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ALMOST HERMITIAN SUBMERSIONS WHOSE TOTAL MANIFOLDS ADMIT A RICCI SOLITON

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Abstract. The object of the present paper is to study the almost Hermitian submersion from an almost Hermitian manifold admits a Ricci soliton. Where, we investigate any fibre of such a submersion is a Ricci soliton or Einstein. We also get necessary conditions for which the base manifold of an almost Hermitian submersion is a Ricci soliton or Einstein. Moreover, we obtain the harmonicity of an almost Hermitian submersion from a Ricci soliton to an almost Hermitian manifold.

1. Introduction

A smooth vector field η on a Riemannian manifold (N, g) is said to define a Ricci soliton if it satisfies

$$\frac{1}{2}\mathcal{L}_\eta g + \tau + \mu g = 0, \tag{1}$$

where $\mathcal{L}_\eta g$ is the Lie-derivative of the metric tensor g with respect to η, τ is the Ricci tensor of (N, g) and μ is a constant. Ricci soliton is a natural generalization of the Einstein metric (that is, $\tau = bg$, for some constant b), and is a special self similar solution of the Hamilton's Ricci flow (see [17] $\frac{\partial}{\partial x} g(x) = -2\tau(x)$ with initial condition $g(0) = g$. We shall denote a Ricci soliton by (N, g, η, μ). We say that the Ricci soliton is shrinking when $\mu < 0$, steady when $\mu = 0$, and expanding when $\mu > 0$. Pigola et al. defined a new class of Ricci solitons by taking μ is a variable function instead of the constant and then, the Ricci soliton is called an almost Ricci soliton [23]. The study of Ricci solitons has a long history, and a lot of conclusions were acquired, see ([2, 3, 4, 5, 8, 9, 21, 24]) etc.

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On the other hand, the theory of Riemannian submersion was introduced by O'Neill and Gray in [22] and [13], respectively. Later, Riemannian submersions were considered between almost complex manifolds by Watson in [26] under the name of almost Hermitian submersion. He showed that if the total manifold is a Kähler manifold, the base manifold is also a Kähler manifold. Since then, Riemannian submersions have been used as an effective tool to describe the structure of a Riemannian manifold equipped with a differentiable structure. The study of (semi-)Riemannian submersions has a long history, and a lot of conclusions were acquired, see ([1, 7, 11, 14, 15, 16, 19, 20, 25]) etc.
Motivated by the above studies the object of the present paper is to study the almost Hermitian submersion from an almost Hermitian manifold admits a Ricci soliton. The organization of the paper is as follows. After recalling some basic definitions and formulas in the second part, we investigate any fibre of such a submersion is a Ricci soliton or Einstein and then, we get necessary conditions for which the base manifold of an almost Hermitian submersion is a Ricci soliton or Einstein in the last part.

2. Preliminaries

In this section we are going to recall main definitions and properties of almost Hermitian manifolds and Riemannian submersions.

2.1. Almost Hermitian manifolds

In this subsection we recall some basic notions about almost Hermitian submersions from ([27]):
A $(1,1)-$tensor field J on an $2n-$dimensional smooth manifold N is said to be an almost complex structure if $J^2 = -I$. An almost Hermitian manifold (N, g, J) is a smooth manifold endowed with an almost complex structure J and a Riemannian metric g compatible in the sense that

$$g(JX_1, X_2) + g(X_1, JX_2) = 0, \quad X_1, X_2 \in \chi(N).$$

The Kähler form of the almost Hermitian manifold is defined by $\Phi(X_1, X_2) = g(X_1, JX_2)$.

An almost Hermitian manifold is called

(i) *Kähler*, if $\nabla J = 0$;
(ii) *Quasi Kähler* , if $(\nabla_{JX_1} J)JX_2 + (\nabla_{X_1} J)X_2 = 0$;
(iii) *nearly Kähler*, if $(\nabla_{X_1} J)X_1 = 0$;

(iv) *almost Kähler,* if $d\Phi = 0$.

2.2. Almost Hermitian submersions

In this subsection we recall some basic notions about almost Hermitian submersions from ([11, 12, 18, 26]):

Let (N_1, g_1) and (N_2, g_2) be Riemannian manifolds. A Riemannian submersion is a smooth map $\phi : N_1 \to N_2$ which is onto and satisfies the following conditions:
(i) $\phi_{*p} : T_pN_1 \to T_{\phi(p)}N_2$ is onto for all $p \in N_1$;
(ii) The fibres $\psi_x, x \in N_2$, are Riemannian submanifolds of N_1;
(iii) ϕ_{*p} preserves the length of the horizontal vectors.
The vectors tangent to the fibres are called vertical and those normal to the fibres are called horizontal. The tangent bundle of N_1 splits as the Whitney sum of two distributions, the vertical one $ker\phi_* = \mathcal{V}$ and the orthogonal complementary distribution $(ker\phi_*)^\perp = \mathcal{H}$ called horizontal, and we denote by v and h the vertical and horizontal projections. A horizontal vector field X_1 on N_1 is said to be basic if X_1 is $\phi-$related to a vector field X_{1*} on N_2. A Riemannian submersion $\phi : N_1 \to N_2$ determines two $(1, 2)$ tensor fields $\mathcal{T}$ and $\mathcal{A}$ on N_1, by the formulas:

$$\mathcal{T}(X_1, X_2) = \mathcal{T}_{X_1}X_2 = h\nabla^1_{vX_1}vX_2 + v\nabla^1_{vX_1}hX_2 \tag{2}$$

and

$$\mathcal{A}(X_1, X_2) = \mathcal{A}_{X_1}X_2 = v\nabla^1_{hX_1}hX_2 + h\nabla^1_{hX_1}vX_2 \tag{3}$$

for any $X_1, X_2 \in \Gamma(TN_1)$.
It is known that $\mathcal{A}$ is alternating on the horizontal distribution:

$$\mathcal{A}_{X_1}X_2 = -\mathcal{A}_{X_2}X_1,$$

for $X_1, X_2 \in \Gamma((ker\phi_*)^\perp)$ and $\mathcal{T}$ is symmetric on the vertical distribution:

$$\mathcal{T}_{U_1}U_2 = \mathcal{T}_{U_2}U_1,$$

for $U_1, U_2 \in \Gamma(ker\phi_*)$.

Let (N_1, g_1, J_1) and (N_2, g_2, J_2) be almost Hermitian manifolds. A Riemannian submersion $\phi : N_1 \to N_2$ is said an almost Hermitian submersion if ϕ is an almost complex map, i.e. if $\phi_* \circ J_1 = J_2 \circ \phi_*$ holds. Moreover, an almost Hermitian submersion is called a nearly Kähler, a Kähler, an almost Kähler or a quasi Kähler if the total space is nearly Kähler, Kähler, almost Kähler or quasi Kähler, respectively. It is well known that the fibres and the base manifold belong to the same class as the total space N_1.

Let $\phi : N_1 \to N_2$ be an almost Kähler, a Kähler or a nearly Kähler submersion. Then the horizontal distribution is integrable ($\mathcal{A} = 0$). Moreover the horizontal and vertical distributions determined by ϕ are J_1-invariant, i.e. $J_1 ker\phi_* = ker\phi_*$ and $J_1(ker\phi_*)^\perp = (ker\phi_*)^\perp$.

We now recall the following result which will be useful for later.

Lemma 2.1. *If $\phi : N_1 \to N_2$ is an almost Hermitian submersion and X_1, X_2 basic vector fields on N_1, $\phi-$related to X_{1*} and X_{2*} on N_2 then we have the following properties*

1. $g_1(X_1, X_2) = g_2(X_{1*}, X_{2*}) \circ \phi$,
2. *$J_1 X_1$ is the basic vector field associated to $J_2 X_{1*}$,*
3. *$h[X_1, X_2]$ is the basic vector field associated to $\pi_*[X_{1*}, X_{2*}] \circ \phi$,*
4. *$h(\nabla^1_{X_1} X_2)$ is the basic vector field $\phi-$related to $(\nabla^2_{X_{1*}} X_{2*})$, where ∇^1 and ∇^2 are the Levi-Civita connection on N_1 and N_2.*

$\bar{\nabla}$ denoting the Levi-Civita connection on ψ_x the fibre ϕ. For a Kähler submersion, the conditions $\bar{\nabla}\bar{J} = 0$ and $\nabla_2 J_2 = 0$ imply the vanishing of $\nabla^1 J_1$ on the pairs of vector fields which are both vertical or horizontal. Firstly we relate the Ricci tensors of the manifolds (N_1, g_1, J_1), (N_2, g_2, J_2) and of the fibres, respectively denoted by $\tau_1, \tau_2, \bar{\tau}$.
We recall that the Ricci tensor of a nearly Kähler or Kähler manifold is J_1-invariant, i.e.

$$\tau_1(X_1, X_2) = \tau_1(J_1 X_1, J_1 X_2).$$

The same property does not hold for almost Kähler manifolds. The Ricci tensor of an almost Hermitian manifold (N_1, g_1, J_1) is defined by

$$\tau_1^*(X_1, X_2) = \Sigma_{i=1}^{2n} R(X_1, e_i, J_1 X_2, J_1 e_i), \{e_i\}_{1 \le i \le 2n}$$

being a local orthonormal basis, and even if τ_1^* is J_1-invariant, in general τ_1^* is not symmetric.

Given a Kähler submersion, for any $U_1, U_2 \in \Gamma(ker\phi_*)$, we have

$$\tau_1(U_1, U_2) = \bar{\tau}(U_1, U_2). \tag{4}$$

For an almost Kähler submersion, we get

$$\tau_1(U_1, U_2) + \tau_1(J_1 U_1, J_1 U_2) = \bar{\tau}(U_1, U_2) + \bar{\tau}(\bar{J} U_1, \bar{J} U_2), \quad U_1, U_2 \in \Gamma(ker\phi_*). \tag{5}$$

If ϕ is a nearly Kähler submersion, for any $X_1, X_2 \in \Gamma((ker\phi_*)^{\perp})$, we obtain

$$(\tau_1 - \tau_1^*)(X_1, X_2) = (\tau_2 - \tau_2^*)(X_{1*}, X_{2*}) \circ \phi. \tag{6}$$

Let $\phi : (N_1, g_1, J_1) \to (N_2, g_2, J_2)$ be a quasi Kähler submersion with integrable horizontal distribution. The Ricci tensors τ_2^* of N_2 and τ_1^* of N_1 are related by

$$\tau_1^*(X_1, X_2) = \tau_2^*(X_{1*}, X_{2*}) \circ \phi - \Sigma_{j=1}^{2r} g_1(\mathcal{T}_{U_j} X_1, \mathcal{T}_{J_1 U_j} J_1 X_2). \tag{7}$$

for any $X_1, X_2 \in \Gamma((ker\phi_*)^{\perp})$.

Let $\phi : (N_1^4, g_1, J_1) \to (N_2^2, g_2, J_2)$ be an almost Kähler submersion with fibres $(\psi^2, \bar{g}, \bar{J})$. Suppose $\{e, Je\}$ is an orthonormal J_1-basis for the vertical local vector fields on N_1. Then,

(8)

$$\tau_1^*(X_1, X_2) = \tau_2^*(X_{1*}, X_{2*}) \circ \phi + g_1(\mathcal{T}_e X_1, \mathcal{T}_{J_1 e} J_1 X_2) - g_1(\mathcal{T}_{J_1 e} X_1, \mathcal{T}_e J_1 X_2).$$

3. Almost Hermitian submersions from almost Hermitian manifolds admit a Ricci soliton

First we prove the following result.

Theorem 3.1. *Let $(N_1, g_1, J_1, \eta, \mu)$ be a Ricci soliton with the potential field $\eta \in \Gamma(TN_1)$ and $\phi : (N_1, g_1, J_1) \to (N_2, g_2, J_2)$ be a Kähler submersion. Then, we have:*

(i) *If the vector field η is vertical, then any fibre $(\psi, \bar{g}, \bar{J})$ is a Ricci soliton with potential field η,*

(ii) *If the vertical tensor field $\mathcal{T}$ vanishes, then any fibre $(\psi, \bar{g}, \bar{J})$ is an Einstein manifold, for $\eta \in \Gamma((ker\phi_*)^{\perp})$.*

Proof. Since (N_1, g_1, J_1) is a Ricci soliton, for any $U_1, U_2 \in \Gamma(ker\phi_*)$ we obtain

$$\frac{1}{2}(\mathcal{L}_\eta g_1)(U_1, U_2) + \tau_1(U_1, U_2) + \mu g_1(U_1, U_2) = 0.$$

Using (4) we get

$$\frac{1}{2}\{g_1(\nabla^1_{U_1}\eta, U_2) + g_1(\nabla^1_{U_2}\eta, U_1)\} + \bar{\tau}(U_1, U_2) + \mu\bar{g}(U_1, U_2) = 0. \tag{9}$$

(i) Let η be vertical. From (2) and (9), we have

$$\frac{1}{2}\{g_1(v\nabla^1_{U_1}\eta, U_2) + g_1(v\nabla^1_{U_2}\eta, U_1)\} + \bar{\tau}(U_1, U_2) + \mu\bar{g}(U_1, U_2) = 0.$$

Thus, $\frac{1}{2}(\mathcal{L}_\eta \bar{g})(U_1, U_2) + \bar{\tau}(U_1, U_2) + \mu \bar{g}(U_1, U_2) = 0$, which means fibres of ϕ are a Ricci soliton.

(ii) Let η be horizontal. Applying (2) to above equation (9), we arrive at

$$\frac{1}{2}\{g_1(\mathcal{T}_{U_1}\eta, U_2) + g_1(\mathcal{T}_{U_2}\eta, U_1)\} + \bar{\tau}(U_1, U_2) + \mu \bar{g}(U_1, U_2) = 0.$$

Since $\mathcal{T}$ vanishes, we get $\bar{\tau}(U_1, U_2) + \mu \bar{g}(U_1, U_2) = 0$. Thus, any fibre is an Einstein manifold. □

In a similar way, using (5) we get the following result.

Proposition 3.2. *Let $(N_1, g_1, J_1, \eta, \mu)$ be a Ricci soliton with the potential field $\eta \in \Gamma(TN_1)$ and $\phi : (N_1, g_1, J_1) \to (N_2, g_2, J_2)$ be an almost Kähler submersion. Then, we have:*

(i) *If the vector field η is vertical, then any fibre $(\psi, \bar{g}, \bar{J})$ is a Ricci soliton with potential field η,*

(ii) *If the vertical distribution is totally geodesic, then any fibre $(\psi, \bar{g}, \bar{J})$ is an Einstein, for $\eta \in \Gamma((ker\phi_*)^\perp)$.*

For an almost Kähler submersion, $\mathcal{A}$ vanishes.

Theorem 3.3. *Let $(N_1^4, g_1, J_1, \eta, \mu)$ be a Ricci soliton with the potential field $\eta \in \Gamma(TN_1)$ and $\phi : (N_1^4, g_1, J_1) \to (N_2^2, g_2, J_2)$ be an almost Kähler submersion. Then, we have:*

(i) *If the vector field η is basic and $\mathcal{T}$ vanishes, then the base manifold (N_2, g_2, J_2) is a Ricci soliton with potential field η,*

(ii) *If the vector field η is vertical and $\mathcal{T}$ vanishes, then the base manifold (N_2, g_2, J_2) is an Einstein manifold.*

Proof. i) For any $X_1, X_2, \eta \in \Gamma((ker\phi_*)^\perp)$. Since $(N_1^4, g_1, J_1, \eta, \mu)$ is a Ricci soliton, using (1), we obtain

$$\frac{1}{2}(\mathcal{L}_\eta g_1)(X_1, X_2) + \tau_1^*(X_1, X_2) + \mu g_1(X_1, X_2) = 0. \tag{10}$$

On the other hand, let X_1, X_2 and η be basic vector fields on N_1, $\phi-$related to X_{1*}, X_{2*} and η_* on N_2. Applying Lemma 2.1, we get

$$\begin{aligned}(\mathcal{L}_\eta g_1)(X_1,X_2) &= g_1(\nabla^1_{X_1}\eta, X_2) + g_1(\nabla^1_{X_2}\eta, X_1)\\ &= g_1(h\nabla^1_{X_1}\eta + \mathcal{A}_{X_1}\eta, X_2) + g_1(h\nabla^1_{X_2}\eta + \mathcal{A}_{X_2}\eta, X_1)\\ &= g_2(\phi_*(h\nabla^1_{X_1}\eta + \mathcal{A}_{X_1}\eta), \phi_* X_2)\\ &\qquad + g_2(\phi_*(h\nabla^1_{X_2}\eta + \mathcal{A}_{X_2}\eta), \phi_* X_1)\\ &= g_2(\nabla^2_{X_{1*}}\eta_*, X_{2*}) + g_2(\nabla^2_{X_{2*}}\eta_*, X_{1*}).\end{aligned}$$

Thus, we get

$$\phi_*(\frac{1}{2}(\mathcal{L}_\eta g_1)(X_1,X_2)) = \frac{1}{2}(\mathcal{L}_{\eta_*} g_2)(X_{1*}, X_{2*}). \tag{11}$$

From (8), (10) and (11), we have

$$(\frac{1}{2}(\mathcal{L}_{\eta_*} g_2)(X_{1*}, X_{2*}))\circ\phi + \tau_2^*(X_{1*}, X_{2*})\circ\phi + \mu g_2(X_{1*}, X_{2*})\circ\phi = 0,$$

which means the almost Kähler manifold (N_2^2, g_2, J_2) is a Ricci soliton with potential field η_*.

(ii) Let η be vertical. Since $\mathcal{A} = 0$, using(3) we get

$$\begin{aligned}(\mathcal{L}_\eta g_1)(X_1,X_2) &= g_1(\nabla^1_{X_1}\eta, X_2) + g_1(\nabla^1_{X_2}\eta, X_1)\\ &= g_1(v\nabla^1_{X_1}\eta + \mathcal{A}_{X_1}\eta, X_2) + g_1(v\nabla^1_{X_2}\eta + \mathcal{A}_{X_2}\eta, X_1)\\ &= 0,\end{aligned}$$

for any $X_1, X_2, \eta \in \Gamma((ker\phi_*)^\perp)$.
From (1) and (8), using Lemma 2.1 we arrive at

$$\tau_2^*(X_{1*}, X_{2*})\circ\phi + \mu g_2(X_{1*}, X_{2*})\circ\phi = 0,$$

Thus, the base manifold N_2 is an Einstein manifold. □

In a similar way, using (7) we obtain the following result.

Corollary 3.4. *Let $(N_1, g_1, J_1, \eta, \mu)$ be a Ricci soliton with the potential field $\eta \in \Gamma((ker\phi_*)^\perp))$ and $\phi : (N_1, g_1, J_1) \to (N_2, g_2, J_2)$ be a quasi Kähler submersion. If the vertical tensor field $\mathcal{T}$ vanishes , then the base manifold (N_2, g_2, J_2) is a Ricci soliton with potential field η_**

From (6) we have:

Corollary 3.5. *Let $(N_1, g_1, J_1, \eta, \mu)$ be a Ricci soliton with the potential field $\eta \in \Gamma((ker\phi_*)^\perp))$ and $\phi : (N_1, g_1, J_1) \to (N_2, g_2, J_2)$ be a nearly Kähler submersion. Then, the base manifold (N_2, g_2, J_2) is a Ricci soliton with potential field η_*.*

Now, we deal with the harmonicity of nearly Kähler submersion from a Ricci soliton to a nearly Kähler manifold.

Let (N_1, g_1) and (N_2, g_2) be Riemannian manifolds and suppose that $\phi : N_1 \to N_2$ is a smooth map between them. Then, the second fundamental form of ϕ is given by

$$(\nabla\phi_*)(X_1, X_2) = \nabla^{\phi}_{X_1}\phi_*(X_2) - \phi_*(\nabla^1_{X_1}X_2), \tag{12}$$

for any $X_1, X_2 \in \chi(N_1)$. A smooth map $\phi : N_1 \to N_2$ is said to be harmonic if $trace(\nabla\phi_*) = 0$.
Also the tension field of ϕ is given by

$$\sigma(\phi) = div\phi_* = \sum_{i=1}^{m}(\nabla\phi_*)(E_i, E_i), \tag{13}$$

where $E_1, ..., E_m$ is the orthonormal frame on N_1. Then, it follows that ϕ is harmonic if and only if $\sigma(\phi) = 0$, for details, see [10].

Let $\phi : (N_1, g_1, J_1) \to (N_2, g_2, J_2)$ be a nearly Kähler submersion. Since N_1 is a nearly Kähler manifold and using(2), we obtain

$$\mathcal{T}_{U_1}J_1U_2 = J_1\mathcal{T}_{U_1}U_2$$

and

$$\mathcal{T}_{J_1U_1}U_2 = J_1\mathcal{T}_{U_1}U_2$$

for any $U_1, U_2 \in \Gamma(ker\phi_*)$. Then, by direct calculations we have

$$\mathcal{T}_{U_1}U_1 + \mathcal{T}_{J_1U_1}JU_1 = 0. \tag{14}$$

Let $\{E_1, ..., E_k, J_1E_1, ..., J_1E_k\}$ be a the orthonormal frame on $ker\phi_*$. The mean curvature vector field, $\mathcal{H}$, of the fibres is given by

$$\mathcal{H} = \sum_{i=1}^{k}\{\mathcal{T}_{E_i}E_i + \mathcal{T}_{J_1E_i}JE_i\}.$$

Using (14) we have $\mathcal{H} = 0$. Thus the fibre submanifolds are minimally immersed.
From (2), (12), (13) and (14) we get

$$\sigma(\phi) = -\phi_*(\sum_{i=1}^{k}\mathcal{T}_{E_i}E_i + \mathcal{T}_{J_1E_i}JE_i) = 0. \tag{15}$$

Then, ϕ is a harmonic.
From(15) we have the following result.

Theorem 3.6. *Let $(N_1, g_1, J_1, \eta, \mu)$ be a Ricci soliton with the vertical potential field η and $\phi : (N_1, g_1, J_1) \to (N_2, g_2, J_2)$ be a nearly Kähler submersion. Then the nearly Kähler submersion ϕ is harmonic.*

Example 3.7. *Let N_1 be a $4-$dimensional Euclidean space given by $N_1 = \{(x, y, z, w) \in \mathcal{R}^4 : y \in \mathcal{R} - \{k\frac{\pi}{2}, k\pi\}, k \in \mathcal{Z}$ and $x \neq 0\}$. We define the Kähler structure (J_1, g_1) on N_1 given by*

$$g_1 = e^{-2x}((dx)^2 + (dy)^2) + (dz)^2 + (dw)^2$$

and

$$J_1(b_1, b_2, b_3, b_4) = (b_2, -b_1, b_4, -b_3).$$

Let N_2 be $\{(x, v) \in \mathcal{R}^2 : x \neq 0\}$. We choose the Kähler structure (J_2, g_2) on N_2 in the following form

$$g_2 = e^{-4x}((dx)^2 + (dv)^2) \text{ and } J_2(b_1, b_2) = (b_2, -b_1).$$

Now we define the map $\phi : (N_1, g_1, J_1) \to (N_2, g_2, J_2)$ by

$$\phi(x, y, z, w) = (e^x \cos y, e^x \sin y).$$

Then the kernel of ϕ_* is

$$ker\phi_* = Span\{U_1 = \frac{\partial}{\partial z}, U_2 = \frac{\partial}{\partial w}\}$$

and the horizontal distribution is spanned by

$$(ker\phi_*)^\perp = Span\{X_1 = e^x \cos y \frac{\partial}{\partial x} - e^x \sin y \frac{\partial}{\partial y},$$
$$X_2 = e^x \sin y \frac{\partial}{\partial x} + e^x \cos y \frac{\partial}{\partial y}\}.$$

Hence we have

$$g_1(X_1, X_1) = g_2(\phi_* X_1, \phi_* X_2) = 1, \quad g_1(X_2, X_2) = g_2(\phi_* X_2, \phi_* X_2) = 1.$$

Thus, ϕ is a Riemannnian submersion. Moreover, we can easily obtain that ϕ satisfies

$$\phi_* J_1 X_1 = J_2 \phi_* X_1 = -e^{2x} \frac{\partial}{\partial v}$$

and

$$\phi_* J_1 X_2 = J_2 \phi_* X_2 = e^{2x} \frac{\partial}{\partial x}.$$

Thus, ϕ is a Kähler submersion.
By direct calculations we obtain

$$\phi_* X_1 = E_{1*} = e^{2x} \frac{\partial}{\partial x}, \quad \phi_* X_2 = E_{2*} = e^{2x} \frac{\partial}{\partial v}, \; [E_{1*}, E_{2*}] = 2e^{2x} E_{2*}.$$

On the other hand using Koszul's formula for the Riemannian metric g_2, *we get*

$$\nabla^2_{E_{1*}} E_{1*} = \nabla^2_{E_{1*}} E_{2*} = 0,\ \nabla^2_{E_{2*}} E_{1*} = -2e^{2x} E_{2*},\ \nabla^2_{E_{2*}} E_{2*} = 2e^{2x} E_{2*}.$$

For $\eta = U_1 - U_2 \in \Gamma(ker\phi_*)$ *and* $X_* = E_{1*} - E_{2*}$, $Y_* = E_{1*} - 2E_{2*} \in \Gamma((ker\phi_*)^{\perp})$, *by direct calculations we have* $(\frac{1}{2}(\mathcal{L}_\eta g_2)(X_*, Y_*)) \circ \phi = 0$ *and* $\tau_2^*(X_*, Y_*) \circ \phi = 0$.
That is, it is Ricci-flat. Thus, we get $\mu = 0$. *Thus, for a Kähler submersion* ϕ, *the Ricci soliton* $(N_2, g_2, J_2, \eta, \mu)$ *is a steady.*

Definition 3.8. *A vector field* η *on a Riemannian manifold is called concurrent if it satisfies*

$$\nabla_U \eta = kU \tag{16}$$

for any tangent vector U, *where* k *ia a non-zero constant([6]).*

Theorem 3.9. $\phi : (N_1, g_1, J_1) \to (N_2, g_2, J_2)$ *be a Kähler submersion. Then, we get the following.*
(i) For any concurrent vector field η *on* $(ker\phi_*)^{\perp}$, $J_1\eta$ *is not a concurrent vector field on* $(ker\phi_*)^{\perp}$.
(ii) For concurrent vector field $\eta \in \Gamma((ker\phi_*)^{\perp})$, $[\eta, J\eta]$ *vanishes.*

Proof. i) For concurrent vector field $\eta \in \Gamma((ker\phi_*)^{\perp})$, since N_1 is a Kähler manifold, we obtain

$$\nabla^1_X J_1\eta = J_1 \nabla^1_X \eta,\ \ X \in \Gamma((ker\phi_*)^{\perp}).$$

Using (3), we have

$$\nabla^1_X J_1\eta = J_1(h\nabla^1_X \eta + \mathcal{A}_X \eta).$$

From (16) and $\mathcal{A} = 0$ we get

$$\nabla^1_X J_1\eta = J_1(h\nabla^1_X \eta) = kJ_1X \tag{17}$$

which implies that $J_1\eta$ is never a concurrent vector field.
From (3), (16) and (17), we arrive at

$$h\nabla^1_{J_1\eta}\eta = h\nabla^1_{J_1} J_1\eta = kJ_1\eta.$$

Thus, we get $[\eta, J\eta] = 0$. □

Theorem 3.10. *Let* $(N_1, g_1, J_1, \eta, \mu)$ *be a Ricci soliton with the concurrent potential field* $\eta \in \Gamma(TN_1)$ *and* $\phi : (N_1, g_1, J_1) \to (N_2, g_2, J_2)$ *be a Kähler submersion. If the concurrent potential field* η *is vertical and* $(k + \mu) \neq 0$, *then any fibre* $(\psi, \bar{g}, \bar{J})$ *is an Einstein manifold.*

Proof. Since $(N_1, g_1, J_1, \eta, \mu)$ is a Ricci soliton, for any $U_1, U_2 \in \Gamma(ker\phi_*)$ we obtain

$$\frac{1}{2}(\mathcal{L}_\eta g_1)(U_1, U_2) + \tau_1(U_1, U_2) + \mu g_1(U_1, U_2) = 0.$$

Using (4) we get

$$\frac{1}{2}\{g_1(\nabla^1_{U_1}\eta, U_2) + g_1(\nabla^1_{U_2}\eta, U_1)\} + \bar{\tau}(U_1, U_2) + \mu\bar{g}(U_1, U_2) = 0. \tag{18}$$

Let η be vertical. From (2) and (18), we have

$$\frac{1}{2}\{g_1(\mathcal{T}_{U_1}\eta + v\nabla^1_{U_1}\eta, U_2) + g_1(\mathcal{T}_{U_2}\eta + v\nabla^1_{U_2}\eta, U_1)\}$$
$$+ \bar{\tau}(U_1, U_2) + \mu\bar{g}(U_1, U_2) = 0.$$

Using (16), we get

$$\frac{1}{2}\{g_1(kU_1, U_2) + g_1(kU_2, U_1)\} + \bar{\tau}(U_1, U_2) + \mu\bar{g}(U_1, U_2) = 0.$$

Thus, $\bar{\tau}(U_1, U_2) + (k + \mu)\bar{g}(U_1, U_2) = 0$, which means any fibre of ϕ is an Einstein manifold. □

From Theorem 3.10 we have:

Corollary 3.11. *Let $(N_1, g_1, J_1, \eta, \mu)$ be a Ricci soliton with the concurrent potential field $\eta \in \Gamma(TN_1)$ and $\phi : (N_1, g_1, J_1) \to (N_2, g_2, J_2)$ be an almost Kähler submersion. If the concurrent potential field η is vertical and $(k+\mu) \neq 0$, then any fibre $(\psi, \bar{g}, \bar{J})$ is an Einstein manifold.*

Corollary 3.12. *Let $(N_1, g_1, J_1, \eta, \mu)$ be a Ricci soliton with the concurrent potential field $\eta \in \Gamma(TN_1)$ and $\phi : (N_1, g_1, J_1) \to (N_2, g_2, J_2)$ be a Kähler or an almost Kähler submersion. If the concurrent potential field η is vertical and $(k + \mu) \neq 0$, then the scalar curvature of any fibre is a constant.*

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