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SOME DISTORTION THEOREMS FOR NEW SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS

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Abstract. We introduced and studied a new class of harmonic univalent functions on unit disc $\mathbb{U}$. Also we provided coefficient conditions, extreme points and convolution conditions for that class of harmonic univalent functions.

1. Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic. In any simply connected domain $B \subset \mathbb{C}$, we can write $f = h + g$, where h and g are analytic in B. We call h and g are analytic part and co-analytic part of f respectively. Clunie and Sheil-Small [5] observed that a necessary and sufficient condition for the harmonic functions $f = h + g$ to be locally univalent and sense-preserving in B is that $|h'(z)| > |g'(z)|$ for all $z \in B$. Denote by $\mathcal{S}_H$ the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$.

In 1984, Clunie and Sheil-Small [5] investigated the class $\mathcal{S}_H$ as well as its geometric subclasses and obtained some coefficient bounds. They proved that although $\mathcal{S}_H$ is not compact, it is normal with respect to the topology of uniform convergence on compact subsets of $\mathbb{U}$. Meanwhile the subclass $\mathcal{S}_H^0$ of $\mathcal{S}_H$ consisting of the functions having the property $f_{\bar{z}}(0) = 0$ is compact.

Some of the subclasses studied are as follows:

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(i) The classes $\mathcal{S}_H(1,0;0,0) = \mathcal{S}_H$ and $\mathcal{S}_H(2,1;0,0) = \mathcal{C}_H$ which is studied by Avci and Zlotkiewicz in [4].
(ii) The classes $\mathcal{S}_H(1,0;\alpha,0) = \mathcal{S}_H(\alpha)$ and $\mathcal{S}_H(2,1;\alpha,0) = \mathcal{C}_H(\alpha)$ which is studied by Özturk and Yalçin in [13].
(iii) The class $\mathcal{S}_H(m,n;\alpha,0) = \mathcal{S}_H(m,n,\alpha)$ which is studied by Dixit *et al.* in [8].
(iv) The class $\mathcal{S}_H(1,0;\alpha,\beta) = \mathcal{S}_H(\alpha,\beta)$ which is studied by Seoudy in [16].
(v) The class $\mathcal{S}_H(n+1,n;\alpha,0) = \mathcal{S}_H(\alpha,n)$ which is studied by Aouf *et al. in [15].*

In section 2, we denote some fundamental definitions, theorems, and lemmas and in section 3, we investigate several properties of the classes $\mathcal{S}_H(m,n;\alpha,\beta)$ and $\mathcal{S}^0_H(m,n;\alpha,\beta)$. Also, we generalize, improve, and correct some results of Özturk and Yalçin [13]. More recent works in this area can be found in [1] and [2].

2. Preliminaries and Definitions

We begin with the basic definition on harmonic univalent functions.

Definition 2.1. *A harmonic, complex-valued, orientation preserving, univalent mapping f defined on $\mathbb{U}$ can be written as:*

$$f = h + \overline{g}, \tag{1}$$

where

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{2}$$

We call h the analytic part and g the co-analytic part of f.

Denote by $\mathcal{S}_H(m,n;\alpha,\beta)$ the class of all functions of the form (1) that satisfy the following inequality:

$$\sum_{k=2}^{\infty} (k^m - \alpha k^n)(1 - \beta + \beta k)(|a_k| + |b_k|) \le (1-\alpha)(1-|b_1|), \tag{3}$$

where $m \in \mathbb{N}, n \in \mathbb{N}_0, m > n, 0 \le \alpha < 1, \beta \ge 0$ and $0 \le |b_1| < 1$.

The class $\mathcal{S}_H(m,n;\alpha,\beta)$ with $|b_1| = 0$ will be denoted by $\mathcal{S}^0_H(m,n;\alpha,\beta)$.

Definition 2.2. *If h, g are the form (2) and if $f = h+\overline{g}, F = H+\overline{G}$, then the convolution of f and F is defined to be the function*

$$(f * F)(z) = z + \sum_{k=2}^{\infty} a_k A_k z^k + \overline{\sum_{k=1}^{\infty} b_k B_k z^k}, \tag{4}$$

and the integral convolution is defined by:

$$(f \Diamond F)(z) = z + \sum_{k=2}^{\infty} \frac{a_k A_k}{k} z^k + \overline{\sum_{k=1}^{\infty} \frac{b_k B_k}{k} z^k}, \tag{5}$$

where

$$H(z) = z + \sum_{k=2}^{\infty} A_k z^k, \quad G(z) = \sum_{k=1}^{\infty} B_k z^k. \tag{6}$$

Özturk and Yalçin [13], defined the generalized δ-neighborhood of f to be the set:

$$N(f) = \left\{ F : \sum_{k=2}^{\infty} (k-\alpha)(|a_k - A_k| + |b_k - B_k|) + (1-\alpha)|b_1 - B_1| \leq \delta(1-\alpha) \right\},$$

where $F = H + \overline{G}$, and H, G are the form (6).

Theorem 2.3. *For harmonic univalent mapping f as above, $|a_1| \leq 2$ and ,*

$$|a_n| = |b_n| \leq 2/n, \quad \text{for all} \quad n \geq 2.$$

Complete proof is explained in theorem 4.7 of [11].

3. Main results

We start this section with the most important following theorem.

Theorem 3.1. *For $0 \leq \alpha_1 \leq \alpha_2 < 1$, we have*

$$\mathcal{S}_H(m, n; \alpha_2, \beta) \subseteq \mathcal{S}_H(m, n; \alpha_1, \beta),$$

and

$$\mathcal{S}_H^0(m, n; \alpha_2, \beta) \subseteq \mathcal{S}_H^0(m, n; \alpha_1, \beta).$$

In particular, we have

$$\mathcal{S}_H(m, n; \alpha, \beta) \subseteq \mathcal{S}_H(m, n; 0, \beta),$$

and

$$\mathcal{S}_H^0(m,n;\alpha,\beta) \subseteq \mathcal{S}_H^0(m,n;0,\beta),$$

where $m \in \mathbb{N}, n \in \mathbb{N}_0, m > n$ *and* $\beta \geq 0$.

Proof. Let $f \in \mathcal{S}_H(m,n;\alpha_2,\beta)$ and

$$A_1^i = \sum_{k=2}^{\infty} \frac{(k^m - \alpha_i k^n)(1-\beta+\beta k)}{1-\alpha_i}.$$

Thus we have

$$A_1^2(|a_k| + |b_k|) \leq 1 - |b_1|. \tag{7}$$

Now, using (7), we have

$$A_1^1(|a_k| + |b_k|) \leq A_1^2(|a_k| + |b_k|) \leq 1 - |b_1|.$$

□

The following theorem provides a more particular characterization.

Theorem 3.2. *The following statements hold:*

(i) $\mathcal{S}_H(m,n;\alpha,\beta) \subseteq \mathcal{S}_H(\alpha)$ *for* $m,n \in \mathbb{N}, m > n, 0 \leq \alpha < 1, \beta \geq 0$.
(ii) $\mathcal{S}_H(m,n;\alpha,\beta) \subseteq \mathcal{C}_H(\alpha)$ *if* $\beta \geq 1$ *or* $m \geq 2, n \geq 1$ *and* $m > n$.

Proof. Let $f \in \mathcal{S}_H(m,n;\alpha,\beta)$, Then we have

$$\sum_{k=2}^{\infty} \frac{(k^m - \alpha k^n)(1-\beta+\beta k)}{1-\alpha}(|a_k| + |b_k|) \leq 1 - |b_1|. \tag{8}$$

Now, using (8), we have

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{k-\alpha}{1-\alpha}(|a_k| + |b_k|) \\
&\leq \sum_{k=2}^{\infty} \frac{(k^m - \alpha k^n)(1-\beta+\beta k)}{1-\alpha}(|a_k| + |b_k|) \\
&\leq 1 - |b_1|.
\end{aligned}$$

Therefore, $f \in \mathcal{S}_H(\alpha)$ and we take $\mathcal{S}_H(m,n;\alpha,\beta) \subseteq \mathcal{S}_H(\alpha)$.
We have to show that $\mathcal{S}_H(m,n;\alpha,\beta) \subseteq \mathcal{C}_H(\alpha)$. By using the inequality (8), we have

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{k(k-\alpha)}{1-\alpha}(|a_k|+|b_k|) \\
&\leq \sum_{k=2}^{\infty} \frac{(k^m-\alpha k^n)(1-\beta+\beta k)}{1-\alpha}(|a_k|+|b_k|) \\
&\leq 1-|b_1|,
\end{aligned}$$

where $\beta \geq 1$ or $m \geq 2, n \geq 1$ and $m > n$. Therefore, $f \in \mathcal{C}_H(\alpha)$ and we get $\mathcal{S}_H(m,n;\alpha,\beta) \subseteq \mathcal{C}_H(\alpha)$. □

The next worthy point is to check the preserving property.

Theorem 3.3. *The class $\mathcal{S}_H(m,n;\alpha,\beta)$ consists of univalent sense preserving harmonic mappings.*

Proof. If $z_1 \neq z_2$ then we have

$$\begin{aligned}
\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| \\
&= 1-\left|\frac{\sum\limits_{k=1}^{\infty} b_k(z_1^k-z_2^k)}{(z_1-z_2)+\sum\limits_{k=2}^{\infty} a_k(z_1^k-z_2^k)}\right| \\
&> 1-\left|\frac{\sum\limits_{k=1}^{\infty} k|b_k|}{1-\sum\limits_{k=2}^{\infty} k|a_k|}\right| \\
&\geq 1-\frac{\sum\limits_{k=1}^{\infty} \frac{(k^m-\alpha k^n)(1-\beta+\beta k)}{1-\alpha}|b_k|}{1-\sum\limits_{k=2}^{\infty} \frac{(k^m-\alpha k^n)(1-\beta+\beta k)}{1-\alpha}|a_k|} \geq 0,
\end{aligned}$$

which proves f is a univalent function. Note that f is sense preserving in $\mathbb{U}$ because

$$\begin{aligned}
|h'(z)| &\geq 1 - \sum_{k=2}^{\infty} k|a_k||z|^k \\
&> 1 - \sum_{k=2}^{\infty} \frac{(k^m - \alpha k^n)(1-\beta+\beta k)}{1-\alpha}|a_k| \\
&\geq \sum_{k=2}^{\infty} \frac{(k^m - \alpha k^n)(1-\beta+\beta k)}{1-\alpha}|b_k| \\
&> \sum_{k=2}^{\infty} \frac{(k^m - \alpha k^n)(1-\beta+\beta k)}{1-\alpha}|b_k||z|^{k-1} \\
&\geq \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \geq |g'(z)|.
\end{aligned}$$

In this way, the proof of Theorem 3.3 is completed. □

One of the interesting problems in this part is study the bounds of $|f(z)|$.

Theorem 3.4. *If $f \in \mathcal{S}_H(m,n;\alpha,\beta)$, then we have*

$$(1-|b_1|)(|z| - \psi|z|^2) \leq |f(z)| \leq (1+|b_1|)|z| + \psi(1-|b_1|)|z|^2, \tag{9}$$

where $\psi = \dfrac{1-\alpha}{(1+\beta)(2^m - 2^n\alpha)}$, *and equalities are attained by the functions:*

$$f_\theta(z) = z + |b_1|e^{i\theta}\bar{z} + (1-|b_1|)\psi z^2, \tag{10}$$

and

$$f_\theta(z) = z + |b_1|e^{i\theta}\bar{z} + (1-|b_1|)\psi \bar{z}^2, \tag{11}$$

for properly chosen real θ.

Proof. We have

(12)

$$\begin{aligned}|f(z)| &\le (1+|b_1|)|z| + |z|^2 \sum_{k=2}^{\infty}(|a_k|+|b_k|)\\ &\le (1+|b_1|)|z|\\ &\quad + |z|^2 \frac{1-\alpha}{(1+\beta)(2^m-2^n\alpha)} \sum_{k=2}^{\infty} \frac{(1-\beta+\beta k)(k^m-k^n\alpha)}{1-\alpha}(|a_k|+|b_k|)\\ &\le (1+|b_1|)|z| + |z|^2 \frac{1-\alpha}{(1+\beta)(2^m-2^n\alpha)}(1-|b_1|).\end{aligned}$$

and

(13)

$$\begin{aligned}|f(z)| &\ge (1-|b_1|)|z| - \sum_{k=2}^{\infty}(|a_k|+|b_k|)|z|^k\\ &\ge (1-|b_1|)|z| - |z|^2 \sum_{k=2}^{\infty}(|a_k|+|b_k|)\\ &\ge (1-|b_1|)|z|\\ &\quad - |z|^2 \frac{1-\alpha}{(1+\beta)(2^m-2^n\alpha)} \sum_{k=2}^{\infty} \frac{(1-\beta+\beta k)(k^m-k^n\alpha)}{1-\alpha}(|a_k|+|b_k|)\\ &\ge (1-|b_1|)|z| - |z|^2 \frac{1-\alpha}{(1+\beta)(2^m-2^n\alpha)}(1-|b_1|)\\ &= (1-|b_1|)\Big(|z| - |z|^2 \frac{1-\alpha}{(1+\beta)(2^m-2^n\alpha)}\Big).\end{aligned}$$

We define $\psi := \dfrac{1-\alpha}{(1+\beta)(2^m-2^n\alpha)}$ in inequalities (12) and (13), and so attained (9). It can be easily seen that the functions $f_\theta(z)$ defined by (10) and (11) are extremal for Theorem 3.4.
Thus the class $\mathcal{S}_H(m,n;\alpha,\beta)$ is uniformly bounded, and hence it is normal by Montel's Theorem. □

Putting $m=1, n=0$ and $\beta=0$ in Theorem 3.4, we obtain the following result which correct the result of Özturk and Yalçin [13, Theorem 3.6].

Corollary 3.5. *If $f \in \mathcal{S}_H(\alpha)$, then we have*

$$(1-|b_1|)(|z|-\frac{1-\alpha}{2-\alpha}|z|^2) \leq |f(z)| \leq (1+|b_1|)|z| + \frac{(1-\alpha)(1-|b_1|)}{2-\alpha}|z|^2. \tag{14}$$

Equalities are attained by the functions:

$$f_\theta(z) = z + |b_1|e^{i\theta}\bar{z} + \frac{(1-\alpha)(1-|b_1|)}{2-\alpha}z^2, \tag{15}$$

and

$$f_\theta(z) = z + |b_1|e^{i\theta}\bar{z} + \frac{(1-\alpha)(1-|b_1|)}{2-\alpha}\bar{z}^2, \tag{16}$$

for properly chosen real θ.

Remark 3.6. *The above result is different from that of Özturk and Yalçin [13, Theorem 3.6]. Also, our result gives a better estimate than that of [13] because*

$$\begin{aligned} |f(z)| &\leq (1+|b_1|)|z| + \frac{(1-\alpha)(1-|b_1|)}{2-\alpha}|z|^2 \\ &\leq (1+|b_1|)|z| + \frac{(1-\alpha^2)(1-|b_1|)}{2}|z|^2, \end{aligned}$$

and

$$|f(z)| \geq (1-|b_1|)(|z|-\frac{1-\alpha}{2-\alpha}|z|^2) \geq (1-|b_1|)\Big(|z|-\frac{1-\alpha^2}{2}|z|^2\Big).$$

Although, Özturk and Yalçin [13] state that the result is sharp for the function

$$f_\theta(z) = z + |b_1|e^{i\theta}\bar{z} + \frac{(1-\alpha^2)(1-|b_1|)}{2}\bar{z}^2.$$

It can be easily seen that the function $f_\theta(z)$ does not satisfy the coefficient condition for the class $f \in \mathcal{S}_H(\alpha)$ defined by them. Hence, the function $f_\theta(z)$ does not belong to the class $f \in \mathcal{S}_H(\alpha)$. Therefore the result of Özturk and Yalçin [13, Theorem 3.6] is incorrect. The modified result is mentioned in (14) and that is sharp for functions given by (15) and (16), respectively.

Putting $m = 2, n = 1$ and $\beta = 0$ in Theorem 3.4, we obtain the following result which correct the result of Özturk and Yalçin [13, Theorem 3.6].

Corollary 3.7. *If $f \in \mathcal{C}_H(\alpha)$, then we have*

$$(1-|b_1|)(|z| - \frac{1-\alpha}{2(2-\alpha)}|z|^2) \leq |f(z)|$$

$$\leq (1+|b_1|)|z| + \frac{(1-\alpha)(1-|b_1|)}{2(2-\alpha)}|z|^2. \tag{17}$$

In particular, the equalities are attained by the functions:

$$f_\theta(z) = z + |b_1|e^{i\theta}\bar{z} + \frac{(1-\alpha)(1-|b_1|)}{2(2-\alpha)}z^2, \tag{18}$$

and

$$f_\theta(z) = z + |b_1|e^{i\theta}\bar{z} + \frac{(1-\alpha)(1-|b_1|)}{2(2-\alpha)}\bar{z}^2, \tag{19}$$

for properly chosen real θ.

Remark 3.8. *This result is different from the result of Özturk and Yalçin [13, Theorem 3.8], and it can be easily seen that our result gives a better estimate. Also, it can be easily verified that the sharp result for [13, Theorem 3.8] is given by the function*

$$f_\theta(z) = z + |b_1|e^{i\theta}\bar{z} + \frac{3-\alpha-2\alpha}{2\alpha}\bar{z}^2,$$

does not belong to the class $f \in \mathcal{C}_H(\alpha)$. Hence the result of Özturk and Yalçin [13] is incorrect. The correct result is given by corollary 3.7.

Theorem 3.9. *Let $f = h + \bar{g}$, where h and g are of the form (2). Then $f \in \mathcal{S}_H^0(m,n;\alpha,\beta)$ if and only if*

$$f(z) = \sum_{k=1}^{\infty} [x_k h_k(z) + y_k g_k(z)],$$

where

$$h_1(z) = z, \quad h_k(z) = z + \frac{1-\alpha}{(1+\beta)(2^m - 2^n\alpha)}z^k, \quad \text{for} \quad k = 2,3,\ldots,$$

$$g_k(z) = z + \frac{1-\alpha}{(1+\beta)(2^m - 2^n\alpha)}\bar{z}^k, \quad \text{for} \quad k = 2,3,\ldots,$$

and

$$x_k, y_k \geq 0, \qquad y_1 = 0, \qquad x_1 = 1 - \sum_{k=2}^{\infty}(x_k + y_k).$$

In particular, the extreme points of the class $\mathcal{S}_H^0(m,n;\alpha,\beta)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. Suppose that

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty} (x_k h_k(z) + y_k g_k(z)) \\ &= z + \sum_{k=2}^{\infty} \Big(\frac{1-\alpha}{(1-\beta+\beta z)(k^m - k^n\alpha)} x_k z^k + \frac{1-\alpha}{(1-\beta+\beta k)(k^m - k^n\alpha)} y_k \bar{z}^k \Big). \end{aligned}$$

Then we have

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{(1-\beta+\beta z)(k^m - k^n\alpha)}{1-\alpha} \Big(\frac{1-\alpha}{(1-\beta+\beta z)(k^m - k^n\alpha)} x_k z^k \\ &\quad + \frac{1-\alpha}{(1-\beta+\beta k)(k^m - k^n\alpha)} y_k \bar{z}^k \Big) = \sum_{k=2}^{\infty} x_k + \sum_{k=2}^{\infty} y_k = 1 - x_1 \le 1. \end{aligned}$$

Therefore $f \in \overline{\mathcal{S}_H^0(m,n;\alpha,\beta)}$. Conversely, if $f \in \overline{\mathcal{S}_H^0(m,n;\alpha,\beta)}$. Set

$$\begin{aligned} x_k &= \frac{1-\alpha}{(1-\beta+\beta k)(k^m - k^n\alpha)} |a_k|, \\ & \qquad\qquad\qquad\qquad\qquad\qquad \textit{for} \quad k = 2,3,4,\ldots. \\ y_k &= \frac{1-\alpha}{(1-\beta+\beta k)(k^m - k^n\alpha)} |b_k|, \end{aligned}$$

Note that $0 \le x_k, y_k \le 1$ for integer $k \ge 2$. We define

$$x_1 = 1 - \sum_{k=2}^{\infty} x_k - \sum_{k=2}^{\infty} y_k, x_1 \ge 0, \quad \textit{and} \quad y_1 = 0.$$

Consequently, we obtain required representation, since

$$\begin{aligned} f(z) &= z + \sum_{k=2}^{\infty} (|a_k| z^k + |b_k| \bar{z}^k) \\ &= z + \sum_{k=2}^{\infty} \Big(\frac{1-\alpha}{(1+\beta)(2^m - 2^n\alpha)} x_k z^k + \frac{1-\alpha}{(1+\beta)(2^m - 2^n\alpha)} y_k \bar{z}^k \Big) \\ &= z + \sum_{k=2}^{\infty} \Big((h_k(z) - z) x_k + (g_k(z) - z) y_k \Big) \\ &= \Big(1 - \sum_{k=2}^{\infty} x_k - \sum_{k=2}^{\infty} y_k \Big) z + \sum_{k=2}^{\infty} h_k(z) x_k + \sum_{k=2}^{\infty} g_k(z) y_k \\ &= \sum_{k=1}^{\infty} [x_k h_k(z) + y_k g_k(z)]. \end{aligned}$$

This completes the proof of Theorem 3.9. □

Remark 3.10. *The following statements are significant:*

(i) Putting $m = 1, n = 0$ and $\beta = 0$ in Theorem 3.9, we obtain the extreme points of the class $\mathcal{S}_H^0(\alpha)$.
(ii) Putting $m = 2, n = 1$ and $\beta = 0$ in Theorem 3.9, we obtain the extreme points of the class $\mathcal{C}_H^0(\alpha)$.

Let $\mathcal{K}_H^0$ denote the class of harmonic univalent functions of the form (1) with $b_1 = 0$ that map $\mathbb{U}$ on to convex domains. It is known [5, Theorem 5.10], that the sharp inequalities $|A_k| \leq \dfrac{k+1}{2}, |B_k| \leq \dfrac{k-1}{2}$ hold. These results will be used in the next theorem.

From several researches (see [6,7]), recall that a function $f : \mathbb{U} \to \mathbb{C}$ is subordinate to a function $F : \mathbb{U} \to \mathbb{C}$ and write $f(z) \prec F(z)$ (or simply $f \prec F$), if there exists a complex-valued function ω which maps $\mathbb{U}$ into oneself with $\omega(0) = 0$, such that

$$f(z) = F(\omega(z)), \quad z \in \mathbb{U}.$$

Furthermore, if the function F is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec F(z) \Longleftrightarrow f(0) = F(0), \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}). \tag{20}$$

Theorem 3.11. *Suppose that $F(z) = z + \sum\limits_{k=1}^{\infty} (A_k z^k + \overline{B_k z^k})$ belongs to the class $\mathcal{K}_H^0$. If $f \in \mathcal{S}_H^0(m, n; \alpha, \beta)$, then $f * F \in \mathcal{S}_H^0(m-1, n-1; \alpha, \beta)$ if $n > 1$ and $f \Diamond F \in \mathcal{S}_H^0(m, n; \alpha, \beta)$. Moreover $f \Diamond F \prec f$.*

Proof. Since $f \in \mathcal{S}_H^0(m, n; \alpha, \beta)$, then we have

$$\sum_{k=2}^{\infty} (k^m - \alpha k^n)(1 - \beta + \beta k)(|a_k| + |b_k|) \leq (1 - \alpha). \tag{21}$$

Using (21), we have

$$\begin{aligned}
&\sum_{k=2}^{\infty}(k^{m-1}-\alpha k^{n-1})(1-\beta+\beta k)(|a_kA_k|+|b_kB_k|)\\
&\quad=\sum_{k=2}^{\infty}(k^{m}-\alpha k^{n})(1-\beta+\beta k)\Big(|a_k|\Big|\frac{A_k}{k}\Big|+|b_k|\Big|\frac{B_k}{k}\Big|\Big)\\
&\quad\leq\sum_{k=2}^{\infty}(k^{m}-\alpha k^{n})(1-\beta+\beta k)\Big(|a_k|\Big|\frac{k+1}{2k}\Big|+|b_k|\Big|\frac{k-1}{2k}\Big|\Big)\\
&\quad\leq\sum_{k=2}^{\infty}(k^{m}-\alpha k^{n})(1-\beta+\beta k)(|a_k|+|b_k|)\leq(1-\alpha).
\end{aligned}$$

It follows that $f*F\in\mathcal{S}_H^0(m-1,n-1;\alpha,\beta)$. By using (21) again, we have

$$\begin{aligned}
&\sum_{k=2}^{\infty}(k^{m}-\alpha k^{n})(1-\beta+\beta k)\Big(\Big|\frac{a_kA_k}{k}\Big|+\Big|\frac{b_kB_k}{k}\Big|\Big)\\
&\quad\leq\sum_{k=2}^{\infty}(k^{m}-\alpha k^{n})(1-\beta+\beta k)\Big(|a_k|\Big|\frac{k+1}{2k}\Big|+|b_k|\Big|\frac{k-1}{2k}\Big|\Big)\\
&\quad\leq\sum_{k=2}^{\infty}(k^{m}-\alpha k^{n})(1-\beta+\beta k)(|a_k|+|b_k|)\leq(1-\alpha).
\end{aligned}$$

Therefore $f\Diamond F\in\mathcal{S}_H^0(m,n;\alpha,\beta)$. Moreover, using the equivalent condition (20) implies the subordinating relation. This completes the proof of Theorem 3.11. □

Let $\mathcal{S}$ denotes the class of analytic univalent functions of the following form

$$F(z)=z+\sum_{k=2}^{\infty}A_kz^k.$$

It is well known that the sharp inequality $|A_k|\leq k$ is true. We use this fact in the next theorem.

Theorem 3.12. *Let $f\in\mathcal{S}_H^0(m,n;\alpha,\beta)$ and $F\in\mathcal{S}$. If $n>1$ then we have*

$$f*(F+\zeta\overline{F})\in\mathcal{S}_H^0(m-1,n-1;\alpha,\beta),\quad \textit{for}\quad |\zeta|\leq 1.$$

Moreover $f*(F+\zeta\overline{F})\prec f$.

Proof. Let $f \in \mathcal{S}_H^0(m,n;\alpha,\beta)$, then we have

$$\sum_{k=2}^{\infty}(k^m-\alpha k^n)(1-\beta+\beta k)(|a_k|+|b_k|) \le (1-\alpha). \tag{22}$$

Using (22), we have

$$\begin{aligned}&\sum_{k=2}^{\infty}(k^{m-1}-\alpha k^{n-1})(1-\beta+\beta k)(|a_kA_k|+|\zeta b_k\overline{A_k}|)\\ &\quad\le \sum_{k=2}^{\infty}(k^m-\alpha k^n)(1-\beta+\beta k)(|a_k|+|b_k|) \le (1-\alpha).\end{aligned} \tag{23}$$

It follows that $f*(F+\zeta\overline{F}) \in \mathcal{S}_H^0(m-1,n-1;\alpha,\beta)$ if $n>1$. Also, by (23) with equivalent condition (20), it is easy to see that the subordinating relation holds. □

Let $\mathcal{P}_H^0$ denote the class of complex and harmonic functions F in $\mathbb{U}$, then we have the decomposition $F = H + \overline{G}$ such that $Re\{F(z)\} > 0, z \in \mathbb{U}$ and

$$H(z) = 1 + \sum_{k=1}^{\infty} A_k z^k, G(z) = \sum_{k=2}^{\infty} B_k z^k.$$

It is known [12, Theorem 3] that the sharp inequalities $|A_k| < k + 1, |B_k| < k-1$ are true.

Theorem 3.13. *Suppose that* $F(z) = 1 + \sum_{k=2}^{\infty}(A_kz^k + \overline{B_kz^k})$ *belong to class* $\mathcal{P}_H^0$. *Let* $f \in \mathcal{S}_H^0(m,n;\alpha,\beta)$, $\frac{2}{3} \le |A_1| \le 2$ *and* $\frac{1}{A_1} f\Diamond F \in \mathcal{S}_H^0(m,n;\alpha,\beta)$. *Then*

$$\frac{1}{A_1} f * F \in \mathcal{S}_H^0(m-1,n-1;\alpha,\beta), \quad \textit{for} \quad n \ge 1.$$

Proof. Since $f \in \mathcal{S}_H^0(m,n;\alpha,\beta)$, then we have

$$\sum_{k=2}^{\infty}(k^m-\alpha k^n)(1-\beta+\beta k)(|a_k|+|b_k|) \le (1-\alpha). \tag{24}$$

Using (24), we have

$$\begin{aligned}
&\sum_{k=2}^{\infty}(k^{m-1}-\alpha k^{n-1})(1-\beta+\beta k)\Big(\Big|\frac{a_kA_k}{A_1}\Big|+\Big|\frac{b_kB_k}{A_1}\Big|\Big)\\
&\quad\le\sum_{k=2}^{\infty}(k^m-\alpha k^n)(1-\beta+\beta k)\Big(\Big|\frac{a_k}{A_1}\Big|\frac{k+1}{k}+\Big|\frac{b_k}{A_1}\Big|\frac{k-1}{k}\Big)\\
&\quad\le\sum_{k=2}^{\infty}(k^m-\alpha k^n)(1-\beta+\beta k)(|a_k|+|b_k|)\le(1-\alpha).
\end{aligned}$$

Therefore $\frac{1}{A_1}f*F\in\mathcal{S}_H^0(m-1,n-1;\alpha,\beta)$ for $n\ge 1$.
Similarly, we can show that $\frac{1}{A_1}f\Diamond F\in\mathcal{S}_H^0(m,n;\alpha,\beta)$. □

Theorem 3.14. *Let $f(z)$ of the form*

$$f(z)=z+\overline{b_1z}+\sum_{k=2}^{\infty}(a_kz^k+\overline{b_kz^k}),$$

be a member of the class $\mathcal{S}_H(m,n;\alpha,\beta)$. If we have

$$\delta\le\frac{(1-|b_1|)((2^m-2^n\alpha)(1+\beta)-1)}{(2^m-2^n\alpha)(1+\beta)},$$

then $N(f)\subset\mathcal{S}_H(\alpha)$.

Proof. Let $f \in \mathcal{S}_H(m,n;\alpha,\beta)$ and $F(z) = z+\overline{B_1 z}+\sum_{k=2}^{\infty}(A_k z^k + \overline{B_k z^k})$ belong to $N(f)$. We have

$$\begin{aligned}
&(1-\alpha)|B_1| + \sum_{k=2}^{\infty}(k-\alpha)(|A_k|+|B_k|) \\
&\leq (1-\alpha)|B_1 - b_1| + \sum_{k=2}^{\infty}(k-\alpha)(|A_k - a_k| + |B_k - b_k|) \\
&+ (1-\alpha)|b_1| + \sum_{k=2}^{\infty}(k-\alpha)(|a_k|+|b_k|) \\
&\leq (1-\alpha)\delta + (1-\alpha)|b_1| \\
&\quad + \frac{1}{(1+\beta)(2^m - 2^n\alpha)}\sum_{k=2}^{\infty}(1-\beta+\beta k)(k^m - k^n\alpha)(|a_k|+|b_k|) \\
&\leq (1-\alpha)\delta + (1-\alpha)|b_1| + \frac{(1-\alpha)(1-|b_1|)}{(1+\beta)(2^m-2^n\alpha)} \leq 1-\alpha,
\end{aligned}$$

if $\delta \leq \dfrac{(1-|b_1|)((2^m-2^n\alpha)(1+\beta)-1)}{(2^m-2^n\alpha)(1+\beta)}$. Therefore $F(z) \in \mathcal{S}_H(\alpha)$. □

Conclusion

In this paper, we introduced the subclass $\mathcal{S}_H^0(m,n;\alpha,\beta)$ of harmonic functions which is an extension of the special subclasses, studied before by Özturk and Yalçin in a collection of their published papers. Our most important achievement is to developed and refine their results in that subclass, particularly in theorems 3.11-3.14. We also, modify some of their results in a sharper form in theorems 3.2-3.6. We also recommend the readers to study similar theorems in reproducing kernel Hilbert spaces (see [9, 10]).

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