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INEQUALITIES FOR THE ARGUMENTS LYING ON LINEAR AND CURVED PATH

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Abstract. The mathematical proof for establishing some new inequalities involving arithmetic, geometric, harmonic means for the arguments lying on the paths of triangular wave function (linear) and new parabolic function (curved) over the interval $(0,1)$ are discussed. The results representing an extension as well as strengthening of Ky Fan Type inequalities.

1. Introduction

The Hand book of Means and their Inequalities, by Bullen [1], gave the tremendous work on Mathematical means and the corresponding inequalities involving huge number of means. The authors in [3, 4] discussed about the relations between the well known means and series. The generalization of the means are discussed in [5, 17]. Relevant to this paper the authors in [11, 18] established the good number of inequalities and double inequalities. In [8] authors introduced new homogeneous function, as an application inequalities involving means are obtained. The set of arbitrary non-negative real numbers $y_i \in (0, \frac{1}{2}]$ and $y'_i = (1 - y_i) \in [\frac{1}{2}, 1)$ is represented as a function in the form given by [1].

$$f(y) = \begin{cases} y, & 0 < y \leq \frac{1}{2} \\ (1-y), & \frac{1}{2} \leq y < 1 \end{cases}$$

The following are the few definitions of means extracted from the above survey papers. For given n arbitrary non-negative real numbers $y_1, y_2, \cdots$

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$, y_n \in (0, \frac{1}{2}]$, the standard notations for the un-weighted arithmetic, geometric and harmonic means are represented by A_n, G_n and H_n are respectively given by

$$A_n = \frac{1}{n}\sum_{i=1}^{n} y_i, \qquad G_n = \prod_{i=1}^{n} \sqrt[n]{y_i} \qquad \text{and} \qquad H_n = \frac{n}{\sum_{i=1}^{n} \frac{1}{y_i}}$$

Also, the arithmetic, geometric and harmonic means of the set of elements $1 - y_1, 1 - y_2, \cdots, 1 - y_n$ represented by A'_n, G'_n and H'_n are respectively given by

$$A'_n = \frac{1}{n}\sum_{i=1}^{n}(1 - y_i), \qquad G'_n = \prod_{i=1}^{n} \sqrt[n]{1 - y_i} \qquad \text{and} \qquad H'_n = \frac{n}{\sum_{i=1}^{n} \frac{1}{(1-y_i)}}$$

Ky Fan initiated the popular inequalities involving them and later on strengthened by several authors namely Rooin et al. and Sandoor et al. [16, 18]. The inequalities obtained in this paper have numerous applications in the study of porous medium problems [2]. This work motivate us to develop two double inequalities in this paper.

For two positive arguments e and f, the above means are given by;

$$A = \frac{e + f}{2}, \quad \text{and} \qquad A' = \frac{(1 - e) + (1 - f)}{2}$$

$$G = \sqrt{ef} \qquad \text{and} \qquad G' = \sqrt{(1 - f)(1 - e)}$$

$$H = \frac{2ef}{e + f} \qquad \text{and} \qquad H' = \frac{2(1 - f)(1 - e)}{(1 - e) + (1 - f)}$$

The motivation of the work carried out by the eminent researchers and discussion with experts, results in study of a function which is symmetric about the point $\frac{1}{2}$ and is similar to parabolic curve in nature given below:

$$f^*(y) = \begin{cases} 2y^2, & 0 < y \leq \frac{1}{2} \\ 2(1 - y)^2, & \frac{1}{2} \leq y < 1 \end{cases}$$

The functions $f(y)$ and $f^*(y)$ are graphically represented as shown below:

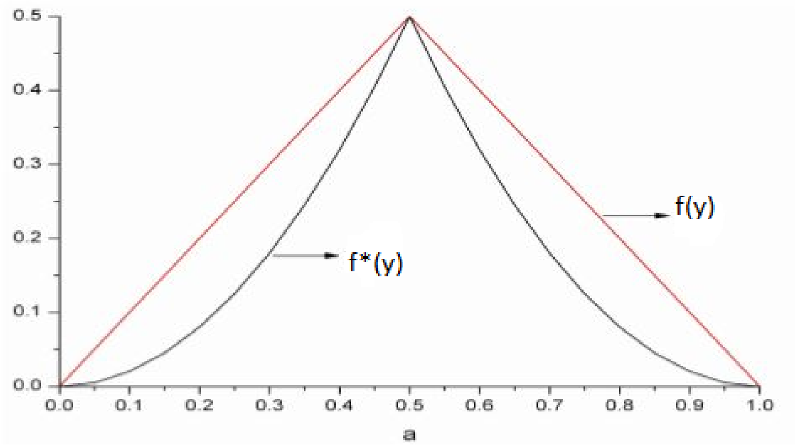

FIGURE 1. Graph of $f(y)$ and $f^*(y)$ over [0,1]

The corresponding arithmetic, geometric and harmonic means A^*, G^* and H^* are considered for the arguments lying on the curved path of $f^*(y)$ and some important inequality chains involving them are established.

$$A_n^* = \frac{1}{n}\sum_{i=1}^{n} 2e_i^2 \qquad \text{and} \qquad (A_n^*)' = \frac{1}{n}\sum_{i=1}^{n} 2(1-e_i)^2$$

$$G_n^* = \prod_{i=1}^{n} \sqrt[n]{2^n(e_i^2)} \quad \text{and} \quad (G_n^*)' = \prod_{i=1}^{n} \sqrt[n]{2^n(1-e_i)^2}$$

$$H_n^* = \frac{n}{\sum_{i=1}^{n} \frac{1}{2e_i^2}} \qquad \text{and} \qquad (H_n^*)' = \frac{n}{\sum_{i=1}^{n} \frac{1}{2(1-e_i)^2}}$$

For two positive arguments e and f, above said means takes the form;

$$A^* = \frac{2e^2+2f^2}{2} \quad \text{and} \quad (A^*)' = \frac{2(1-e)^2+2(1-f)^2}{2}$$

$$G^* = \sqrt{[2e^2][2f^2]} \qquad \text{and} \qquad (G^*)' = \sqrt{[2(1-e)^2][2(1-f)^2]}$$

$$H^* = \frac{4e^2f^2}{e^2+f^2} \quad \text{and} \quad (H^*)' = \frac{4(1-e)^2(1-f)^2}{(1-e)^2+(1-f)^2}$$

2. Results

In this section, some inequalities involving arithmetic, geometric and harmonic means for the arguments lying on linear and curved path are established.

Theorem 2.1. *For $e, f \in (0, \frac{1}{2}]$, the following mean inequality chain holds;*

$$(G^*)' > (H^*)' > G^* > H^*$$

Proof. The proof is discussed in following three cases.
Case[*i*]: The geometric mean $(G^*)'$ is greater than harmonic mean $(H^*)'$,

where $(G^*)' = 2(1-f)(1-e)$ and $(H^*)' = \dfrac{4(1-f)^2(1-e)^2}{(1-f)^2+(1-e)^2}$

Consider for two positive arguments,

$$\begin{aligned}(G^*)' - (H^*)' &= 2(1-f)(1-e) - \frac{4(1-f)^2(1-e)^2}{(1-f)^2+(1-e)^2} \\ &= 2(1-f)(1-e)\left[1 - \frac{2(1-f)(1-e)}{(1-f)^2+(1-e)^2}\right] \\ &= 2(1-f)(1-e)\left[\frac{(1-f)^2+(1-e)^2-2(1-f)(1-e)}{(1-f)^2+(1-e)^2}\right]\end{aligned}$$

Now we consider

$$\left[\frac{(1-f)^2+(1-e)^2-2(1-e)(1-f)}{(1-f)^2+(1-e)^2}\right]$$
$$= 1-2e+e^2+f^2+1-2f-2+2f+2e-2ef = e^2+f^2-2ef$$
$$= 4(A^2-G^2) > 0.$$

Also, $(1-e) > 0$ and $(1-f) > 0$, then,

$$(G^*)' - (H^*)' = 2(1-e)(1-f)\left[\frac{(1-f)^2+(1-e)^2-2(1-e)(1-f)}{(1-f)^2+(1-e)^2}\right] > 0$$

Hence, $(G^*)' - (H^*)' > 0$, this proves that $(G^*)' > (H^*)'$.
Case[*ii*]: To prove $(H^*)' > G^*$,
where $(H^*)' = \frac{4(1-e)^2(1-f)^2}{(1-e)^2+(1-f)^2}$ and $G^* = 2ef$.

Consider $(H^*)' - G^* = \dfrac{4(1-f)^2(1-e)^2}{(1-f)^2+(1-e)^2} - 2ef$

$$= 2[2(1-e)^2(1-f)^2 - ef[1+e^2-2e+1+f^2-2f]]$$

$$= 2 + 2(f+e)^2 + 2ef - 2ef(f+e) - 4(f+e) + 2e^2f^2 - ef(e^2+f^2)$$
$$= 2+2(f+e)^2+2ef-2ef(f+e)-4(f+e)+2e^2f^2-ef(f+e)^2+2e^2f^2$$
$$= 2ef(1-f-e) + 2(f+e)^2(2-fe) + 2(1-2e-2f) + 4e^2f^2 > 0.$$

Since, $2ef(1-e-f) > 0$ and $(e+f)^2(2-ef) = (e+f)^2(2-G^2) > 0.$

We have $\dfrac{4(1-f)^2(1-e)^2}{(1-f)^2+(1-e)^2} - 2ef > 0.$

Hence, $(H^*)' - G^* > 0,$ this proves that $(H^*)' > G^*.$
Case[*iii*]: In order to prove $G^* > H^*$,
where $G^* = \sqrt{(2e^2)(2f^2)}$ and $H^* = \frac{4e^2f^2}{e^2+f^2}.$
Consider

$$G^* - H^* = \sqrt{(2e^2)(2f^2)} - \frac{4e^2f^2}{e^2+f^2} = 2ef - \frac{4e^2f^2}{e^2+f^2} = 2ef\left[1 - \frac{2ef}{e^2+f^2}\right]$$

$$= \frac{2ef}{e^2+f^2}\left[(e+f)^2 - 4ef\right] = \frac{8ef}{e^2+f^2}\left[A^2 - G^2\right] > 0.$$

Then $G^* - H^* > 0,$ this proves that $G^* > H^*.$
Thus the Theorem 2.1 completes. □

Theorem 2.2. *For $e, f \in (0, \frac{1}{2}]$, the mean inequality $G' > H' > G^*$ holds.*

Proof. The proof is discussed in following two cases.
Case[*i*]: The geometric mean (G') is greater than harmonic mean (H'),
where $G' = \sqrt{(e-1)(f-1)}$ and $H' = \frac{2(f-1)(e-1)}{2-f-e}.$
Consider $G' - H' = \sqrt{(f-1)(1-e)} - \frac{2(1-e)(1-f)}{2-f-e}$
With assumption that $\sqrt{(1-f)(1-e)} > \frac{2(1-f)(1-e)}{2-f-e}$
Then $(1-e)(1-f) > \frac{4(1-f)^2(1-e)^2}{(2-f-e)^2}$
Which is equivalent to $(1-f)(1-e)\left[\frac{(2-f-e)^2-4(1-e)(1-f)}{(2-f-e)^2}\right] > 0$
Now consider

$$(2-f-e)^2 - 4(1-e)(1-f) = 4+e^2-4e+f^2-4f+2ef-4+4e+4f-4ef$$
$$= e^2+f^2-2ef = (e+f)^2 - 4ef$$
$$= 4(A^2 - G^2) > 0$$

This proves that $G' > H'.$
Case[*ii*]: To prove $H' > G^*$,
where $H' = \frac{2(1-e)(1-f)}{2-e-f}$ and $G^* = \sqrt{(2e^2)(2f^2)}$
Consider $H' - G^* = \frac{2(1-e)(1-f)}{2-f-e} - \sqrt{(2e^2)(2f^2)}$

With assumption that $\frac{(1-e)(1-f)2}{2-e-f} > \sqrt{(2e^2)(2f^2)}$
Equivalently $\frac{2(1-f)(1-e)}{2-f-e} > 2ef$ and $1-f-e+ef > ef(2-e-f)$

$$\implies \quad 1+ef-e-f-2ef+e^2f+ef^2 > 0$$
$$\implies \quad 1-(e+f)-ef+ef(e+f) > 0$$
$$\implies \quad 1-ef+(ef-1)(e+f) > 0$$
$$\implies \quad (1-ef)[1-e-f] > 0$$
$$\implies \quad (1-ef) > 0 \quad \text{and} \quad (1-e-f) > 0$$

Therefore our assumption is true and hence $H' - G^* > 0$ this proves that $H' > G^*$.
Thus the Theorem 2.2 completes. □

Theorem 2.3. *For $e, f \in (0, \frac{1}{2}]$, the following mean inequality chain holds;*

$$A' > H' > H > H^*$$

Proof. The proof is discussed in following three cases.
Case[*i*]: To prove $A' > H'$, where $A' = \frac{2-e-f}{2}$ and $H = \frac{2ef}{e+f}$.
Consider $A' - H' = \frac{2-f-e}{2} - \frac{2ef}{e+f} = \frac{(2-e-f)(f+e)-4ef}{2(f+e)}$

$$= \frac{2e+2f-e^2-ef-ef-e^2-4ef}{2(f+e)} = \frac{2(e+f)-(f+e)^2-4ef}{2(e+f)}$$

$$= \frac{(e+f)-(e+f)^2+e+f-4ef}{2(e+f)} = \frac{(1-e-f)(e+f)+(2A-4G^2]}{2f+e}.$$

Since $0 < 1-f-e$ and $2(A-2G^2) > 0$
We have $A' - H' = \frac{2-f-e}{2} - \frac{2ef}{f+e} > 0.$
Hence $A' - H' > 0$, this proves that $A' > H'$.
Case[ii]: To prove $H' > H$, where $H' = \frac{2(1-e)(1-f)}{2-e-f}$ and $H = \frac{2ef}{e+f}$.
Consider $H' - H = \frac{2(1-e)(1-f)}{2-f-e} - \frac{2ef}{f+e} = 2\left[\frac{(1-e)(e+f)(1-f)-ef(2-f-e)}{(2-e-f)(e+f)}\right].$
Now we consider $(1-e)(1-f)(e+f) - ef(2-e-f)$

$$= f+e-e^2-ef-f^2-ef+e^2f-2ef+ef^2+e^2f+f^2e$$
$$= 2e^2f+2ef^2-4ef+e-e^2+f-f^2$$
$$= 2ef(f+e)+e+f-2ef-(e+f)^2$$
$$= [2ef-(e+f)](e+f)+(e+f-2ef)$$
$$= [(f+e-2ef)(1-f-e)]$$
$$= 2(A-G^2)(1-2A) > 0$$

Then $H' - H = \frac{2(1-e)(1-f)}{2-e-f} - \frac{2ef}{e+f} > 0.$
Hence $H' - H > 0$, this proves that $H' > H$.
Case[iii]: To prove $H > H^*$, where $H = \frac{2ef}{e+f}$ and $H' = \frac{4e^2f^2}{e^2+f^2}$.
Consider $H - H^* = \frac{2ef}{f+e} - \frac{4e^2f^2}{f^2+e^2}$

$= 2fe\frac{4A^2-2G^2-4AG^2}{(f+e)(f^2+e^2)}$
$= 2ef\frac{2(A^2-G^2)+2(A^2-2G^2)}{(e+f)(e^2+f^2)}.$

Since $(G^2 < A^2)$ and $(A^2 - 2G^2) > 0$
We have $H - H^* > 0$, this proves that $H > H^*$.
Thus the Theorem 2.3 completes. □

Theorem 2.4. *For $e, f \in (0, \frac{1}{2}]$, the mean inequality $G^{*\prime} > H > G^*$ holds.*

Proof. The proof is discussed in following two cases.
Case[*i*]: The geometric mean $(G^{*\prime})$ is greater than harmonic mean (H), where $G^{*\prime} = 2(1-e)(1-f)$ and $H = \frac{2fe}{f+e}$.
Consider $G^{*\prime} - H = 2(1-f)(1-e) - \frac{2ef}{f+e}$

$$= \frac{2(1-e)(1-f)(f+e) - 2ef}{f+e}$$
$$= 2\left[\frac{f+e-e^2-4fe-f^2+2e^2f+2ef^2}{(f+e)}\right]$$
$$= (f+e) - (f+e)^2 - 2ef + 2ef(f+e)$$
$$= (1-e-f)(f+e-2ef)$$

Since, $(f+e-2ef) = 2A - 2G^2 = 2(A - G^2) > 0$ and $(1-e-f) > 0$.
Hence $G^{*\prime} - H = 2(1-e)(1-f) - \frac{2ef}{e+f} > 0$.
Then, $G^{*\prime} - H > 0$, this proves that $G^{*\prime} > H$.
Case[ii]: To prove $H > G^*$, where $H = \frac{2ef}{e+f}$ and $G^* = 2ef$.
Consider $H - G^* = \frac{2ef}{e+f} - 2ef = \frac{2ef}{e+f}[1-e-f]$
Since, $1 - e - f > 0$.
Then, $H - G^* = \frac{2ef}{e+f} - 2ef > 0$.
$H - G^*$, this proves that $H > G^*$.
Thus the Theorem 2.4 completes. □

Conclusion. New inequality chains involving arithmetic, geometric and harmonic means are established by considering the arguments lying in the triangular wave function $f(y)$ and new parabolic function $f^*(y)$ in (0, 1). Further investigation to be carried out on more parabolic functions based on their nature and verifying the properties of means.

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