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CERTAIN GEOMETRIC PROPERTIES OF MODIFIED LOMMEL FUNCTIONS

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Abstract. In this article, we find some sufficient conditions under which the modified Lommel function is close-to-convex with respect to $-\log(1-z)$ and $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$. Starlikeness, convexity and uniformly close-to-convexity of the modified Lommel function are also discussed. Some results related to the H. Silverman are also the part of our investigation.

1. Introduction and preliminaries

The Lommel function of the first kind $s_{\kappa,\eta}(z)$ is a particular solution of the in-homogeneous Bessel differential equation

$$z^2w''(z)+zw'(z)+(z^2-\eta^2)w(z)=z^{\kappa+1}, \tag{1}$$

and it can be expressed in terms of hypergeometric series

$$s_{\kappa,\eta}(z)=\frac{z^{\kappa+1}}{(\kappa-\eta+1)(\kappa+\eta+1)}\ {}_1F_2\left(1;\frac{\kappa-\eta+3}{2},\frac{\kappa+\eta+3}{2};-\frac{z^2}{4}\right),$$

where $\kappa\pm\eta$ is not negative odd integer. It is observed that, Lommel function $s_{\kappa,\eta}$ does not belong to the class $\mathcal{A}$. Thus, the normalized Lommel function of first kind is defined as:

$$\mathcal{L}_{\kappa,\eta}(z) = (\kappa-\eta+1)(\kappa+\eta+1)\,z^{\frac{1-\kappa}{2}}s_{\kappa,\eta}(\sqrt{z}) \tag{2}$$

$$\mathcal{L}_{\kappa,\eta}(z) = \sum_{n=0}^{\infty}\frac{\left(\frac{-1}{4}\right)^n}{(\mathcal{B})_n(\mathcal{V})_n}z^{n+1}, \tag{3}$$

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where $\mathcal{B} = \frac{\kappa-\eta+3}{2}$, $\mathcal{V} = \frac{\kappa+\eta+3}{2}$ and $(a)_n$ shows the Appell symbol which is defined in terms of Eulers gamma functions such that $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)...(a+n-1)$. Clearly the function $\mathcal{L}_{\kappa,\eta}$ belongs to the class $\mathcal{A}$. To discuss the certain properties of normalized Lommel functions with different aspects, here we define modified form of the normalized Lommel functions

$$\begin{aligned} \mathbb{L}(z) &= \frac{z}{1+z} * \mathcal{L}_{\kappa,\eta}(z) = \sum_{n=0}^{\infty} \frac{1}{4^n (\mathcal{B})_n (\mathcal{V})_n} z^{n+1} \\ &= z + \sum_{n=2}^{\infty} \frac{1}{4^{n-1} (\mathcal{B})_{n-1} (\mathcal{V})_{n-1}} z^n. \end{aligned} \tag{4}$$

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{5}$$

analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}^*(\alpha)$, $\mathcal{C}(\alpha)$ and $\mathcal{K}(\alpha)$ denote the classes of starlike, convex and close-to-convex functions of order α respectively and are defined as:

$$\begin{aligned} \mathcal{S}^*(\alpha) &= \left\{ f : f \in \mathcal{A} \text{ and } \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \mathcal{U}, \, \alpha \in [0,1) \right\}, \\ \mathcal{C}(\alpha) &= \left\{ f : f \in \mathcal{A} \text{ and } \Re\left(\frac{(zf'(z))'}{f'(z)}\right) > \alpha, \quad z \in \mathcal{U}, \, \alpha \in [0,1) \right\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{K}(\alpha) = \Bigg\{ f : f \in \mathcal{A} \text{ and } \Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha, \\ z \in \mathcal{U}, \, \alpha \in [0,1), \; g \in \mathcal{S}^*(0) :\equiv \mathcal{S}^* \Bigg\}. \end{aligned}$$

Recently, many researchers studied the geometric properties of some special functions with different approaches. Special functions have great importance in pure and applied mathematics. The wide use of these functions have attracted many researchers to work on the different directions. For details we refer here [2, 3, 7, 10, 14, 15]. Certain conditions for close-to-convexity of some special functions like Bessel functions, q-Mittag-Leffler functions, Wright functions, Dini functions have determined by many mathematicians with different methods (for details see

[2, 4, 11, 12, 13]). To prove our main results, we need the following lemmas:

Lemma 1.1. [20] *If the function* $f(z) = z + a_2z^2 + ... + a_nz^n + ...$ *is analytic in* $\mathcal{U}$ *and in addition* $1 \geq 2a_2 \geq ... \geq na_n \geq ... \geq 0$ *or* $1 \leq 2a_2 \leq ... \leq na_n ... \leq 2$, *then* $f(z)$ *is close-to-convex function with respect to the convex function* $z \to -Log\,(1-z)$.

Lemma 1.2. [20] *If the odd function* $g(z) = z + b_3z^3 + ... + b_{2n-1}z^{2n-1} + ...$ *is analytic in* $\mathcal{U}$ *and if* $1 \geq 3b_3 \geq ... \geq (2n+1)b_{2n+1} ... \geq 0$ *or* $1 \leq 3b_3 \leq ... \leq (2n+1)b_{2n+1} ... \leq 2$, *then* $g(z)$ *is univalent in* $\mathcal{U}$.

We can verify directly that if a function $f : \mathcal{U} \to \mathbb{C}$ satisfies the hypothesis of Lemma 1.1, then it is close-to-convex with respect to the convex function

$$z \mapsto \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

Lemma 1.3. [18] *A function* f *defined in* (5) *belongs to the classes* $\mathcal{S}^*(\alpha)$ *and* $\mathcal{C}(\alpha)$, *if it satisfies the following conditions*

$$\begin{aligned}\sum_{n=2}^{\infty}(n-\alpha)\,|a_n| &\leq 1-\alpha, \qquad \alpha \in [0,1),\\ \sum_{n=2}^{\infty}n\,(n-\alpha)\,|a_n| &\leq 1-\alpha, \qquad \alpha \in [0,1)\end{aligned}$$

respectively.

Lemma 1.4. [6] *If* $f \in \mathcal{A}$ *satisfy* $|f'(z) - 1| < 1$ *for each* $z \in \mathcal{U}$, *then* f *is convex in* $\mathcal{U}_{1/2} = \left\{z : |z| < \frac{1}{2}\right\}$.

Lemma 1.5. [8] *If* $f \in \mathcal{A}$ *satisfy* $\left|\frac{f(z)}{z} - 1\right| < 1$ *for each* $z \in \mathcal{U}$, *then* f *is starlike in* $\mathcal{U}_{1/2} = \left\{z : |z| < \frac{1}{2}\right\}$.

Lemma 1.6. [9] *If* $f \in \mathcal{A}$ *satisfy* $|f'(z) - 1| < \frac{2}{\sqrt{5}}$ *for each* $z \in \mathcal{U}$, *then* f *is starlike in* $\mathcal{U}_{1/2} = \left\{z : |z| < \frac{1}{2}\right\}$.

Lemma 1.7. [16] *Let* $\beta \in \mathbb{C}$ *with* $\Re(\beta) > 0$, $c \in \mathbb{C}$ *with* $|c| \leq 1$, $c \neq -1$. *If* $h \in \mathcal{A}$ *satisfies*

$$\left|c\,|z|^{2\beta} + \left(1 - |z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right| \leq 1, \quad z \in \mathcal{U},$$

then the integral operator

$$C_\beta(z) = \left\{ \beta \int\limits_0^z t^{\beta-1} h'(t)dt \right\}^{1/\beta}, \quad z \in \mathcal{U},$$

is analytic and univalent in $\mathcal{U}$.

Lemma 1.8. [19] *If* $f \in \mathcal{A}$ *satisfies* $\left| \frac{zf''(z)}{f'(z)} \right| < \frac{1}{2}$, *then* $f \in \mathcal{UCV}$.

Lemma 1.9. [5] *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of non negative real numbers such that* $a_1 = 1$. *If* $\{a_n\}_{n=2}^{\infty}$ *is convex decreasing. i.e.* $0 \geq a_{n+2} - a_{n+1} \geq a_{n+1} - a_n$, *then*

$$\Re \left\{ \sum_{n=1}^{\infty} a_n z^{n-1} \right\} > \frac{1}{2}, \quad (z \in \mathcal{U}).$$

2. Close to Convexity of Modified Lommel Functions with Respect to Certain Functions

In this section we will discuss some conditions on the parameters μ and v under which the modified Lommel functions are assured Close-to-Convex with respect to the functions

$$-\log(1-z) \text{ and } \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

By using the Lemma 1.1, we will get the following results.

Theorem 2.1. *If* $\kappa, \eta \in \mathbb{R}^+$ *and* $\kappa \geq \eta$ *with inequality*

$$\mathcal{B} \geq \frac{3}{8(\mathcal{V}+1)} - 1,$$

then $z \to \mathbb{L}(z)$ *is close-to-convex with respect to convex function* $-\log(1-z)$.

Proof. Set

$$\mathbb{L}(z) = z + \sum_{n=2}^{\infty} a_{n-1} z^n.$$

Here $a_{n-1} = \frac{1}{4^{n-1}(\mathcal{B})_{n-1}(\mathcal{V})_{n-1}}$, we have $a_{n-1} > 0$ for all $n \geq 2$ and $a_1 = \frac{1}{4\mathcal{B}\mathcal{V}} < 1$.

From (4) , we have

$$\begin{aligned}\underline{\Omega}a_n &= na_{n-1} - (n+1)a_n \\ &= \frac{4n(\mathcal{B}+n-1)(\mathcal{V}+n-1)}{4^n(\mathcal{B})_n(\mathcal{V})_n}\left\{1 - \frac{n+1}{4n(\mathcal{B}+n-1)(\mathcal{V}+n-1)}\right\} \\ &= \frac{4n(\mathcal{B}+n-1)(\mathcal{V}+n-1)}{4^n(\mathcal{B})_n(\mathcal{V})_n}\varkappa(n),\end{aligned}$$

where

$$\varkappa(n) = 1 - \frac{n+1}{4n(\mathcal{B}+n-1)(\mathcal{V}+n-1)}.$$

In view of Lemma 1.1, we have to show that $\varkappa(n) \geq 0$. By using the inequality $4n \geq n+1$, $\forall n \geq 1$ it is observed that $\varkappa(n) \geq 0$, since $(\mathcal{B}+n-1) \geq 0$ and $(\mathcal{V}+n-1) \geq 0$, $\forall n \geq 2$. Thus $\mathbb{L}(z)$ is close-to-convex with respect to convex function $-\log(1-z)$ with $\mathcal{B} \geq \frac{3}{8(\mathcal{V}+1)} - 1$. □

Theorem 2.2. *If $\kappa, \eta \in \mathbb{R}^+$ and $\kappa \geq \eta$ with inequality*

$$\mathcal{B} \geq \frac{5}{12(\mathcal{V}+1)} - 1,$$

then $z \to \mathbb{L}(z)$ is close-to-convex with respect to convex function $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$.

Proof. Set

$$f(z) = z\mathbb{L}(z^2) = z + \sum_{n=2}^{\infty} A_{2n-1}z^{2n-1}.$$

Here $A_{2n-1} = a_{n-1} = \frac{1}{4^{n-1}(\mathcal{B})_{n-1}(\mathcal{V})_{n-1}}$, therefore we have $a_1 = a_1 = \frac{1}{4\mathcal{B}\mathcal{V}} \leq 1$ and $A_{2n-1} > 0$ for all $n \geq 2$. To prove our main result we will prove that $\{(2n-1)a_{n-1}\}_{n\geq 2}$ is a decreasing sequence. By a short computation we obtain

$$\begin{aligned}\overline{\Omega}a_n &= (2n-1)a_{n-1} - (2n+1)a_n \\ &= \frac{4(2n-1)(\mathcal{B}+n-1)(\mathcal{V}+n-1)}{4^n(\mathcal{B})_n(\mathcal{V})_n} \\ &\qquad \left\{1 - \frac{2n+1}{4(2n-1)(\mathcal{B}+n-1)(\mathcal{V}+n-1)}\right\} \\ &= \frac{4(2n-1)(\mathcal{B}+n-1)(\mathcal{V}+n-1)}{4^n(\mathcal{B})_n(\mathcal{V})_n}\phi(n),\end{aligned}$$

where

$$\phi(n) = 1 - \frac{2n+1}{4(2n-1)(\mathcal{B}+n-1)(\mathcal{V}+n-1)}.$$

In view of Lemma 1.2, we have to show that $\phi(n) \geq 0$. By using the inequality $4(2n-1) \geq 2n+1$, $\forall n \geq 1$ it is observed that $\phi(n) \geq 0$, $\forall n \geq 2$. Thus $\mathbb{L}(z)$ is close-to-convex with respect to convex function $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ with $\mathcal{B} \geq \frac{5}{12(\mathcal{V}+1)} - 1$. □

3. Sufficient Conditions Involving Results of H. Silverman

In this section, we are interested to find out the conditions on the parameters of modified lommel functions, under these conditions $\mathbb{L}$ defined in (4) belongs to the various subclasses of starlike and convex functions. For similar results involving Gaussian hypergeometric functions and Bessel functions, we refer [1, 17].

Theorem 3.1. *If $\kappa, \eta \in \mathbb{R}^+$, $\alpha \in [0,1)$ and $\kappa \geq \eta$ with inequality*

$$\Phi\{\Phi(2-\alpha)+\alpha-1\} \leq 4\Phi\mathcal{B}\mathcal{V}(\Phi-1)(1-\alpha),$$

where, $\Phi = (\mathcal{B}+1)(\mathcal{V}+1)$, then $\mathbb{L}$ defined in (4) belongs to the class $\mathcal{S}^*(\alpha)$.

Proof. Set

$$\mathbb{L}(z) = z + \sum_{n=2}^{\infty} a_{n-1} z^n,$$

here $a_{n-1} = \frac{1}{4^{n-1}(\mathcal{B})_{n-1}(\mathcal{V})_{n-1}}$,we have $a_{n-1} > 0$ for all $n \geq 2$ and $a_1 = \frac{1}{4\mathcal{B}\mathcal{V}} < 1$. From Lemma 1.3 we need only show that

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \leq 1-\alpha.$$

Consider

(6)

$$\sum_{n=2}^{\infty}(n-\alpha)a_{n-1} = \sum_{n=2}^{\infty}\frac{n-1}{4^{n-1}(\mathcal{B})_{n-1}(\mathcal{V})_{n-1}} + \sum_{n=2}^{\infty}\frac{1-\alpha}{4^{n-1}(\mathcal{B})_{n-1}(\mathcal{V})_{n-1}}.$$

By using the expression $(\Psi)_{n-1} = \Psi\,(\Psi+1)_{n-2}$, we may write (6) as follows

$$\begin{aligned}
&\sum_{n=2}^{\infty}(n-\alpha)\,a_{n-1} \\
(7)\qquad &= \frac{1}{4}\sum_{n=2}^{\infty}\frac{4\,(n-1)}{4^{n-1}\mathcal{B}\,(\mathcal{B}+1)_{n-2}\,\mathcal{V}\,(\mathcal{V}+1)_{n-2}} \\
&\qquad + \sum_{n=2}^{\infty}\frac{1-\alpha}{4^{n-1}\mathcal{B}\,(\mathcal{B}+1)_{n-2}\,\mathcal{V}\,(\mathcal{V}+1)_{n-2}}.
\end{aligned}$$

By using the inequalities

$$\begin{aligned}
4^{n-1} &\geq 4\,(n-1)\,,\ n\in\mathbb{N}, \\
(\Psi+1)_{n-1} &\geq (\Psi+1)^{n-1}\,,\ n\in\mathbb{N}
\end{aligned}$$

the expression (7) becomes

$$\begin{aligned}
\sum_{n=2}^{\infty}(n-\alpha)\,a_{n-1} &\leq \frac{1}{4\mathcal{B}\mathcal{V}}\left\{\sum_{n=2}^{\infty}\left(\frac{1}{(\mathcal{B}+1)\,(\mathcal{V}+1)}\right)^{n-2}\right. \\
&\qquad \left. + (1-\alpha)\sum_{n=2}^{\infty}\left(\frac{1}{4\,(\mathcal{B}+1)\,(\mathcal{V}+1)}\right)^{n-2}\right\} \\
&= \frac{\Phi}{4\mathcal{B}\mathcal{V}}\left\{\frac{1}{\Phi-1}+\frac{1-\alpha}{\Phi}\right\}.
\end{aligned}$$

This sum is bounded above by $1-\alpha$ if and only if $\Phi\,\{\Phi\,(2-\alpha)+\alpha-1\} \leq 4\Phi\mathcal{B}\mathcal{V}\,(\Phi-1)\,(1-\alpha)$ holds. □

Theorem 3.2. *If $\kappa,\eta\in\mathbb{R}^{+}$, $\alpha\in[0,1)$ and $\kappa\geq\eta$ with inequality*

$$\Phi\,(2-\alpha) \leq 2\mathcal{B}\mathcal{V}\,(1-\alpha)\,(\Phi-1)\,,$$

where, $\Phi = (\mathcal{B}+1)\,(\mathcal{V}+1)$, then $\mathbb{L}$ defined in (4) belongs to the class $\mathcal{C}\,(\alpha)$.

Proof. Set

$$\mathbb{L}(z) = z + \sum_{n=2}^{\infty}a_{n-1}z^{n},$$

here $a_{n-1} = \frac{1}{4^{n-1}(\mathcal{B})_{n-1}(\mathcal{V})_{n-1}}$,we have $a_{n-1} > 0$ for all $n \geq 2$. From Lemma 1.3 we need only show that

$$\sum_{n=2}^{\infty}n\,(n-\alpha)\,|a_n| \leq 1-\alpha.$$

Consider

$$\sum_{n=2}^{\infty} n\,(n-\alpha)\,a_{n-1} = \sum_{n=0}^{\infty} (n+2)\,(n+2-\alpha)\,a_{n+1} = \rho - \alpha\sigma, \tag{8}$$

where

$$\rho = \sum_{n=0}^{\infty} (n+2)^2\, a_{n+1} \quad \text{and} \quad \sigma = \sum_{n=0}^{\infty} (n+2)\, a_{n+1}.$$

Thus, we have to calculate the conditions under which the $\rho - \alpha\sigma \leq 1-\alpha$. For this by using the inequality $4^{n+1} \geq (n+2)^2$, $\forall\ n \geq 0$ and the expression $(\Psi)_{n-1} = \Psi\,(\Psi+1)_{n-2}$ we obtain

$$\begin{aligned} \rho &\leq \frac{1}{\mathcal{B}\mathcal{V}} \sum_{n=0}^{\infty} \left(\frac{1}{(\mathcal{B}+1)\,(\mathcal{V}+1)} \right)^n \\ &= \frac{(\mathcal{B}+1)\,(\mathcal{V}+1)}{\mathcal{B}\mathcal{V}\,\{(\mathcal{B}+1)\,(\mathcal{V}+1)-1\}}. \end{aligned} \tag{9}$$

Analogously, by using the inequality $4^{n+1} \geq 2\,(n+2)$, $\forall\ n \geq 0$ and the expression $(\Psi)_{n-1} = \Psi\,(\Psi+1)_{n-2}$ we obtain

$$\begin{aligned} \sigma &\leq \frac{1}{2\mathcal{B}\mathcal{V}} \sum_{n=0}^{\infty} \left(\frac{1}{(\mathcal{B}+1)\,(\mathcal{V}+1)} \right)^n \\ &= \frac{(\mathcal{B}+1)\,(\mathcal{V}+1)}{2\mathcal{B}\mathcal{V}\,\{(\mathcal{B}+1)\,(\mathcal{V}+1)-1\}}. \end{aligned} \tag{10}$$

Therefore the expression $\rho - \alpha\sigma$ with the help of (9) and (10) becomes

$$\rho - \alpha\sigma \leq \frac{(\mathcal{B}+1)\,(\mathcal{V}+1)}{\mathcal{B}\mathcal{V}\,\{(\mathcal{B}+1)\,(\mathcal{V}+1)-1\}} - \frac{(\mathcal{B}+1)\,(\mathcal{V}+1)}{2\mathcal{B}\mathcal{V}\,\{(\mathcal{B}+1)\,(\mathcal{V}+1)-1\}},$$

is bounded above by $1-\alpha$ if and only if $\Phi\,(2-\alpha) \leq 2\mathcal{B}\mathcal{V}\,(1-\alpha)\,(\Phi-1)$, where, $\Phi = (\mathcal{B}+1)\,(\mathcal{V}+1)$ holds. □

Theorem 3.3. *If $\kappa, \eta \in \mathbb{R}^+$ and $\kappa \geq \eta$ with inequalities*

$$\begin{aligned} 4\,(\mathcal{B}+n-2)\,(\mathcal{V}+n-2) &\geq 1, \\ (\mathcal{B}+n-2)\,(\mathcal{V}+n-2)\,(\mathcal{B}+n-1)\,(\mathcal{V}+n-1)+1 &\geq 8\,(\mathcal{B}+n-1) \\ &\qquad (\mathcal{V}+n-1), \end{aligned}$$

then

$$\Re\left\{ \frac{\mathbb{L}(z)}{z} \right\} > \frac{1}{2}, \quad \forall\ z \in \mathcal{U}.$$

Proof. To prove this, we have to show that

$$\{a_n\}_{n=1}^{\infty} = \left\{ \frac{1}{4^{n-1} (\mathcal{B})_{n-1} (\mathcal{V})_{n-1}} \right\}_{n=1}^{\infty} \tag{11}$$

is a convex decreasing sequence. For this

$$\begin{aligned} a_n - a_{n+1} &= \frac{1}{4^{n-1}} \left\{ \frac{1}{\mathcal{B}(\mathcal{B}+1)_{n-2}\, \mathcal{V}(\mathcal{V}+1)_{n-2}} \right. \\ &\qquad \left. - \frac{1}{4\mathcal{B}(\mathcal{B}+1)_{n-1}\, \mathcal{V}(\mathcal{V}+1)_{n-1}} \right\} \\ &= \frac{1}{4^{n-1}\mathcal{B}\mathcal{V}} \left\{ \frac{4(\mathcal{B}+n-2)(\mathcal{V}+n-2) - 1}{4(\mathcal{B}+1)_{n-1}(\mathcal{V}+1)_{n-1}} \right\} > 0, \end{aligned}$$

where $\forall\ m \geq 1$, $\mathcal{B} \geq \frac{3}{2}$, $\mathcal{V} \geq \frac{3}{2}$ and $4(\mathcal{B}+n-2)(\mathcal{V}+n-2) \geq 1$. Now, to show that the sequence defined in (11) is convex decreasing, we prove that $a_n - 2a_{n+1} + a_{n+2} \geq 0$.

Take

$$\begin{aligned} & a_n - 2a_{n+1} + a_{n+2} \\ = & \frac{1}{4^{n-1}(\mathcal{B})_{n-1}(\mathcal{V})_{n-1}} - \frac{2}{4^n (\mathcal{B})_n (\mathcal{V})_n} + \frac{1}{4^{n+1}(\mathcal{B})_{n+1}(\mathcal{V})_{n+1}} \\ = & \frac{1}{4^{n-1}\mathcal{B}\mathcal{V}} \left\{ \frac{\begin{array}{c}(\mathcal{B}+n-2)(\mathcal{V}+n-2)(\mathcal{B}+n-1)(\mathcal{V}+n-1) \\ -8(\mathcal{B}+n-1)(\mathcal{V}+n-1)+1\end{array}}{16(\mathcal{B}+1)_n(\mathcal{V}+1)_n} \right\} > 0. \end{aligned}$$

The above expression is non-negative $\forall\ m \geq 1$, $\mathcal{B} \geq \frac{3}{2}$, $\mathcal{V} \geq \frac{3}{2}$ and

$$\begin{aligned} (\mathcal{B}+n-2)(\mathcal{V}+n-2)(\mathcal{B}+n-1)(\mathcal{V}+n-1) + 1 \\ \geq 8(\mathcal{B}+n-1)(\mathcal{V}+n-1), \end{aligned}$$

which shows that the sequence defined in (11) is decreasing and convex. From Lemma 1.9

$$\Re\left(\sum_{n=1}^{\infty} a_n z^{n-1}\right) > 1/2, \quad \forall\ z \in \mathcal{U},$$

which is equivalent to

$$\Re\left\{\frac{\mathbb{L}(z)}{z}\right\} > \frac{1}{2}, \quad \forall\ z \in \mathcal{U}.$$

□

4. Starlikeness and Convexity of Modified Lommel Functions in $\mathcal{U}_{1/2}$

Theorem 4.1. *Let $\kappa, \eta \in \mathbb{R}^+$ and $\kappa \geq \eta$. Then the following assertions are true:*

(i) If $\frac{(\mathcal{B}+1)(\mathcal{V}+1)}{4(\mathcal{B}+1)(\mathcal{V}+1)-1} < \mathcal{BV}$, then $\mathbb{L}_{\kappa,\eta}$ is starlike in $\mathcal{U}_{1/2}$.

(ii) If $\frac{(\mathcal{B}+1)(\mathcal{V}+1)}{(\mathcal{B}+1)(\mathcal{V}+1)-1} < 2\mathcal{BV}$, then $\mathbb{L}_{\kappa,\eta}$ is convex in $\mathcal{U}_{1/2}$.

Proof. (i) By using the well-known triangle inequality and the inequality $(\mathcal{B})_n \geq (\mathcal{B})^n$, for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned}\left|\frac{\mathbb{L}_{\kappa,\eta}(z)}{z} - 1\right| &\leq \left|\sum_{n=1}^{\infty} \frac{1}{4^n (\mathcal{B})_n (\mathcal{V})_n} z^n\right| \\ &\leq \frac{1}{4\mathcal{BV}} \sum_{n=1}^{\infty} \left(\frac{1}{4(\mathcal{B}+1)(\mathcal{V}+1)}\right)^{n-1} \\ &\leq \frac{(\mathcal{B}+1)(\mathcal{V}+1)}{\mathcal{BV}\{4(\mathcal{B}+1)(\mathcal{V}+1)-1\}}.\end{aligned}$$

In view of Lemma 1.5, $\mathbb{L}_{\kappa,\eta}$ is starlike in $\mathcal{U}_{1/2}$, if $\frac{(\mathcal{B}+1)(\mathcal{V}+1)}{\mathcal{BV}\{4(\mathcal{B}+1)(\mathcal{V}+1)-1\}} < 1$, but this is true under the hypothesis.

(ii) Conisder,

$$\left|\mathbb{L}'_{\kappa,\eta}(z) - 1\right| \leq \sum_{n=1}^{\infty} \frac{n+1}{4^n (\mathcal{B})_n (\mathcal{V})_n}.$$

Since, $4^n \geq 2(n+1)$, for all $n \in \mathbb{N}$ and $(\mathcal{B})_n \geq (\mathcal{B})^n$, for all $n \in \mathbb{N}$. Therefore

$$\begin{aligned}\left|\mathbb{L}'_{\kappa,\eta}(z) - 1\right| &\leq \frac{1}{2\mathcal{BV}} \sum_{n=1}^{\infty} \left(\frac{1}{(\mathcal{B}+1)(\mathcal{V}+1)}\right)^{n-1} \\ &= \frac{(\mathcal{B}+1)(\mathcal{V}+1)}{2\mathcal{BV}\{(\mathcal{B}+1)(\mathcal{V}+1)-1\}}.\end{aligned}$$

In view of Lemma 1.4, $\mathbb{L}_{\kappa,\eta}$ is convex in $\mathcal{U}_{1/2}$, if $\frac{(\mathcal{B}+1)(\mathcal{V}+1)}{2\mathcal{BV}\{(\mathcal{B}+1)(\mathcal{V}+1)-1\}} < 1$, but this is true under the hypothesis. □

Theorem 4.2. *Let $\alpha \in [0,1)$, $\kappa, \eta \in \mathbb{R}^+$ and $\kappa \geq \eta$,*

$$\mathcal{G} = 4(\mathcal{B}+1)(\mathcal{V}+1) = (\kappa+5)^2 - \eta^2$$

and $\mathcal{H} = 4\mathcal{BV} = (\kappa+3)^2 - \eta^2$. Then for all $z \in \mathcal{U}$ the following assertions are true:

(i) If $\kappa > -5+\sqrt{2+\eta^2}$ *and* $\frac{\mathcal{G}(\mathcal{G}-1)}{(\mathcal{G}-2)(\mathcal{GH}-\mathcal{G}-\mathcal{H})} < 1-\alpha$, *then* $\mathbb{L}_{\kappa,\eta} \in \mathcal{S}^*(\alpha)$.

(ii) If $\kappa > -5+\sqrt{3+\eta^2}$ *and* $\frac{2\mathcal{G}(2\mathcal{G}-3)}{(\mathcal{G}-3)(2\mathcal{GH}-4\mathcal{G}-3\mathcal{H})} < 1-\alpha$, *then* $\mathbb{L}_{\kappa,\eta} \in \mathcal{C}^*(\alpha)$.

Proof. The proof of the above Theorem is similar as Theorem 2.1 in [21]. □

Consider the integral operator $\mathcal{F}_\beta : \mathcal{U} \to \mathbb{C}$, where $\beta \in \mathbb{C}$, $\beta \neq 0$,

$$\mathcal{F}_\beta(z) = \left\{ \beta \int_0^z t^{\beta-2}\mathbb{L}_{\kappa,\eta}(t)\,dt \right\}^{\frac{1}{\beta}}, \qquad z \in \mathcal{U}.$$

Here $\mathcal{F}_\beta \in \mathcal{A}$. In the next theorem, we obtain the conditions so that $\mathcal{F}_\beta$ is univalent in $\mathcal{U}$.

Theorem 4.3. *Let* $\kappa, \eta \in \mathbb{R}^+$ *and* $\kappa \geq \eta$. *Let* $(\mathcal{G}-2)(\mathcal{GH}-\mathcal{G}-\mathcal{H}) > \mathcal{G}(\mathcal{G}-1)$ *and suppose that* M *is a positive real number such that* $|\mathbb{L}_{\kappa,\eta}(z)| \leq M$ *in the open unit disc. If*

$$|\beta-1| + \frac{\mathcal{G}(\mathcal{G}-1)}{(\mathcal{G}-2)(\mathcal{GH}-\mathcal{G}-\mathcal{H})} + \frac{M}{|\beta|} \leq 1,$$

then $\mathcal{F}_\beta$ *is univalent in* $\mathcal{U}$.

Proof. A calculations gives us

$$\frac{z\mathcal{F}_\beta''(z)}{\mathcal{F}_\beta'(z)} = \frac{z\mathbb{L}_{\kappa,\eta}'(z)}{\mathbb{L}_{\kappa,\eta}(z)} + \frac{z^{\beta-1}}{\beta}\mathbb{L}_{\kappa,\eta}(z) + \beta - 2, \ z \in \mathcal{U}.$$

Since $\mathbb{L}_{\kappa,\eta} \in \mathcal{A}$, then by the Schwarz Lemma, triangle inequality and Theorem (4.2), we obtain

$$\begin{aligned}
&\left(1-|z|^2\right)\left|\frac{z\mathcal{F}_\beta''(z)}{\mathcal{F}_\beta'(z)}\right| \\
\leq\ & \left(1-|z|^2\right)\left\{|\beta-1| + \left|\frac{z\mathbb{L}_{\kappa,\eta}'(z)}{\mathbb{L}_{\kappa,\eta}(z)} - 1\right| + \frac{|z|^{\mathcal{R}(\beta)}}{|\beta|}\left|\frac{\mathbb{L}_{\kappa,\eta}(z)}{z}\right|\right\} \\
\leq\ & \left(1-|z|^2\right)\left\{|\beta-1| + \frac{\mathcal{G}(\mathcal{G}-1)}{(\mathcal{G}-2)(\mathcal{GH}-\mathcal{G}-\mathcal{H})} + \frac{M}{|\beta|}\right\} \\
\leq\ & 1.
\end{aligned}$$

This shows that the given integral operator satisfies the Becker's criterion for univalence, hence $\mathcal{F}_\beta$ is univalent in $\mathcal{U}$. □

5. Uniformly Convexity of Modified Lommel Functions

Theorem 5.1. *Let* $\kappa, \eta \in \mathbb{R}^{+}$ *and* $\kappa \geq \eta$. *If* $(\mathcal{G}-3)(2\mathcal{G}\mathcal{H}-4\mathcal{G}-3\mathcal{H}) > 4\mathcal{G}(2\mathcal{G}-3)$, *then* $\mathbb{L}_{\kappa,\eta} \in \mathcal{UCV}$.

Proof. Since from Theorem 4.2

$$\left|\frac{z\mathbb{L}''_{\kappa,\eta}(z)}{\mathbb{L}'_{\kappa,\eta}(z)}\right| \leq \frac{2\mathcal{G}(2\mathcal{G}-3)}{(\mathcal{G}-3)(2\mathcal{G}\mathcal{H}-4\mathcal{G}-3\mathcal{H})}.$$

By using Lemma 1.8, we have

$$\left|\frac{z\mathbb{L}''_{\kappa,\eta}(z)}{\mathbb{L}'_{\kappa,\eta}(z)}\right| < \frac{1}{2},$$

if

$$\frac{2\mathcal{G}(2\mathcal{G}-3)}{(\mathcal{G}-3)(2\mathcal{G}\mathcal{H}-4\mathcal{G}-3\mathcal{H})} < \frac{1}{2}.$$

This implies that

$$(\mathcal{G}-3)(2\mathcal{G}\mathcal{H}-4\mathcal{G}-3\mathcal{H}) > 4\mathcal{G}(2\mathcal{G}-3).$$

Hence the required result. □

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