D with respect to μ and c, we obtain

$$\frac{\partial D}{\partial \mu} = \frac{1}{2304}\left[(4-c^2)(9c^2 + 4c^2\mu + 32\mu - 24c\mu)\right]$$

and

$$\frac{\partial D}{\partial c} = \frac{1}{2304}\left[36c^3 - 8c^3\mu^2 - 36c^3\mu - 36c^2 \right.$$
$$\left. +36c^2\mu^2 + 72c\mu - 16c\mu^2 - 48\mu^2 + 48\right].$$

The condition $\frac{\partial D}{\partial \mu} = 0$ gives $c = \pm 2$ or $\mu = -\frac{9c^2}{4(8-6c+c^2)}$ in Ω. Points (c, μ) satisfying such conditions are not interior point of Ω. So the maximum cannot attain in the interior of Ω. Now to see on the boundary, taking the boundary line $L_1 = \{(0, \mu) : 0 \leq \mu \leq 1\}$, we have $D(0, \mu) = \mu^2/36$, and its maximum on this line is equal to $1/36$, which is attained at the point $(0, 1)$. On the boundary line $L_2 = \{(2, \mu) : 0 \leq \mu \leq 1\}$, we have $D(2, \mu) = 1/16$, which is a constant. On the boundary line $L_3 = \{(c, 0) : 0 \leq c \leq 2\}$, we have $D(c, 0) = (9c^4 - 12c^3 + 48c)/2304$, and its maximum on this line is equal to $1/16$, which is attained at the point $(2, 0)$. On the line $L_4 = \{(c, 1) : 0 \leq c \leq 2\}$, we have $D(c, 1) = (-2c^4 + 28c^2 + 64)/2304$, and its maximum on this line is $1/16$, which is attained at the point $(2, 1)$. Comparing these results, we get

$$\max_{\Omega} D(c, \mu) = D(2, \mu) = 1/16.$$

This completes the proof of Theorem 2.4. □

Theorem 2.5. *Let $f \in \mathcal{G}$ be of the form* (1) *and its inverse f^{-1} be given by* (2)*. Then*

$$|H_{3,1}(f^{-1})| \leq \frac{31}{64}. \tag{10}$$

The equality in (10) *is attained by the inverse of function $f_0(z) = z - z^2/2$.*

Proof. By using Theorem 2.1, Corollary 2.3, Theorem 2.4 and the triangle inequality, we get

$$|H_{3,1}(f^{-1})| \leq |\gamma_3||\gamma_2\gamma_4 - \gamma_3^2| + |\gamma_4||\gamma_2\gamma_3 - \gamma_4| + |\gamma_5||\gamma_3 - \gamma_2^2| = \frac{31}{64}.$$

This completes the proof of Theorem 2.5. □

Theorem 2.6. *Let $f \in \mathcal{F}$ be of the form* (1) *and its inverse f^{-1} be given by* (2)*. Then*

$$|\gamma_2| \leq \frac{3}{2}, \quad |\gamma_3| \leq \frac{5}{2}, \quad |\gamma_4| \leq \frac{35}{8}, \quad \textit{and} \quad |\gamma_5| \leq \frac{63}{8}. \tag{11}$$

The equalities in (11) *hold for the inverse of function* $f_1(z) = \dfrac{z - z^2/2}{(1-z)^2}$ *and its rotation.*

Proof. Let $g(z) = zf'(z)$, where $f \in \mathcal{F}$. Then it is clear that

$$\frac{zg'(z)}{g(z)} = \frac{3}{2}p(z) - \frac{1}{2}, \tag{12}$$

where $p \in \mathcal{P}$, and $g(z)$ is given by (2). Substituting (8) and (2) in (12), we get

$$\begin{aligned} &a_2 = \tfrac{3}{4}c_1,\ a_3 = \tfrac{1}{8}(3c_1^2 + 2c_2),\ a_4 = \tfrac{1}{64}(9c_1^3 + 18c_1c_2 + 8c_3) \\ &\text{and} \\ &a_5 = \tfrac{3}{640}\left(16c_4 + 32c_1c_3 + 36c_1^2c_2 + 12c_2^2 + 9c_1^4\right). \end{aligned} \tag{13}$$

Again, by using (4) in (13), we obtain

$$\begin{aligned} &\gamma_2 = -\tfrac{3}{4}c_1,\ \gamma_3 = \tfrac{1}{4}(3c_1^2 - c_2),\ \gamma_4 = -\tfrac{1}{32}(27c_1^3 + 4c_3 - 21c_1c_2) \\ &\text{and} \\ &\gamma_5 = -\tfrac{3}{160}\left(4c_4 + 69c_1^2c_2 - 22c_1c_3 - 7c_2^2 - 54c_1^4\right). \end{aligned} \tag{14}$$

By Lemma 1.1, it is clear that $|\gamma_2| \le 3/2$. Using Lemma 1.1, we can get the bounds on the remaining coefficients, but these bounds can be improved again by using Lemma 1.3. Hence, by using Lemma 1.3 and Lemma 1.1, we can see easily that

$$\begin{aligned} &|\gamma_3| \le \tfrac{1}{4}\left(|c_2^*| + 2|c_1|^2\right) \le 5/2 \\ &|\gamma_4| \le \tfrac{1}{32}\left(4|c_3^*| + 13|c_2^*||c_1| + 10|c_1|^3\right) \le 35/8 \\ &\text{and} \\ &|\gamma_5| \le \tfrac{3}{160}\left[4|c_4^*| + 14|c_1||c_3^*| + 36|c_1|^2|c_2^*| + 7|c_2||c_1^2| + 3|c_2|^2\right] \le 63/8. \end{aligned}$$

To show the equalities in (11), we consider the inverse of $f_1(z)$. For this, we may write

$$f_1(z) = \frac{z - z^2/2}{(1-z)^2} = \frac{1}{2}\left(\frac{z}{1-z} + \frac{z}{(1-z)^2}\right) = z + \sum_{n=2}^{\infty} \frac{n+1}{2} z^n.$$

Now by using (4), we find that

$$\gamma_2 = -a_2 = -3/2,\ \gamma_3 = 2a_2^2 - a_3 = 5/2,\ \gamma_4 = 5a_2a_3 - 5a_2^3 - a_4 = -35/8$$

and

$$\gamma_5 = 14a_2^4 - 21a_2^2a_3 + 6a_2a_4 + 3a_3^2 - a_5 = 63/8.$$

These values of γ_i $(i = 2, 3, 4, 5)$ showing the equalities in (11). This completes the proof of Theorem 2.6. □

Theorem 2.7. *Let $f \in \mathcal{F}$ be of the form* (1) *and its inverse f^{-1} be given by* (2)*. Then for any complex number μ, we have*

$$|\gamma_3 - \mu\gamma_2^2| \leq \frac{1}{2}\, max\left\{1, \frac{|10-9\mu|}{2}\right\}. \tag{15}$$

The equality in (15) *is attained by the inverse function of $f_1(z) = \dfrac{z - z^2/2}{(1-z)^2}$.*

Proof. By using (14), we get

$$|\gamma_3 - \mu\gamma_2^2| = \frac{1}{4}\left|c_2 - \frac{12-9\mu}{4}c_1^2\right|.$$

The result now follows from the application of Lemma 1.1. □

If we take $\mu = 1$ in Theorem 2.7, we obtain the following result.

Corollary 2.8. *Let $f \in \mathcal{F}$ be of the form* (1) *and its inverse f^{-1} be given by* (2)*. Then*

$$|\gamma_3 - \gamma_2^2| \leq \frac{1}{2}.$$

Theorem 2.9. *Let $f \in \mathcal{F}$ be of the form* (1) *and its inverse f^{-1} be given by* (2)*. Then*

$$|\gamma_2\gamma_3 - \gamma_4| \leq \frac{13\sqrt{78}}{144} \quad \text{and} \quad |\gamma_2\gamma_4 - \gamma_3^2| \leq \frac{209}{512}. \tag{16}$$

Proof. By using (14), we obtain

$$\begin{aligned} &|\gamma_2\gamma_3 - \gamma_4| = \frac{1}{32}\left|9c_1^3 - 15c_1c_2 + 4c_3\right| \\ &\text{and} \\ &|\gamma_2\gamma_4 - \gamma_3^2| = \frac{1}{128}\left|9c_1^4 - 15c_1^2c_2 - 8c_2^2 + 12c_1c_3\right|. \end{aligned} \tag{17}$$

Using Lemma 1.2 in (17), we obtain

$$\begin{aligned} &|\gamma_2\gamma_3 - \gamma_4| = \frac{1}{64}\left|5c_1^3 + (4-c_1^2)\{-11c_1x - 2c_1x^2 + 4(1-|x|^2)z\}\right| \\ &\text{and} \\ &|\gamma_2\gamma_4 - \gamma_3^2| = \frac{1}{256}\left|5c_1^4 + (4-c_1^2)\{-11c_1^2x - 4x^2(4-c_1^2) \right.\\ &\qquad\qquad \left. -6c_1^2x^2 + 12c_1(1-|x|^2)z\}\right|. \end{aligned}$$

As $|c_1| \leq 2$, therefore, letting $c_1 = c$, we may assume without restriction that $c \in [0, 2]$. Thus, applying triangle inequality with $\mu = |x|$, we obtain

$$|\gamma_2\gamma_3 - \gamma_4| \leq \frac{1}{64}\left[5c^3 + (4-c^2)\{11c\mu + 2c\mu^2 + 4(1-\mu^2)\}\right] := E(c,\mu)$$

and

$$|\gamma_2\gamma_4 - \gamma_3^2| \leq \frac{1}{256}\left[5c^4+(4-c^2)\{11c^2\mu+2c^2\mu^2+12c(1-\mu^2)+16\mu^2)\}\right] := F(c,\mu).$$

Now to prove our results, we need to maximize the values of E and F over the region $\Omega = \{(c,\mu) : 0 \leq c \leq 2, 0 \leq \mu \leq 1\}$. For this, first differentiating E with respect to μ and c, we obtain

$$\frac{\partial E}{\partial \mu} = \frac{1}{64}\left[(4-c^2)(11c + 4c\mu - 8\mu)\right]$$

and

$$\frac{\partial E}{\partial c} = \frac{1}{64}\left[15c^2 - 6c^2\mu^2 - 33c^2\mu + 8c\mu^2 - 8c + 44\mu + 8\mu^2\right].$$

The condition $\frac{\partial E}{\partial \mu} = 0$ gives $c = \pm 2$ or $\mu = \frac{11c}{8-4c}$, and such points (c,μ) are not interior point of Ω. So the maximum cannot attain in the interior of Ω. Now to see on the boundary, taking the boundary line $L_1 = \{(0,\mu) : 0 \leq \mu \leq 1\}$, we have $E(0,\mu) = (1-\mu^2)/4$, and its maximum on this line is equal to $1/4$, which is attained at the point $(0,0)$. On the boundary line $L_2 = \{(2,\mu) : 0 \leq \mu \leq 1\}$, we have $E(2,\mu) = 5/8$, which is a constant. On the boundary line $L_3 = \{(c,0) : 0 \leq c \leq 2\}$, we have $E(c,0) = (5c^3 - 4c^2 + 16)/64$, and its maximum on this line is $5/8$, which is attained at the point $(2,0)$. On the line $L_4 = \{(c,1) : 0 \leq c \leq 2\}$, we have $E(c,1) = (52c - 8c^3)/64$, and its maximum on this line is $13\sqrt{78}/144$, which is attained at the point $(\sqrt{13/6}, 1)$. Comparing these results, we get

$$\max_{\Omega} E(c,\mu) = E\left(\sqrt{13/6}, 1\right) = \frac{13\sqrt{78}}{144}.$$

Further, differentiating F with respect to μ and c, we obtain

$$\frac{\partial F}{\partial \mu} = \frac{1}{256}\left[(4-c^2)(11c^2 + 4c^2\mu + 32\mu - 24c\mu)\right]$$

and

$$\frac{\partial F}{\partial c} = \frac{1}{256}\left[20c^3 - 8c^3\mu^2 - 44c^3\mu - 36c^2 + 36c^2\mu^2 + 8c\mu - 16c\mu^2 - 48\mu^2 + 48\right].$$

The condition $\frac{\partial F}{\partial \mu} = 0$ gives $c = \pm 2$ or $\mu = -\frac{11c^2}{4(8-6c+c^2)}$, and such points (c, μ) are not interior point of Ω. So the maximum cannot attain in the interior of Ω. Now to see on the boundary, taking the boundary line $L_1 = \{(0, \mu) : 0 \leq \mu \leq 1\}$, we have $F(0, \mu) = \mu^2/4$, and its maximum on this line is equal to $1/4$, which is attained at the point $(0, 1)$. On the boundary line $L_2 = \{(2, \mu) : 0 \leq \mu \leq 1\}$, we have $F(2, \mu) = 5/16$, which is a constant. On the boundary line $L_3 = \{(c, 0) : 0 \leq c \leq 2\}$, we have $F(c, 0) = (5c^4 - 12c^3 + 48c)/256$, and its maximum on this line is $5/16$, which is attained at the point $(2, 0)$. On the line $L_4 = \{(c, 1) : 0 \leq c \leq 2\}$, we have $F(c, 1) = (-8c^4 + 36c^2 + 64)/256$, and its maximum on this line is $209/512$, which is attained at the point $(3/2, 1)$. Comparing these results, we get

$$\max_{\Omega} F(c, \mu) = F(3/2, 1) = 209/512.$$

This completes the proof of Theorem 2.9. □

Theorem 2.10. *Let $f \in \mathcal{F}$ be of the form* (1) *and its inverse f^{-1} be given by* (2). *Then*

$$|H_{3,1}(f^{-1})| \leq \frac{45693 + 3640\sqrt{78}}{9216}.$$

Proof. By using Theorem 2.6, Corollary 2.8, Theorem 2.9 and the triangle inequality, we get

$$\begin{aligned} |H_{3,1}(f^{-1})| &\leq |\gamma_3||\gamma_2\gamma_4 - \gamma_3^2| + |\gamma_4||\gamma_2\gamma_3 - \gamma_4| + |\gamma_5||\gamma_3 - \gamma_2^2| \\ &= \frac{45693 + 3640\sqrt{78}}{9216}. \end{aligned}$$

This completes the proof of Theorem 2.10. □

Open Problem

Löwner [21] proved that, if $f \in \mathcal{S}$ and its inverse is given by (2), then the sharp estimate $|\gamma_n| \leq \frac{\Gamma(2n+1)}{\Gamma(n+1)\Gamma(n+2)}$ holds and the inverse of the Koebe function $k(z) = z/(1-z)^2$ provides the equality bounds for all $|\gamma_n|$ $(n = 2, 3, \cdots)$. But still there are many important subclasses of class $\mathcal{S}$ like class of starlike functions, class of convex functions etc. for which sharp upper bounds of $|\gamma_n|$ are unknown.

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Sudhananda Maharana
P.G. Department of Mathematics, Utkal University, Vani Vihar,
Bhubaneswar-751004, Odisha, India.
E-mail: snmmath@gmail.com

Jugal K. Prajapat
Department of Mathematics, Central University of Rajasthan,
Bandarsindri, Kishangarh-305817, Dist.-Ajmer, Rajasthan, India.
E-mail: jkprajapat@curaj.ac.in

Deepak Bansal
Department of Mathematics,
University College of Engineering and Technology,
Bikaner Technical University, Bikaner-334004, Rajasthan, India.
E-mail: deepakbansal_79@yahoo.com