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PERIODICITY AND POSITIVITY IN NEUTRAL NONLINEAR LEVIN-NOHEL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. Our paper deals with the following neutral nonlinear Levin-Nohel integro-differential with variable delay

$$\frac{d}{dt}x(t)+\int_{t-\tau(t)}^{t}a(t,s)\,x(s)\,ds+\frac{d}{dt}g(t,x(t-\tau(t)))=0.$$

By using Krasnoselskii's fixed point theorem we obtain the existence of periodic and positive periodic solutions and by contraction mapping principle we obtain the existence of a unique periodic solution. An example is given to illustrate this work.

1. Introduction

Delay differential equations arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, medicine, biology, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of linear and nonlinear delay differential equations have received the attention of many authors (see [1]–[31], [33]–[40] and the references therein).

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In this paper, we consider the following neutral nonlinear Levin-Nohel integro-differential equation with variable delay

$$\frac{d}{dt}x(t) + \int_{t-\tau(t)}^{t} a(t,s)\,x(s)\,ds + \frac{d}{dt}g(t, x(t-\tau(t))) = 0, \tag{1}$$

where a, τ and g are continuous functions with $\tau(t) > 0$. Equation (1) has a long history and the simpler form of it was considered in 1928 by Volterra with a biological application in mind (see [12, 38]). In the case $g(t,0) = 0$, the authors in [13] used the contraction mapping principle to show the asymptotic stability of the zero solution for (1). The purpose of this paper is to transform (1) into an integral equation and then use the Krasnoselskii's fixed point theorem to show the existence of periodic and positive periodic solutions. The obtained integral equation is the sum of two mappings; one is a contraction and the other is compact. Also by employing the contraction mapping principle, the existence of a unique periodic solution has been established. An example is also given to illustrate this work.

2. Existence and uniqueness of periodic solutions

For $T > 0$ let P_T be the set of all continuous scalar functions x, periodic in t of period T. Then $(P_T, \|.\|)$ is a Banach space with the supremum norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,T]} |x(t)|.$$

Since we are searching for the existence of periodic solutions for (1), it is natural to assume that

$$a(t+T, s+T) = a(t,s),\ \ \tau(t+T) = \tau(t), \tag{2}$$

with τ being scalar, continuous and $\tau(t) \geq \tau^* > 0$. Also, we assume

$$\int_0^T A(z)\,dz > 0,\ \ A(t) = \int_{t-\tau(t)}^{t} a(t,s)\,ds. \tag{3}$$

The function $g(t,x)$ is periodic in t of period T, it is also globally Lipschitz continuous in x. That is

$$g(t+T, x) = g(t,x), \tag{4}$$

and there is positive constant E such that

$$|g(t,x) - g(t,y)| \leq E\,\|x-y\|. \tag{5}$$

The proof of the following lemma is close to the proof of Lemma 2.2 given in [13].

Lemma 2.1. *Suppose (2)–(4) hold. If $x \in P_T$, then x is a solution of equation (1) if and only if*

$$x(t) = -g(t, x(t-\tau(t))) - \left(1 - e^{-\int_{t-T}^{t} A(z)dz}\right)^{-1} \times \int_{t-T}^{t} \left[L_x(s) - A(s) g(s, x(s-\tau(s)))\right] e^{-\int_s^t A(z)dz} ds, \tag{6}$$

where

$$L_x(t) = \int_{t-\tau(t)}^{t} a(t,s) \int_s^t \left(\int_{u-\tau(u)}^{u} a(u,\nu) x(\nu) d\nu\right) duds + \int_{t-\tau(t)}^{t} a(t,s) \left(g(t, x(t-\tau(t))) - g(s, x(s-\tau(s)))\right) ds. \tag{7}$$

Proof. Obviously, we have

$$x(s) = x(t) - \int_s^t \frac{\partial}{\partial u} x(u) du.$$

Inserting these relation into (1), we get

$$\frac{d}{dt} x(t) + \int_{t-\tau(t)}^{t} a(t,s) \left(x(t) - \int_s^t \frac{\partial}{\partial u} x(u) du\right) ds + \frac{d}{dt} g(t, x(t-\tau(t))) = 0,$$

or equivalently

$$\frac{d}{dt} x(t) + x(t) \int_{t-\tau(t)}^{t} a(t,s) ds - \int_{t-\tau(t)}^{t} a(t,s) \left(\int_s^t \frac{\partial}{\partial u} x(u) du\right) ds + \frac{d}{dt} g(t, x(t-\tau(t))) = 0.$$

After substituting $\frac{\partial x}{\partial u}$ from (1), we obtain

$$\frac{d}{dt} x(t) + x(t) \int_{t-\tau(t)}^{t} a(t,s) ds + \frac{d}{dt} g(t, x(t-\tau(t))) + \int_{t-\tau(t)}^{t} a(t,s) \left(\int_s^t \left(\int_{u-\tau(u)}^{u} a(u,\nu) x(\nu) d\nu + \frac{\partial}{\partial u} g(u, x(u-\tau(u)))\right) du\right) ds = 0. \tag{8}$$

By performing the integration, we have

$$\int_s^t \frac{\partial}{\partial u} g\left(u, x\left(u-\tau\left(u\right)\right)\right) du = g\left(t, x\left(t-\tau\left(t\right)\right)\right) - g\left(s, x\left(s-\tau\left(s\right)\right)\right). \tag{9}$$

Substituting (9) into (8), we have

$$\frac{d}{dt}x\left(t\right) + A\left(t\right)x\left(t\right) + L_x\left(t\right) + \frac{d}{dt}g\left(t, x\left(t-\tau\left(t\right)\right)\right) = 0,$$

where A and L_x are given by (3) and (7), respectively. We rewrite this equation as

$$\begin{aligned} &\frac{d}{dt}\left\{x\left(t\right) + g\left(t, x\left(t-\tau\left(t\right)\right)\right)\right\} \\ &= -A\left(t\right)\left(x\left(t\right) + g\left(t, x\left(t-\tau\left(t\right)\right)\right)\right) \\ &+ A\left(t\right)g\left(t, x\left(t-\tau\left(t\right)\right)\right) - L_x\left(t\right). \end{aligned} \tag{10}$$

Multiply both sides of (10) with $e^{\int_0^t A(z)dz}$ and then integrate from $t-T$ to t to obtain

$$\begin{aligned} &\int_{t-T}^t \frac{d}{ds}\left[\left(x\left(s\right) + g\left(s, x\left(s-\tau\left(s\right)\right)\right)\right)e^{\int_0^s A(z)dz}\right] ds \\ &= -\int_{t-T}^t \left[L_x\left(s\right) - A\left(s\right)g\left(s, x\left(s-\tau\left(s\right)\right)\right)\right] e^{\int_0^s A(z)dz} ds. \end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned} &\left(x\left(t\right) + g\left(t, x\left(t-\tau\left(t\right)\right)\right)\right)e^{\int_0^t A(z)dz} \\ &- \left(x\left(t-T\right) + g\left(t-T, x\left(t-T-\tau\left(t-T\right)\right)\right)\right)e^{\int_0^{t-T} A(z)dz} \\ &= -\int_{t-T}^t \left[L_x\left(s\right) - A\left(s\right)g\left(s, x\left(s-\tau\left(s\right)\right)\right)\right] e^{\int_0^s A(z)dz} ds. \end{aligned}$$

Dividing both sides of the above equation by $e^{\int_0^t A(z)dz}$ and the fact that $x\left(t-T\right) = x\left(t\right)$, we obtain

$$\begin{aligned} &x\left(t\right) + g\left(t, x\left(t-\tau\left(t\right)\right)\right) \\ &= -\left(1 - e^{-\int_{t-T}^t A(z)dz}\right)^{-1} \\ &\times \int_{t-T}^t \left[L_x\left(s\right) - A\left(s\right)g\left(s, x\left(s-\tau\left(s\right)\right)\right)\right] e^{-\int_s^t A(z)dz} ds. \end{aligned}$$

Since each step is reversible, the converse follows easily. This completes the proof. □

Define a mapping H by

$$(H\varphi)(t) = -g(t, \varphi(t-\tau(t))) - \left(1 - e^{-\int_{t-T}^{t} A(z)dz}\right)^{-1}$$

$$\times \int_{t-T}^{t} [L_{\varphi}(s) - A(s) g(s, \varphi(s-\tau(s)))] e^{-\int_{s}^{t} A(z)dz} ds. \tag{11}$$

Since $a(t+T, s+T) = a(t,s)$ then A is T-periodic. It is clear from (11) that $H : P_T \to P_T$ by the way it was constructed in Lemma 2.1.

Next we state Krasnoselskii's fixed point theorem which enables us to prove the existence of a periodic solution. For the proof of Krasnoselskii's fixed point theorem we refer the reader to [32].

Theorem 2.2 (Krasnoselskii)**.** *Let M be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose that C and B map M into $\mathbb{B}$ such that*

(i) $x, y \in M$, implies $Cx + By \in M$,

(ii) C is continuous and CM is contained in a compact set,

(iii) B is a contraction mapping.

Then there exists $z \in M$ with $z = Cz + Bz$.

We note that to apply the above theorem we need to construct two mappings; one is contraction and the other is compact. Therefore, we express (11) as

$$(H\varphi)(t) = (B\varphi)(t) + (C\varphi)(t),$$

where $C, B : P_T \to P_T$ are given by

$$(B\varphi)(t) = -g(t, \varphi(t-\tau(t))), \tag{12}$$

and

$$(C\varphi)(t) = -\left(1 - e^{-\int_{t-T}^{t} A(z)dz}\right)^{-1}$$

$$\times \int_{t-T}^{t} [L_{\varphi}(s) - A(s) g(s, \varphi(s-\tau(s)))] e^{-\int_{s}^{t} A(z)dz} ds. \tag{13}$$

To simplify notations, we introduce the following constants

$$\eta = \left(1 - e^{-\int_{t-T}^{t} A(z)dz}\right)^{-1}, \ \gamma = \sup_{t\in[0,T]} \left(\sup_{s\in[t-T,t]} e^{-\int_{s}^{t} A(z)dz} \right),$$

$$\rho = \sup_{t\in[0,T]} \left(\sup_{s\in[t-T,t]} \left(\int_{s-\tau(s)}^{s} |a(s,w)| \, dw \right) \right),$$

$$\delta = \sup_{t\in[0,T]} \left(\sup_{s\in[t-T,t]} \left(\sup_{w\in[t-T,t]} \int_{w}^{s} \left(\int_{u-\tau(u)}^{u} |a(u,\nu)| \, d\nu \right) du \right) \right). \tag{14}$$

Lemma 2.3. *Let C be given in (13). Suppose that (2)–(4) hold. Then $C : P_T \to P_T$ is continuous and the image of C contained in a compact set.*

Proof. To see that C is continuous, we let $\varphi, \psi \in P_T$. Given $\epsilon > 0$, take $\beta = \frac{\epsilon}{N}$ with $N = \eta\gamma T\rho\left(\delta + 3E\right)$ where E is given by (5). Now, for $\|\varphi - \psi\| < \beta$, we obtain

$$\begin{aligned}\|C\varphi - C\psi\| &\leq \eta\gamma \int_{t-T}^{t} \left[\rho\delta\left\|\varphi - \psi\right\| + 3\rho E\left\|\varphi - \psi\right\|\right] ds \\ &\leq N\left\|\varphi - \psi\right\| < \epsilon.\end{aligned}$$

This proves that C is continuous. To show that the image of C is contained in a compact set, we consider $D = \{\varphi \in P_T : \|\varphi\| \leq R\}$, where R is a fixed positive constant. Let $\varphi \in D$. Observe that in view of (5) we have

$$\begin{aligned}\left|g\left(t,x\right)\right| &= \left|g\left(t,x\right) - g\left(t,0\right) + g\left(t,0\right)\right| \\ &\leq \left|g\left(t,x\right) - g\left(t,0\right)\right| + \left|g\left(t,0\right)\right| \\ &\leq E\left\|x\right\| + \alpha,\end{aligned}$$

where $\alpha = \sup\limits_{t\in[0,T]} \left|g\left(t,0\right)\right|$. Consequently

$$\begin{aligned}\left|\left(C\varphi\right)\left(t\right)\right| &\leq \eta\gamma \int_{t-T}^{t} \left[\rho\left(\delta + 2\left(ER + \alpha\right)\right) + \rho\left(ER + \alpha\right)\right] ds \\ &\leq \eta\gamma T\left[\rho\left(\delta + 3\left(ER + \alpha\right)\right)\right] = L.\end{aligned}$$

Next we calculate $\left(C\varphi\right)'\left(t\right)$ and show that $C\left(D\right)$ is uniformly bounded. By making use of (2)–(4) we obtain by taking the derivative in (13) that

$$\begin{aligned}\left(C\varphi\right)'\left(t\right) &= \left(1 - e^{-\int_{t-T}^{t} A(z)dz}\right)^{-1} \\ &\quad \times A\left(t\right) \int_{t-T}^{t} \left[L_{\varphi}\left(s\right) - A\left(s\right) g\left(s, \varphi\left(s - \tau\left(s\right)\right)\right)\right] e^{-\int_{s}^{t} A(z)dz} ds \\ &\quad - \left(1 - e^{-\int_{t-T}^{t} A(z)dz}\right)^{-1} \left\{\left(L_{\varphi}\left(t\right) - A\left(t\right) g\left(t, \varphi\left(t - \tau\left(t\right)\right)\right)\right)\right. \\ &\quad \left(L_{\varphi}\left(t - T\right) - A\left(t - T\right)\right. \\ &\quad \left. - \times g\left(t - T, \varphi\left(t - T - \tau\left(t - T\right)\right)\right)\right) e^{-\int_{t-T}^{t} A(z)dz}\Big\} \\ &= -A\left(t\right)\left(C\varphi\right)\left(t\right) - L_{\varphi}\left(t\right) + A\left(t\right) g\left(t, \varphi\left(t - \tau\left(t\right)\right)\right).\end{aligned}$$

Thus, the above expression yields $\left\|\left(C\varphi\right)'\right\| \leq F$, for some positive constant F. Thus $C\left(D\right)$ is uniformly bounded and equicontinuous. Hence by Ascoli-Arzela's theorem $C(D)$ is relatively compact. Then, $C(D)$ is contained in a compact set. $\square$

Lemma 2.4. *If B is given by (12) with $E<1$, then $B : P_T \to P_T$ is a contraction.*

Proof. Let B be defined by (12). Then for $\varphi, \psi \in P_T$ we have

$$\begin{aligned} \|B\varphi - B\psi\| &= \sup_{t\in[0,T]} |(B\varphi)(t) - (B\psi)(t)| \\ &\leq E \sup_{t\in[0,T]} |\varphi(t-\tau(t)) - \psi(t-\tau(t))| \\ &\leq E\|\varphi-\psi\|. \end{aligned}$$

Hence B defines a contraction. $\square$

Our first result is based on Krasnoselskii's fixed point theorem.

Theorem 2.5. *Suppose the hypothesis of Lemma 2.4. Let $\alpha = \sup_{t\in[0,T]} |g(t,0)|$. Suppose (2)–(5) hold. Let J be a positive constant satisfying the inequality*

$$EJ + \alpha + \eta\gamma T(\rho(\delta J + 3(EJ+\alpha))) \leq J. \tag{15}$$

Let $M = \{\varphi \in P_T : \|\varphi\| \leq J\}$. Then equation (1) has a solution in M.

Proof. By Lemma 2.3, $C : M \to P_T$ is continuous and $C(M)$ is contained in a compact set. Also, from Lemma 2.4, the mapping $B : M \to P_T$ is a contraction. Next, we show that if $\varphi, \psi \in M$, we have $\|C\varphi + B\psi\| \leq J$. Let $\varphi, \psi \in M$ with $\|\varphi\|, \|\psi\| \leq J$. Then

$$\begin{aligned} &\|C\varphi + B\psi\| \\ &\leq E\|\psi\| + \alpha + \eta\gamma \int_{t-T}^{t} [\rho(\delta\|\varphi\| + 2(E\|\varphi\|+\alpha)) + \rho(E\|\varphi\|+\alpha)]\,ds \\ &\leq EJ + \alpha + \eta\gamma T(\rho(\delta J + 3(EJ+\alpha))) \leq J. \end{aligned}$$

We now see that all the conditions of Krasnoselskii's theorem are satisfied. Thus there exists a fixed point z in M such that $z = Cz + Bz$. By Lemma 2.1, this fixed point is a solution of (1). Hence (1) has a T-periodic solution. $\square$

Our second result is based on Banach's fixed point theorem.

Theorem 2.6. *Suppose (2)–(5) hold. If*

$$E + \eta\gamma T\rho(\delta + 3E) < 1, \tag{16}$$

then equation (1) has a unique T-periodic solution.

Proof. Let the mapping H be given by (11). For $\varphi, \psi \in P_T$, in view of (11), we have

$$\|H\varphi - H\psi\| \leq (E + \eta\gamma T\rho(\delta + 3E)) \|\varphi - \psi\|.$$

This completes the proof by invoking the contraction mapping principle. □

Remark 2.7. *The condition (16) implies the condition (15).*

Corollary 2.8. *Suppose (2)–(4) hold and let α be the constant defined in Theorem 2.5. Let J be a positive constant and define $M = \{\varphi \in P_T : \|\varphi\| \leq J\}$. Suppose there is positive constant E^* so that for $x, y \in M$ we have*

$$|g(t,x) - g(t,y)| \leq E^* \|x - y\|.$$

If $E^* < 1$ and $\|H\varphi\| \leq J$ for $\varphi \in M$, then (1) has a T-periodic solution in M. Moreover, if

$$E^* + \eta\gamma T\rho(\delta + 3E^*) < 1,$$

then (1) has a unique T-periodic solution in M.

Proof. Let $M = \{\varphi \in P_T : \|\varphi\| \leq J\}$. Let the mapping H be given by (11). Then, the results follow immediately from Theorem 2.5 and Theorem 2.6 □

Example 2.9. *For small positive ϵ_1 and ϵ_2 we consider the nonlinear neutral integro-differential equation with variable delay*

$$\begin{aligned} &\frac{d}{dt}x(t) + \epsilon_1 \int_{t-\frac{\pi}{\omega}}^{t} (1 + \cos\omega(t-s))\, x(s)\, ds \\ &+ \epsilon_2 \frac{d}{dt}\left(\cos(\omega t)\, x^3\left(t - \frac{\pi}{\omega}\right)\right) = 0, \end{aligned} \tag{17}$$

where ω is a positive constant. So, we have $a(t,s) = \epsilon_1(1 + \cos\omega(t-s))$, $\tau(t) = \frac{\pi}{\omega}$, $g(t, x(t - \tau(t))) = \epsilon_2\left(\cos(\omega t)\, x^3\left(t - \frac{\pi}{\omega}\right) + 1\right)$. Define $M = \left\{\varphi \in P_{\frac{2\pi}{\omega}} : \|\varphi\| \leq J\right\}$, where J is a positive constant. For $\varphi \in M$, we *have*

$$
\begin{aligned}
&\left|(H\varphi)(t)\right| \\
&= \Big| -g\left(t,\varphi\left(t-\tau\left(t\right)\right)\right) - \left(1-e^{-\int_{t-T}^{t} A(z)dz}\right)^{-1} \\
&\times \int_{t-T}^{t} \left[L_{\varphi}\left(s\right) - A\left(s\right) g\left(s,\varphi\left(s-\tau\left(s\right)\right)\right)\right] e^{-\int_{s}^{t} A(z)dz} ds \Big| \\
&\leq \epsilon_2 J^3 + \epsilon_2 + \left(1-e^{\frac{-2\epsilon_1\pi^2}{\omega^2}}\right)^{-1} \frac{4\pi^2}{\omega^2}\epsilon_1 \left(\frac{4\epsilon_1\pi^2}{\omega^2}J + 3\epsilon_2 J^3 + 3\epsilon_2\right).
\end{aligned}
$$

Thus, the inequality

$$
\epsilon_2 J^3 + \epsilon_2 + \left(1-e^{\frac{-2\pi^2\epsilon_1}{\omega^2}}\right)^{-1} \frac{4\pi^2}{\omega^2}\epsilon_1 \left(\frac{4\epsilon_1\pi^2}{\omega^2}J + 3\epsilon_2 J^3 + 3\epsilon_2\right) \leq J, \tag{18}
$$

which is satisfied for small ω, ϵ_1 and ϵ_2, implies $\|H\varphi\| \leq J$. Hence, (17) has a $\frac{2\pi}{\omega}$-periodic solution, by Corollary 2.8.

For the uniqueness of the solution we let $\varphi, \psi \in M$. From (17) we see that

$$
\eta = \left(1-e^{\frac{-2\pi^2\epsilon_1}{\omega^2}}\right)^{-1},\ \rho = \frac{2\pi}{\omega}\epsilon_1,\ \gamma \leq 1.
$$

Also $\alpha = \epsilon_2$, $E = 3\epsilon_2 J^2$, where J is given by (18). If

$$
3\epsilon_2 J^2 + \frac{4\pi^2\epsilon_1}{\omega^2}\left(1-e^{\frac{-2\pi^2\epsilon_1}{\omega^2}}\right)^{-1}\left[\frac{4\epsilon_1\pi^2}{\omega^2} + 9\epsilon_2 J^2\right] < 1,
$$

is satisfied for small ϵ_1 and ϵ_2, then (17) has a unique $\frac{2\pi}{\omega}$-periodic solution, by Corollary 2.8.

3. Existence of positive periodic solutions

For some non-negative constant L and a positive constant K, we define the set

$$
\mathcal{M} = \left\{\varphi \in P_T : L \leq \varphi \leq K\right\},
$$

which is a closed convex and bounded subset of the Banach space P_T.

In this section we obtain the existence of a positive periodic solution of (1) by considering the two cases; $g\left(t,x\right) \geq 0$ and $g\left(t,x\right) \leq 0$ for all $t \in \mathbb{R}$, $x \in \mathcal{M}$. To simplify notation, we let

$$
\theta = \max_{t\in[0,T]}\left(\max_{s\in[t-T,t]} e^{-\int_s^t A(z)dz}\right),\ \sigma = \min_{t\in[0,T]}\left(\min_{s\in[t-T,t]} e^{-\int_s^t A(z)dz}\right).
$$

In the case $g(t,x) \leq 0$, we assume that there exist a non-negative constant k_1 and a positive constant k_2 such that

$$k_1 x \leq -g(t,x) \leq k_2 x, \text{ for all } t \in [0,T],\ x \in \mathcal{M}, \tag{19}$$

$$k_2 < 1, \tag{20}$$

and for all $t \in [0,T]$, $x \in \mathcal{M}$

$$\frac{L(1-k_1)}{\eta\sigma T} \leq -L_x(t) + A(t)\, g(t,x) \leq \frac{K(1-k_2)}{\eta\theta T}. \tag{21}$$

Theorem 3.1. *Suppose (2)–(5) and (19)–(21) hold. Then equation (1) has a positive T-periodic solution x in the subset $\mathcal{M}$.*

Proof. By Lemma 2.1 x is a solution of (1) if

$$x = Cx + Bx,$$

where C and B are given by (13), (12) respectively. By Lemma 2.3, $C : \mathcal{M} \to P_T$ is continuous and compact. Also, from Lemma 2.4, the mapping $B : \mathcal{M} \to P_T$ is a contraction. We just need to show that condition (i) of Theorem 2.2 is satisfied. Toward this, let $\varphi, \psi \in \mathcal{M}$, then

$$\begin{aligned}
&(B\psi)(t) + (C\varphi)(t) \\
&= -g(t, \psi(t-\tau(t))) \\
&- \eta \int_{t-T}^{t} \left[L_\varphi(s) - A(s)\, g(s, \varphi(s-\tau(s)))\right] e^{-\int_s^t A(z)dz} ds \\
&\leq k_2 K + \eta\theta T \frac{K(1-k_2)}{\eta\theta T} \leq K.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(B\psi)(t) + (C\varphi)(t) \\
&= -g(t, \psi(t-\tau(t))) \\
&- \eta \int_{t-T}^{t} \left[L_\varphi(s) - A(s)\, g(s, \varphi(s-\tau(s)))\right] e^{-\int_s^t A(z)dz} ds \\
&\geq k_1 L + \eta\sigma T \frac{L(1-k_1)}{\eta\sigma T} \geq L.
\end{aligned}$$

Clearly, all the hypotheses of the Krasnoselskii theorem are satisfied. Thus there exists a fixed point $x \in \mathcal{M}$ such that $x = Bx + Cx$. By Lemma 2.1 this fixed point is a solution of (1) and the proof is complete. $\square$

In the case $g(t,x) \geq 0$, we substitute conditions (19)-(21) with the following conditions respectively. We assume that there exist a negative constant k_3 and a non-positive constant k_4 such that

$$k_3 x \leq -g(t,x) \leq k_4 x, \text{ for all } t \in [0,T], \ x \in \mathcal{M}, \tag{22}$$

$$-k_3 < 1, \tag{23}$$

and for all $t \in [0,T]$, $x \in \mathcal{M}$

$$\frac{L - k_3 K}{\eta \sigma T} \leq -L_x(t) + A(t) g(t,x) \leq \frac{K - k_4 L}{\eta \theta T}. \tag{24}$$

Theorem 3.2. *Suppose (2)–(5) and (22)–(24) hold. Then equation (1) has a positive T-periodic solution x in the subset $\mathcal{M}$.*

The proof follows along the lines of Theorem 3.1, and hence we omit it.

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