Honam Mathematical J. **42** (2020), No. 4, pp. 757–768
https://doi.org/10.5831/HMJ.2020.42.4.757

ON APPROXIMATE MIXED n-JORDAN HOMOMORPHISMS ON BANACH ALGEBRAS

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Abstract. In this paper, the Hyers-Ulam-Rassias stability of mixed n-Jordan homomorphisms on Banach algebras and the superstability of mixed n-Jordan $*$-homomorphism between C^*-algebras are investigated.

1. Introduction

Let X be real normed space and Y be real Banach space. S. M. Ulam [20] posed the problem: When does a linear mapping near an approximately additive mapping $f : X \longrightarrow Y$ exist?

In 1941, Hyers [12] gave an affirmative answer to the question of Ulam for additive Cauchy equation in Banach space.

Let X and Y be two Banach spaces and let $f : X \longrightarrow Y$ be a mapping satisfying:

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon,$$

for all $x, y \in X$ and $\varepsilon > 0$. Then there is a unique additive mapping $F : X \longrightarrow Y$ which satisfies

$$\|F(x) - f(x)\| \leq \varepsilon, \qquad x \in X.$$

Th. M. Rassias [18] considered a generalized version of the Hyers's result which permitted the Cauchy difference to become unbounded. That is, he proved:

Received June 4, 2020. Revised November 20, 2020. Accepted November 20, 2020.

2010 Mathematics Subject Classification. 46H40, 47A10.

Key words and phrases. Mixed n-Jordan homomorphisms, Pseudo n-Jordan homomorphisms, $*$-homomorphism.

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Theorem 1.1. *Let X and Y be two real Banach spaces, $\varepsilon \geq 0$ and $0 \leq p < 1$. If a mapping $f : X \longrightarrow Y$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p),$$

for all $x, y \in X$, then there is a unique additive mapping $F : X \longrightarrow Y$ such that

$$\|F(x) - f(x)\| \leq \frac{2\varepsilon}{|2 - 2^p|}\|x\|^p, \quad x \in X.$$

If, in addition, for each fixed $x \in X$ the function $t \longmapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then F is linear.

This result is called the Hyers-Ulam-Rassias stability of the additive Cauchy equation. In [10], Gajda proved that Theorem 1.1 is valid for $p > 1$, which was raised by Rassias [19]. He also gave an example showing that a similar result to the above does not hold for $p = 1$. If $p < 0$, then $\|x\|^p$ is meaningless for $x = 0$; in this case, if we assume that $\|0\|^p = \infty$, then the proof given in [18] also works for $x \neq 0$. Thus, the Hyers-Ulam-Rassias stability of the additive Cauchy equation holds for $p \in \mathbb{R} \setminus \{1\}$.

An additive mapping $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ between Banach algebras is called n-Jordan homomorphism if $\varphi(a^n) = \varphi(a)^n$, for all $a \in \mathcal{A}$.

If $n = 2$, then φ is called simply a Jordan homomorphism. The concept of n-Jordan homomorphism was dealt with firstly by Herstein in [11]. See also [4], [21] and [22], for characterization of Jordan and 3-Jordan homomorphism.

Badora [2] proved the Hyers-Ulam-Rassias stability of ring homomorphisms, which generalizes the result of Bourgin [5]. The Hyers-Ulam-Rassias stability of Jordan homomorphisms investigated by Miura et al. [14], and it is extended to n-Jordan homomorphisms in [9] and [13].

Let $\mathcal{A}$ and $\mathcal{B}$ be complex algebras and $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear map. Then φ is called an *mixed n-Jordan homomorphism* if for all $a, b \in \mathcal{A}$,

$$\varphi(a^n b) = \varphi(a)^n \varphi(b).$$

A mixed 2-Jordan homomorphism is said to be mixed Jordan homomorphism. The notation of mixed n-Jordan homomorphisms is introduced by Neghabi, Bodaghi and Zivar-Kazempour in [15] for the first time.
The following example which is obtained in [15], proves that the mixed n-Jordan homomorphisms are different from the n-Jordan homomorphisms.

Example 1.2. *Let*

$$\mathcal{A} = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : \quad a, b, c \in \mathbb{R} \right\},$$

and define $\varphi : \mathcal{A} \longrightarrow \mathcal{A}$ *via*

$$\varphi\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, $\varphi(X^2) \neq \varphi(X)^2$, *for all* $X \in \mathcal{A}$. *Hence,* φ *is not Jordan homomorphism, and so it is not homomorphism. But for all* $n \geq 3$ *and for all* $X, Y \in \mathcal{A}$, *we have* $\varphi(X^n Y) = \varphi(X)^n \varphi(Y)$. *Therefore,* φ *is mixed* n*-Jordan homomorphism for all* $n \geq 3$.

Let $\mathcal{A}$ and $\mathcal{B}$ be complex algebras, and let $\mathcal{B}$ be a right [left] $\mathcal{A}$-module. Then a linear map $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is saied to be *pseudo n-Jordan homomorphism* if there exiest an element $w \in \mathcal{A}$ such that for all $a \in \mathcal{A}$,

$$\varphi(a^n w) = \varphi(a)^n \cdot w, \quad [\varphi(wa^n) = w \cdot \varphi(a)^n].$$

The concept of pseudo n-Jordan homomorphism was introduced and studied by Ebadian et al., in [6].

Let $\varphi : \mathcal{A} \longrightarrow \mathcal{A}$ be a mixed n-Jordan homomorphism with a fixed point u. Then for all $a \in \mathcal{A}$,

$$\varphi(a^n u) = \varphi(a)^n \varphi(u) = \varphi(a)^n u.$$

Therefore φ is pseudo n-Jordan homomorphism.

In this paper, we investigate the Hyers-Ulam-Rassias stability of mixed n-Jordan homomorphisms on Banach algebras and the superstability of mixed n-Jordan $*$-homomorphism between C^*-algebras.

2. Stability of Mixed n-Jordan Homomorphisms

We commence with the following characterization of mixed n-Jordan homomorphisms.

Theorem 2.1. *Every mixed n-Jordan homomorphism φ between commutative algebras $\mathcal{A}$ and $\mathcal{B}$ is $(n+1)$-homomorphism.*

Proof. Since every mixed n-Jordan homomorphism is $(n+1)$-Jordan homomorphism, so the result follows from Theorem 2.2 of [3]. □

Theorem 2.2. *Let $\mathcal{A}$ be a unital Banach algebra, $\mathcal{B}$ be a semisimple commutative Banach algebra. Then every mixed n-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is an $(n+1)$-homomorphism.*

Proof. This result follows from Corollary 2.5 of [1]. □

Theorem 2.3. *Let $\mathcal{A}$ be a normed algebra, let $\mathcal{B}$ be a Banach algebra, let δ and ε be nonnegative real numbers, and let p, q be a real numbers such that $(p-1)(q-1) > 0$, $q \geq 0$. Assume that $f : \mathcal{A} \longrightarrow \mathcal{B}$ satisfies*

$$\|f(a+b) - f(a) - f(b)\| \leq \varepsilon\big(\|a\|^p + \|b\|^p\big), \tag{1}$$

$$\|f(a^n b) - f(a)^n f(b)\| \leq \delta \|a\|^{nq} \|b\|, \tag{2}$$

for all $a, b \in \mathcal{A}$. Then, there exists a unique mixed n-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ such that

$$\|\varphi(a) - f(a)\| \leq \frac{2\varepsilon}{|2-2^p|}\|a\|^p, \quad a \in \mathcal{A}. \tag{3}$$

Proof. Put $t := -sgn(p-1)$ and

$$\varphi(x) = \lim_m \frac{1}{2^{tm}} f(2^{tm}x),$$

for all $x \in \mathcal{A}$. It follows from [10] and [18] that φ is additive map satisfies (3). We will show that φ is mixed n-Jordan homomorphism. We have

$$\begin{aligned}
\lim_m \frac{1}{2^{tmn}}\big(\|f(2^{tmn}a^n b) - f(2^{tm}a)^n f(b)\|\big) &\leq \lim_m \frac{\delta}{2^{tmn}} \|2^{tm}a\|^{nq}\|b\| \\
&\leq \lim_m \frac{\delta}{2^{tmn}} 2^{tmnq}\|a\|^{nq}\|b\| \\
&\leq \lim_m 2^{tmn(q-1)}\big(\delta\|a\|^{nq}\|b\|\big) = 0.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\varphi(a^n b) =& \lim_m \frac{1}{2^{tmn}} f(2^{tmn}a^n b) \\
=& \lim_m \frac{1}{2^{tmn}} \{f(2^{tmn}a^n b) - f(2^{tmn}a^n b) + f(2^{tm}a)^n f(b)\} \\
=& \lim_m \frac{1}{2^{tmn}} f(2^{tm}a)^n f(b) \\
=& \varphi(a)^n f(b).
\end{aligned}$$

So $\varphi(a^n b) = \varphi(a)^n f(b)$, for all $a, b \in \mathcal{A}$. Therefore we have

$$\begin{aligned}\|\varphi(a^n b) - \varphi(a)^n \varphi(b)\| &\leq \|\varphi(a^n b) - \varphi(a)^n f(b)\| + \|\varphi(a)^n f(b) - \varphi(a)^n \varphi(b)\| \\ &\leq \|\varphi(a)\|^n \|f(b) - \varphi(b)\| \\ &\leq \frac{2\varepsilon}{|2-2^p|} \|b\|^p \|\varphi(a)\|^n.\end{aligned}$$

Hence

$$\|\varphi(a^n b) - \varphi(a)^n \varphi(b)\| \leq \frac{2\varepsilon}{|2-2^p|} \|b\|^p \|\varphi(a)\|^n. \tag{4}$$

Replacing b by $2^{tm} b$ in (4), gives

$$\|\varphi(a^n b) - \varphi(a)^n \varphi(b)\| \leq \frac{2\varepsilon}{|2-2^p|} \|b\|^p \|\varphi(a)\|^n \lim_m 2^{tm(p-1)} = 0.$$

So $\varphi(a^n b) = \varphi(a)^n \varphi(b)$ and φ is mixed n-Jordan homomorphism. The uniqueness property of φ follows from [10] and [18]. □

The next result follows from above Theorem.

Corollary 2.4. *Let $\mathcal{A}$ be a Banach algebra and let $f : \mathcal{A} \longrightarrow \mathcal{A}$ satisfies in (1) and (2). Assume that f has a fixed point u. Then there exists a unique pseudo n-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{A}$ such that*

$$\|\varphi(u) - u\| \leq \frac{2\varepsilon}{|2-2^p|} \|u\|^p.$$

Theorem 2.5. *Let $\mathcal{A}$ be a normed algebra, let $\mathcal{B}$ be a Banach algebra, let δ and ε be nonnegative real numbers, and let p, q be a real numbers such that $(p-1)(q-1) > 0$, and $q < 0$. Assume that $f : \mathcal{A} \longrightarrow \mathcal{B}$ be a mapping with $f(0) = 0$, such that the inequalities (1) and (2) are hold. Then, there exists a unique mixed n-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ such that*

$$\|\varphi(a) - f(a)\| \leq \frac{2\varepsilon}{|2-2^p|} \|a\|^p, \tag{5}$$

for all $a \in \mathcal{A}$.

Proof. It follows from [18] that there exists an additive map $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ satisfies (5), where we assume that $\|0\|^p = \infty$. We show that

$$\varphi(a^n b) = \varphi(a)^n \varphi(b),$$

for all $a, b \in \mathcal{A}$. Since φ is additive, we have $\varphi(0) = 0$. Hence the result is valid for $a = 0$ or $b = 0$. Suppose that $a, b \in \mathcal{A} \setminus \{0\}$ be arbitrarily. If $a^n \neq 0$ and $b \neq 0$, then the proof of Theorem 2.3 works well, and φ is mixed n-Jordan homomorphism. Now let $a^n = 0$ and $b \neq 0$. It follows from (2), with the hypothesis $f(0) = 0$ that

$$\frac{1}{2^{mn}}\|f(2^m a)^n f(b)\| \leq \frac{\delta}{2^{mn}}\|2^m a\|^{nq}\|b\| = 2^{mn(q-1)}\delta\|a\|^{nq}\|b\|. \tag{6}$$

Since $a, b \in \mathcal{A} \setminus \{0\}$ and $(q-1) < 0$, we get

$$\lim_m \frac{1}{2^{mn}} f(2^m a)^n f(b) = \lim_m 2^{mn(q-1)}\delta\|a\|^{nq}\|b\| = 0. \tag{7}$$

On the other hand, we have

$$\varphi(a) = \lim_m \frac{1}{2^m} f(2^m a), \quad a \in \mathcal{A}. \tag{8}$$

Thus, by (7) and (8),

$$\varphi(a)^n f(b) = \lim_m \{\frac{1}{2^{mn}} f(2^m a)^n\} f(b) = 0.$$

Hence $\varphi(a)^n f(b) = 0$. Now we prove that $\varphi(a)^n\varphi(b) = 0$. To this

$$\begin{aligned} \|\varphi(a)^n\varphi(b)\| &= \|\varphi(a)^n\varphi(b) - \varphi(a)^n f(b)\| \\ &\leq \|\varphi(a)\|^n\|\varphi(b) - f(b)\| \\ &\leq \frac{2\varepsilon}{|2-2^p|}\|b\|^p\|\varphi(a)\|^n. \end{aligned}$$

Consequently,

$$\|\varphi(a)^n\varphi(b)\| \leq \frac{2\varepsilon}{|2-2^p|}\|b\|^p\|\varphi(a)\|^n. \tag{9}$$

Replacing b by $2^{tm}b$ in (9), gives

$$\|\varphi(a)^n\varphi(b)\| \leq \frac{2\varepsilon}{|2-2^p|}\|b\|^p\|\varphi(a)\|^n \lim_m 2^{tm(p-1)} = 0.$$

Therefore $\varphi(a)^n\varphi(b) = 0$, which proves that $\varphi(a)^n\varphi(b) = 0 = \varphi(a^n b)$, whenever $a^n = 0$. This completes the proof. □

As a consequence of Theorem 2.3, 2.5 and Theorem 2.1 we have the following.

Corollary 2.6. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are commutative Banach algebras. Let δ and ε be nonnegative real numbers, and let p, q be a real numbers such that $(p-1)(q-1) > 0$, $q \geq 0$ or $(p-1)(q-1) > 0$, $q < 0$ and $f(0) = 0$. Assume that $f : \mathcal{A} \longrightarrow \mathcal{B}$ satisfies (1) and (2). Then, there exists a unique $(n+1)$-homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ such that*

$$\|\varphi(x) - f(x)\| \leq \frac{2\varepsilon}{|2-2^p|}\|x\|^p, \quad x \in \mathcal{A}.$$

From Theorem 2.3, 2.5 and Theorem 2.2 we have the next result.

Corollary 2.7. *Suppose that $\mathcal{A}$ is a unital Banach algebra, and suppose $\mathcal{B}$ is a semisimple commutative Banach algebra. Let δ and ε be nonnegative real numbers, and let p, q be a real numbers such that $(p-1)(q-1) > 0$, $q \geq 0$ or $(p-1)(q-1) > 0$, $q < 0$ and $f(0) = 0$. Assume that $f : \mathcal{A} \longrightarrow \mathcal{B}$ satisfies (1) and (2). Then, there exists a unique $(n+1)$-homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ such that*

$$\|\varphi(x) - f(x)\| \leq \frac{2\varepsilon}{|2 - 2^p|}\|x\|^p,$$

for all $x \in \mathcal{A}$.

Theorem 2.8. *Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear map such that*

$$\|\varphi(a^n b) - \varphi(a)^n \varphi(b)\| \leqslant \delta(\|a\| \pm \|b\|), \quad a, b \in \mathcal{A}, \tag{10}$$

for some $\delta > 0$. Then φ is $(n+1)$-Jordan homomorphism.

Proof. Suppose that for all $a, b \in \mathcal{A}$,

$$\|\varphi(a^n b) - \varphi(a)^n \varphi(b)\| \leqslant \delta(\|a\| - \|b\|).$$

Replacing b by a, we get $\varphi(a^{n+1}) = \varphi(a)^{n+1}$, for all $a \in \mathcal{A}$. So φ is $(n+1)$-Jordan homomorphism. Now let for all $a, b \in \mathcal{A}$,

$$\|\varphi(a^n b) - \varphi(a)^n \varphi(b)\| \leqslant \delta(\|a\| + \|b\|). \tag{11}$$

Interchanging b by a in (11), gives

$$\|\varphi(a^{n+1}) - \varphi(a)^{n+1}\| \leqslant 2\delta\|a\|. \tag{12}$$

Setting $a = 2^m x$, we get

$$\|\varphi(x^{n+1}) - \varphi(x)^{n+1}\| \leqslant \frac{\delta 2^{m+1}}{2^{m(n+1)}}\|x\|. \tag{13}$$

Letting $m \longrightarrow \infty$, we obtain $\varphi(x^{n+1}) = \varphi(x)^{n+1}$ and hence the result follows. $\square$

3. Superstability of Mixed n-Jordan $*$-Homomorphisms

Throughout this section, assume that $\mathcal{A}$, $\mathcal{B}$ be a C^*-algebras. A mixed n-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is called a mixed (pseudo) n-Jordan $*$-homomorphism if

$$\varphi(a^*) = \varphi(a)^*, \qquad a \in \mathcal{A}.$$

Now we investigate the superstability of mixed n-Jordan $*$-homomorphism between C^*-algebras.

Lemma 3.1. *Let $f : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive mapping such that $f(\lambda a) = \lambda f(a)$ for all $a \in \mathcal{A}$ and for all $\lambda \in \mathbb{T} := \{\alpha \in \mathbb{C} : \ |\alpha| = 1\}$. Then the mapping f is $\mathbb{C}$-linear.*

Proof. See [16] □

Lemma 3.2. *Let $f : \mathcal{A} \longrightarrow \mathcal{B}$ be a mapping such that*

$$\|f(\frac{b-a}{3}) + f(\frac{a-3c}{3}) + f(\frac{3a+3c-b}{3})\| \leq \|f(a)\|,$$

for all $a, b, c \in \mathcal{A}$. Then f is additive.

Proof. See [8] □

Theorem 3.3. *Let $p < 1$, δ be nonnegative real numbers and let $f : \mathcal{A} \longrightarrow \mathcal{B}$ satisfies*

$$\|f(\frac{b-a}{3}\lambda) + f(\frac{a-3c}{3}\lambda) + \lambda f(\frac{3a+3c-b}{3})\| \leq \|f(a)\|, \tag{14}$$

$$\|f(a^n b) - f(a)^n f(b)\| \leq \delta \|a\|^{np} \|b\|^p, \tag{15}$$

$$\|f(a^*) - f(a)^*\| \leq \delta \|a^*\|^p, \tag{16}$$

for all $\lambda \in \mathbb{T} := \{\alpha \in \mathbb{C} : \ |\alpha| = 1\}$ and for all $a, b, c \in \mathcal{A}$. Then f is mixed n-Jordan $*$-homomorphism

Proof. Take $\lambda = 1$ in (14), then by Lemma 3.2 the mapping f is additive and so $f(0) = 0$. Letting $a = b = 0$ in (14), gives

$$\|f(-\lambda c) + \lambda f(c)\| \leq \|f(0)\| = 0.$$

Thus, for all $c \in \mathcal{A}$ and for all $\lambda \in \mathbb{T}$,

$$f(\lambda c) = \lambda f(c).$$

By Lemma 3.1 the mapping $f : \mathcal{A} \longrightarrow \mathcal{B}$ is $\mathbb{C}$-linear. It follows from (15) that

$$\begin{aligned}
\|f(a^n b) - f(a)^n f(b)\| &= \|\frac{1}{m^n} f(m^n a^n b) - (\frac{1}{m} f(ma))^n f(b)\| \\
&\leq \frac{1}{m^n} \|f(m^n a^n b) - f(ma)^n f(b)\| \\
&\leq \frac{\delta}{m^n} m^{np} \|a\|^{np} \|b\|^p,
\end{aligned}$$

for all $a, b \in \mathcal{A}$. Since $p < 1$, by letting $m \longrightarrow \infty$, we get

$$f(a^n b) = f(a)^n f(b),$$

for all $a, b \in \mathcal{A}$. It follows from (15) that

$$\begin{aligned} \|f(a^*) - f(a)^*\| &= \|\frac{1}{m}f(ma^*) - (\frac{1}{m}f(ma))^*\| \\ &\leq \frac{1}{m}\|f(ma^*) - f(ma)^*\| \\ &\leq \frac{\delta}{m}m^p\|a^*\|^p, \end{aligned}$$

for all $a, b \in \mathcal{A}$. Since $p < 1$, by letting $m \longrightarrow \infty$, we get

$$f(a^*) = f(a)^*,$$

for all $a \in \mathcal{A}$. □

Corollary 3.4. *Let $p < 1$, δ be nonnegative real numbers and $f : \mathcal{A} \longrightarrow \mathcal{A}$ satisfies (14) and (16). If f has a fixed point u, such that*

$$\|f(a^n u) - f(a)^n u\| \leq \delta\|a\|^{np}\|u\|^p, \quad a \in \mathcal{A},$$

then f is pseudo n-Jordan $*$-homomorphism.

The proof of the next result is similar to the proof of Theorem 3.3.

Theorem 3.5. *Let $p > 1$, δ be nonnegative real numbers and $f : \mathcal{A} \longrightarrow \mathcal{B}$ satisfies (14), (15) and (16). Then the mapping f is mixed n-Jordan $*$-homomorphism.*

In the next example we show that Theorem 3.3 and Theorem 3.5 are fails for $p = 1$.

Example 3.6. *Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(a) = Ma$ where $M = \frac{1+\sqrt{1+4\delta}}{2}$ and $\delta > 0$. Then for all $a, b \in \mathbb{R}$,*

$$f(a + b) = f(a) + f(b).$$

Hence f is additive. Since $M^2 = M + \delta$, so we get

$$\begin{aligned} \|f(a^2 b) - f(a)^2 f(b)\| &= \|Ma^2 b - (Ma)^2(Mb)\| \\ &= \|Ma^2 b - (M + \delta)Ma^2 b\| \\ &= \| - \delta Ma^2 b - \delta a^2 b\| \\ &= (M + 1)\delta\|a\|^2\|b\|. \end{aligned}$$

Thus, f fulfills in Theorem 3.3 with $n = 2$ and $p = 1$. However, the mapping f is not mixed Jordan $*$-homomorphism unless $\delta = 0$.

A linear mapping $D : \mathcal{A} \longrightarrow \mathcal{A}$ is called a mixed n-Jordan derivation if for all $a, b \in \mathcal{A}$,

$$D(a^n b) = D(a^n)b + a^n D(b).$$

Now we investigate the superstability of mixed n-Jordan derivation.

Theorem 3.7. *Let $p > 1$, δ be nonnegative real numbers and let $f : \mathcal{A} \longrightarrow \mathcal{A}$ be a mapping satisfying (14), such that*

$$\|f(a^n b) - f(a^n)b - a^n f(b)\| \leq \delta \|a\|^{np}\|b\|^p, \tag{17}$$

for all $a, b \in \mathcal{A}$. Then $f : \mathcal{A} \longrightarrow \mathcal{A}$ is a mixed n-Jordan derivation.

Proof. By the same method as in the proof of Theorem 3.3, the mapping $f : \mathcal{A} \longrightarrow \mathcal{A}$ is $\mathbb{C}$-linear. It follows from (17) that

$$\begin{aligned}\|f(a^n b)-f(a^n)b-a^n f(b)\| &= L\|f(\frac{a^n}{2^{mn}}\frac{b}{2^m})-f(\frac{a^n}{2^{mn}})\frac{b}{2^m}-\frac{a^n}{2^{mn}}f(\frac{b}{2^m})\| \\ &\leq L\frac{\delta}{L^p}\|a\|^{np}\|b\|^p,\end{aligned}$$

for all $a, b \in \mathcal{A}$, where $L = 2^{m(n+1)}$. Since $p > 1$, by letting $m \longrightarrow \infty$, we get

$$f(a^n b) = f(a^n)b + a^n f(b), \qquad a, b \in \mathcal{A}.$$

Thus, f is a mixed n-Jordan derivation. □

An analogous result of Theorem 3.7 is also holds for $p < 1$. Moreover, if we take $n = 1$ in preceding result, then we get Theorem 2.5 of [17].

Acknowledgments

The authors would like to express his sincere thanks to the referees for this paper.

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