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FRACTIONAL INEQUALITIES FOR SOME EXPONENTIALLY CONVEX FUNCTIONS

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Abstract. In this paper, we establish new integral inequalities via Riemann-Liouville fractional integrals and Katugampola fractional integrals for the class of functions whose derivatives in absolute value are exponentially convex functions and exponentially s-convex functions in the second sense.

1. Introduction

Study of generalized convex functions is important due to its significant application in integral inequalities. On the other hand, fractional integrals also play role in the advancement of integral inequalities.

The Hermite-Hadamard inequality [9, 8] for a convex function $\mathcal{F}:\mathcal{H}\to\mathbb{R}$ on an interval $\mathcal{H}$ is defined by

$$\mathcal{F}\left(\frac{h_1+h_2}{2}\right)\le\frac{1}{h_2-h_1}\int_{h_1}^{h_2}\mathcal{F}(g)dg\le\frac{\mathcal{F}(h_1)+\mathcal{F}(h_2)}{2}, \tag{1}$$

for all $h_1,h_2\in\mathcal{H}$ with $h_1<h_2$. Inequality (1) is then proved for other generalized convex functions, for instance Du [7], Khan [14, 15] and Khurshid [19] proved several inequalities for generalized convex functions. Also see [1, 3, 5, 6, 24, 23]. While Iqbal [10, 11, 12], Khan [16, 17, 18] and Khurshid [20] gave several Hermite-Hadamard type inequalities for convex functions via generalized fractional integrals.

Awan et al. [1] introduced following new class of convex functions.

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Definition 1.1 ([1]). *A function $\mathcal{F} : \mathcal{H} \subseteq \mathbb{R} \to \mathbb{R}$ is called exponentially convex, if*

$$\mathcal{F}(uh_1 + (1-u)h_2) \leq u\frac{\mathcal{F}(h_1)}{e^{\beta h_1}} + (1-u)\frac{\mathcal{F}(h_2)}{e^{\beta h_2}}, \tag{2}$$

for all $h_1, h_2 \in \mathcal{H}$, $u \in [0,1]$ and $\beta \in \mathbb{R}$. If the inequality (2) is in reversed order then $\mathcal{F}$ is called exponentially concave.

Mehreen and Anwar [24] introduced another class of functions called exponentially s-convex in second sense.

Definition 1.2 ([24]). *Let $s \in (0,1]$ and $\mathcal{H} \subset \mathbb{R}_0$ be an interval. A function $\mathcal{F} : \mathcal{H} \to \mathbb{R}$ is called exponentially s-convex in the second sense, if*

$$\mathcal{F}(uh_1 + (1-u)h_2) \leq u^s\frac{\mathcal{F}(h_1)}{e^{\beta h_1}} + (1-u)^s\frac{\mathcal{F}(h_2)}{e^{\beta h_2}}, \tag{3}$$

for all $h_1, h_2 \in \mathcal{H}$, $u \in [0,1]$ and $\beta \in \mathbb{R}$. If (3) is in reversed order then $\mathcal{F}$ is called exponentially s-concave.

Example 1.3. *A function $\mathcal{F} : [0,\infty) \to \mathbb{R}$, defined by $\mathcal{F}(g) = \ln(g)$ for $s \in (0,1)$ is an exponentially s-convex, for all $\beta \leq -1$.*

Definition 1.4 ([21]). *Let $\mathcal{F} \in L[h_1, h_2]$. The right-hand side and left-hand side Riemann- Liouville fractional integrals $J^{\alpha}_{h_1+}\mathcal{F}$ and $\mathcal{J}^{\alpha}_{h_2-}\mathcal{F}$ of order $\alpha > 0$ with $h_2 > h_1 \geq 0$ are defined by*

$$J^{\alpha}_{h_1+}\mathcal{F}(g) = \frac{1}{\Gamma(\alpha)}\int_{h_1}^{g}(g-t)^{\alpha-1}\mathcal{F}(t)dt, \ g > h_1,$$

and

$$J^{\alpha}_{h_2-}\mathcal{F}(g) = \frac{1}{\Gamma(\alpha)}\int_{g}^{h_2}(t-g)^{\alpha-1}\mathcal{F}(t)dt, \ g < h_2,$$

respectively, where $\Gamma(\cdot)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt$.

In [25] and [26], authors proved the following identity via Riemann-Liouville fractional integrals.

Lemma 1.5 ([25]). *Consider a differentiable mapping $\mathcal{F} : [h_1, h_2] \subset \mathbb{R}_+ \to \mathbb{R}$ on (h_1, h_2) with $h_1 < h_2$. Then the following equality holds:*

$$\begin{aligned} &\frac{\mathcal{F}(h_1) + \mathcal{F}(h_2)}{2} - \frac{\Gamma(\alpha+1)}{2(h_2-h_1)^{\alpha}}\left[J^{\alpha}_{h_1+}\mathcal{F}(h_2^{\rho}) + J^{\alpha}_{h_2-}\mathcal{F}(h_1)\right] \\ &\quad = \frac{h_2-h_1}{2}\int_0^1 \left[(1-u)^{\alpha} - u^{\alpha}\right]\mathcal{F}'(uh_1 + (1-u)h_2)du. \end{aligned} \tag{4}$$

Definition 1.6 ([13]). *Let $[h_1, h_2] \subset \mathbb{R}$ be a finite interval. Then, the left- and right-side Katugampola fractional integrals of order $\alpha(> 0)$ of $\mathcal{F} \in X_v^p(h_1, h_2)$ are defined by,*

$$ {}^{\rho}I_{h_1+}^{\alpha}\mathcal{F}(g) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{h_1}^{g} (g^{\rho} - t^{\rho})^{\alpha-1} t^{\rho-1}\mathcal{F}(t)dt, $$

and

$$ {}^{\rho}I_{h_2-}^{\alpha}\mathcal{F}(g) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{g}^{h_2} (t^{\rho} - g^{\rho})^{\alpha-1} t^{\rho-1}\mathcal{F}(t)dt, $$

with $h_1 < g < h_2$ and $\rho > 0$. Where $X_v^p(h_1, h_2)$ ($v \in \mathbb{R}, 1 \leq p \leq \infty$) is the space of those complex valued Lebesgue measurable functions $\mathcal{F}$ on $[h_1, h_2]$ for which $\|\mathcal{F}\|_{X_v^p} < \infty$, where the norm is defined by,

$$ \|\mathcal{F}\|_{X_v^p} = \left(\int_{h_1}^{h_2} |t^v \mathcal{F}(t)|^p \frac{dt}{t} \right)^{1/p} < \infty, $$

for $1 \leq p < \infty$, $u \in \mathbb{R}$ and for the case $p = \infty$,

$$ \|\mathcal{F}\|_{X_v^{\infty}} = ess\ sup_{h_1 \leq t \leq h_2} [t^u |\mathcal{F}(t)|]. $$

Where ess sup stands for essential supremum.

Chen and Katugampola [2] proved following generalized version of Lemma 1.5.

Lemma 1.7 ([2]). *Let $\alpha > 0$ and $\rho > 0$. Consider a differentiable mapping $\mathcal{F} : [h_1^{\rho}, h_2^{\rho}] \subset \mathbb{R}_+ \to \mathbb{R}$ on (h_1^{ρ}, h_2^{ρ}) with $0 \leq h_1 < h_2$. Then the following equality holds if the fractional integrals exist:*

$$ \frac{\mathcal{F}(h_1^{\rho}) + \mathcal{F}(h_2^{\rho})}{2} - \frac{\alpha\rho^{\alpha}\Gamma(\alpha+1)}{2(h_2^{\rho} - h_1^{\rho})^{\alpha}} \left[{}^{\rho}I_{h_1+}^{\alpha}\mathcal{F}(h_2^{\rho}) + {}^{\rho} I_{h_2-}^{\alpha}\mathcal{F}(h_1^{\rho}) \right] \tag{5} $$
$$ = \frac{h_2^{\rho} - h_1^{\rho}}{2} \int_0^1 \left[(1-u^{\rho})^{\alpha} - (u^{\rho})^{\alpha} \right] u^{\rho-1} \mathcal{F}'(u^{\rho}h_1^{\rho} + (1-u^{\rho})h_2^{\rho}) du. $$

2. Inequalities via Riemann-Liouville fractional integrals

In this section we find inequalities using Riemann-Liouville fractional integrals.

Theorem 2.1. *Consider a differentiable mapping $\mathcal{F} : \mathcal{H} \to \mathbb{R}$ on $\mathcal{H}^{\circ}$ and $h_1, h_2 \in \mathcal{H}$ with $h_1 < h_2$ and $\mathcal{F}' \in L_1[h_1, h_2]$. If $|\mathcal{F}'|^q$, for $q \geq 1$, is* *exponentially convex on* $[h_1, h_2]$*, then the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{\mathcal{F}(h_1)+\mathcal{F}(h_2)}{2} - \frac{\Gamma(\alpha+1)}{2(h_2-h_1)^{\alpha}}\left[J^{\alpha}_{h_1+}\mathcal{F}(h_2^{\rho}) + J^{\alpha}_{h_2-}\mathcal{F}(h_1)\right]\right| \\
&\le \frac{h_2-h_1}{2^{\frac{1}{q}}(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left[\left|\frac{\mathcal{F}'(h_1)}{e^{\beta h_1}}\right|^{q} + \left|\frac{\mathcal{F}'(h_2)}{e^{\beta h_2}}\right|^{q}\right]^{\frac{1}{q}}.
\end{aligned} \tag{6}
$$

Proof. Suppose $q = 1$. Using Lemma 1.5 and exponential convexity of $|\mathcal{F}'|$, we get

$$
\begin{aligned}
&\left|\frac{\mathcal{F}(h_1)+\mathcal{F}(h_2)}{2} - \frac{\Gamma(\alpha+1)}{2(h_2-h_1)^{\alpha}}\left[J^{\alpha}_{h_1+}\mathcal{F}(h_2^{\rho}) + J^{\alpha}_{h_2-}\mathcal{F}(h_1)\right]\right| \\
&= \left|\frac{h_2-h_1}{2}\int_0^1 \left[(1-u)^{\alpha} - u^{\alpha}\right]\mathcal{F}'(uh_1+(1-u)h_2)du\right| \\
&\le \frac{h_2-h_1}{2}\int_0^1 |(1-u)^{\alpha} - u^{\alpha}||\mathcal{F}'(uh_1+(1-u)h_2)|du \\
&\le \frac{h_2-h_1}{2}\int_0^1 |(1-u)^{\alpha} - u^{\alpha}|\left[u\left|\frac{\mathcal{F}'(h_1)}{e^{\beta h_1}}\right| + (1-u)\left|\frac{\mathcal{F}'(h_2)}{e^{\beta h_2}}\right|\right] \\
&= \frac{h_2-h_1}{2}\Bigg\{\int_0^{1/2}\left[(1-u)^{\alpha} - u^{\alpha}\right]\left[u\left|\frac{\mathcal{F}'(h_1)}{e^{\beta h_1}}\right| + (1-u)\left|\frac{\mathcal{F}'(h_2)}{e^{\beta h_2}}\right|\right] \\
&\quad + \int_{1/2}^1 \left[u^{\alpha} - (1-u)^{\alpha}\right]\left[u\left|\frac{\mathcal{F}'(h_1)}{e^{\beta h_1}}\right| + (1-u)\left|\frac{\mathcal{F}'(h_2)}{e^{\beta h_2}}\right|\right]\Bigg\}.
\end{aligned} \tag{7}
$$

Since

$$
\begin{aligned}
\int_0^{1/2} u(1-u)^{\alpha}du - \int_0^{1/2} u^{\alpha+1}du &= \int_{1/2}^1 u^{\alpha}(1-u)du - \int_{1/2}^1 (1-u)^{\alpha+1}du \\
&= \frac{1}{(\alpha+1)(\alpha+2)} - \frac{\frac{1}{2^{\alpha+1}}}{\alpha+1},
\end{aligned} \tag{8}
$$

and

$$
\begin{aligned}
\int_0^{1/2} (1-u)^{\alpha+1}du - \int_0^{1/2} u^{\alpha}(1-u)du &= \int_{1/2}^1 u^{\alpha+1}du - \int_{1/2}^1 u(1-u)^{\alpha}du \\
&= \frac{1}{\alpha+2} - \frac{\frac{1}{2^{\alpha+1}}}{\alpha+1}.
\end{aligned} \tag{9}
$$

Thus by using (8) and (9) in (7), we get (6) for $q = 1$. Now let $q > 1$. Using power mean inequality on Lemma 1.5 and exponential convexity of $|\mathcal{F}'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{\mathcal{F}(h_1)+\mathcal{F}(h_2)}{2}-\frac{\Gamma(\alpha+1)}{2(h_2-h_1)^\alpha}\left[J^\alpha_{h_1+}\mathcal{F}(h_2^\rho)+J^\alpha_{h_2-}\mathcal{F}(h_1)\right]\right|\\
&=\left|\frac{h_2-h_1}{2}\int_0^1\left[(1-u)^\alpha-u^\alpha\right]\mathcal{F}'(uh_1+(1-u)h_2)du\right|\\
&\leq\frac{h_2-h_1}{2}\int_0^1|(1-u)^\alpha-u^\alpha||\mathcal{F}'(uh_1+(1-u)h_2)|du\\
&\leq\frac{h_2-h_1}{2}\left(\int_0^1|(1-u)^\alpha-u^\alpha|du\right)^{1-\frac{1}{q}} \qquad (10)\\
&\qquad\left(\int_0^1|(1-u)^\alpha-u^\alpha||\mathcal{F}'(uh_1+(1-u)h_2)|^q du\right)^{\frac{1}{q}}\\
&=\frac{h_2-h_1}{2^{\frac{1}{q}}(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\left[\left|\frac{\mathcal{F}'(h_1)}{e^{\beta h_1}}\right|^q+\left|\frac{\mathcal{F}'(h_2)}{e^{\beta h_2}}\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Since

$$\begin{aligned}
\int_0^1|(1-u)^\alpha-u^\alpha|du&=\int_0^{1/2}[(1-u)^\alpha-u^\alpha]du+\int_{1/2}^1[u^\alpha-(1-u)^\alpha]du\\
&=\frac{2}{\alpha+1}\left(1-\frac{1}{2^\alpha}\right).
\end{aligned}$$

This completes the proof. □

Remark 2.2. *In Theorem* 2.1.
(1) *By letting* $\beta=0$*, then the inequality* (6) *for* $q=1$ *becomes the inequality* (3.5) *of Theorem* 3 *in* [25].
(2) *By letting* $\beta=0$ *and* $\alpha=1$*, then the inequality* (6) *with* $q=1$ *becomes the inequality* 2.3 *of Theorem* 2.2 *in* [4].

Now we prove inequality for exponentially s-convex function in second sense as follows:

Theorem 2.3. *Consider a differentiable mapping* $\mathcal{F}:\mathcal{H}\subseteq(0,\infty)\to\mathbb{R}$ *on* $\mathcal{H}^\circ$ *and* $h_1,h_2\in\mathcal{H}$ *with* $h_1<h_2$ *and* $\mathcal{F}'\in L_1[h_1,h_2]$*. If* $|\mathcal{F}'|^q$*, for* $q\geq1$*, is exponentially* s*-convex in second sense on* $[h_1,h_2]$*, then the*

following inequality holds:

(11)
$$\left|\frac{\mathcal{F}(h_1)+\mathcal{F}(h_2)}{2}-\frac{\Gamma(\alpha+1)}{2(h_2-h_1)^{\alpha}}\left[J^{\alpha}_{h_1+}\mathcal{F}(h_2^{\rho})+J^{\alpha}_{h_2-}\mathcal{F}(h_1)\right]\right|$$
$$\leq \frac{h_2-h_1}{2(\alpha+1)}\left[\frac{2}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\right]^{1-\frac{1}{q}}\left[\left|\frac{\mathcal{F}'(h_1)}{e^{\beta h_1}}\right|^{q}+\left|\frac{\mathcal{F}'(h_2)}{e^{\beta h_2}}\right|^{q}\right]^{\frac{1}{q}}$$
$$\times\left\{\mathcal{B}(\frac{1}{2};s+1,\alpha+1)-\mathcal{B}(\frac{1}{2};\alpha+1,s+1)+\frac{2^{\alpha+s}-1}{2^{\alpha+s}(\alpha+s+1)}\right\}.$$

Where $\mathcal{B}_v$ *is an incomplete beta function defined by* $\mathcal{B}_v(h_1,h_2)=\int_0^v t^{h_1-1}(1-t)^{h_2-1}dt$*,* $v\in(0,1)$*.*

Proof. Suppose $q=1$. Using Lemma 1.5 and exponential s-convexity of $|\mathcal{F}'|$, we get

(12)
$$\left|\frac{\mathcal{F}(h_1)+\mathcal{F}(h_2)}{2}-\frac{\Gamma(\alpha+1)}{2(h_2-h_1)^{\alpha}}\left[J^{\alpha}_{h_1+}\mathcal{F}(h_2^{\rho})+J^{\alpha}_{h_2-}\mathcal{F}(h_1)\right]\right|$$
$$=\left|\frac{h_2-h_1}{2}\int_0^1\left[(1-u)^{\alpha}-u^{\alpha}\right]\mathcal{F}'(uh_1+(1-u)h_2)du\right|$$
$$\leq\frac{h_2-h_1}{2}\int_0^1|(1-u)^{\alpha}-u^{\alpha}||\mathcal{F}'(uh_1+(1-u)h_2)|du$$
$$\leq\frac{h_2-h_1}{2}\int_0^1|(1-u)^{\alpha}-u^{\alpha}|\left[u^s\left|\frac{\mathcal{F}'(h_1)}{e^{\beta h_1}}\right|+(1-u)^s\left|\frac{\mathcal{F}'(h_2)}{e^{\beta h_2}}\right|\right]$$
$$=\leq\frac{h_2-h_1}{2}\left\{\int_0^{1/2}[(1-u)^{\alpha}-u^{\alpha}]\left[u^s\left|\frac{\mathcal{F}'(h_1)}{e^{\beta h_1}}\right|+(1-u)^s\left|\frac{\mathcal{F}'(h_2)}{e^{\beta h_2}}\right|\right]\right.$$
$$\left.+\int_{1/2}^1[u^{\alpha}-(1-u)^{\alpha}]\left[u^s\left|\frac{\mathcal{F}'(h_1)}{e^{\beta h_1}}\right|+(1-u)^s\left|\frac{\mathcal{F}'(h_2)}{e^{\beta h_2}}\right|\right]\right\}.$$

Since

$$(13)\qquad \int_0^{1/2}u^s(1-u)^{\alpha}du=\int_{1/2}^1u^{\alpha}(1-u)^s du=\mathcal{B}(\frac{1}{2};s+1,\alpha+1),$$

$$(14)\qquad \int_0^{1/2}u^{\alpha}(1-u)^s du=\int_{1/2}^1u^s(1-u)^{\alpha}du=\mathcal{B}(\frac{1}{2};\alpha+1,s+1),$$

$$(15)\qquad \int_0^{1/2}u^{\alpha+s}du=\int_{1/2}^1(1-u)^{\alpha+s}du=\frac{1}{2^{s+\alpha+1}(s+\alpha+1)},$$

and

(16)
$$\int_0^{1/2}(1-u)^{\alpha+s}du=\int_{1/2}^1 u^{\alpha+s}du=\frac{1}{s+\alpha+1}-\frac{1}{2^{s+\alpha+1}(s+\alpha+1)}.$$

Thus by using (13)∼(16) in (12), we get

(17)
$$\begin{aligned}&\left|\frac{\mathcal{F}(h_1)+\mathcal{F}(h_2)}{2}-\frac{\Gamma(\alpha+1)}{2(h_2-h_1)^\alpha}\left[J^\alpha_{h_1+}\mathcal{F}(h_2^\rho)+J^\alpha_{h_2-}\mathcal{F}(h_1)\right]\right|\\&\le\frac{h_2-h_1}{2}\left[\left|\frac{\mathcal{F}'(h_1)}{e^{\beta h_1}}\right|+\left|\frac{\mathcal{F}'(h_2)}{e^{\beta h_2}}\right|\right]\\&\quad\times\left\{\mathcal{B}(\frac{1}{2};s+1,\alpha+1)-\mathcal{B}(\frac{1}{2};\alpha+1,s+1)+\frac{2^{\alpha+s}-1}{2^{\alpha+s}(\alpha+s+1)}\right\}.\end{aligned}$$

Now let $q>1$. Using power mean inequality on Lemma 1.5 and exponential s-convexity of $|\mathcal{F}'|^q$, we obtain

$$\begin{aligned}&\left|\frac{\mathcal{F}(h_1)+\mathcal{F}(h_2)}{2}-\frac{\Gamma(\alpha+1)}{2(h_2-h_1)^\alpha}\left[J^\alpha_{h_1+}\mathcal{F}(h_2^\rho)+J^\alpha_{h_2-}\mathcal{F}(h_1)\right]\right|\\&=\left|\frac{h_2-h_1}{2}\int_0^1\left[(1-u)^\alpha-u^\alpha\right]\mathcal{F}'(uh_1+(1-u)h_2)du\right|\\&\le\frac{h_2-h_1}{2}\int_0^1|(1-u)^\alpha-u^\alpha||\mathcal{F}'(uh_1+(1-u)h_2)|du\\&\le\frac{h_2-h_1}{2}\left(\int_0^1|(1-u)^\alpha-u^\alpha|du\right)^{1-\frac{1}{q}}\\&\qquad\left(\int_0^1|(1-u)^\alpha-u^\alpha||\mathcal{F}'(uh_1+(1-u)h_2)|^q du\right)^{\frac{1}{q}}\\&=\frac{h_2-h_1}{2(\alpha+1)}\left[\frac{2}{\alpha+1}\left(1-\frac{1}{2^\alpha}\right)\right]^{1-\frac{1}{q}}\left[\left|\frac{\mathcal{F}'(h_1)}{e^{\beta h_1}}\right|^q+\left|\frac{\mathcal{F}'(h_2)}{e^{\beta h_2}}\right|^q\right]^{\frac{1}{q}}\\&\quad\times\left\{\mathcal{B}(\frac{1}{2};s+1,\alpha+1)-\mathcal{B}(\frac{1}{2};\alpha+1,s+1)+\frac{2^{\alpha+s}-1}{2^{\alpha+s}(\alpha+s+1)}\right\}.\end{aligned}\tag{18}$$

Since

$$\begin{aligned}\int_0^1|(1-u)^\alpha-u^\alpha|du&=\int_0^{1/2}[(1-u)^\alpha-u^\alpha]du+\int_{1/2}^1[u^\alpha-(1-u)^\alpha]du\\&=\frac{2}{\alpha+1}\left(1-\frac{1}{2^\alpha}\right).\end{aligned}$$

This completes the proof. □

Remark 2.4. *In Theorem* 2.3.
(1) *By letting* $\beta = 0$, *then the inequality* (11) *becomes the inequality* (2.5) *of Theorem* 4 *in* [26].
(2) *By letting* $\beta = 0$ *and* $\alpha = 1$, *then the inequality* (11) *with* $q = 1$ *becomes the inequality of Theorem* 1 *in* [22].

3. Inequalities via Katugampola fractional integrals

First we prove the result for exponentially convex functions.

Theorem 3.1. *Let* $\alpha > 0, \rho > 0$. *Consider a differentiable mapping* $\mathcal{F} : [h_1^\rho, h_2^\rho] \to \mathbb{R}$ *on* (h_1^ρ, h_2^ρ) *with* $h_1^\rho < h_2^\rho$ *and* $\mathcal{F}' \in L_1[h_1^\rho, h_2^\rho]$. *If* $|\mathcal{F}'|$ *is exponentially convex on* $[h_1^\rho, h_2^\rho]$, *then the following inequality holds:*

$$\begin{aligned}(19)\quad &\left|\frac{\mathcal{F}(h_1^\rho)+\mathcal{F}(h_2^\rho)}{2} - \frac{\alpha\rho^\alpha\Gamma(\alpha+1)}{2(h_2^\rho-h_1^\rho)^\alpha}\left[\,{}^\rho I^\alpha_{h_1+}\mathcal{F}(h_2^\rho) + {}^\rho I^\alpha_{h_2-}\mathcal{F}(h_1^\rho)\right]\right| \\ &\le \frac{h_2^\rho - h_1^\rho}{2\rho(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\left[\left|\frac{\mathcal{F}'(h_1^\rho)}{e^{\beta h_1^\rho}}\right| + \left|\frac{\mathcal{F}'(h_2^\rho)}{e^{\beta h_2^\rho}}\right|\right].\end{aligned}$$

Proof. Using Lemma 1.7 and exponential convexity of $|\mathcal{F}'|$, we get

(20)

$$\begin{aligned}&\left|\frac{\mathcal{F}(h_1^\rho)+\mathcal{F}(h_2^\rho)}{2} - \frac{\alpha\rho^\alpha\Gamma(\alpha+1)}{2(h_2^\rho-h_1^\rho)^\alpha}\left[{}^\rho I^\alpha_{h_1+}\mathcal{F}(h_2^\rho) +^\rho I^\alpha_{h_2-}\mathcal{F}(h_1^\rho)\right]\right| \\ &= \left|\frac{h_2^\rho-h_1^\rho}{2}\int_0^1 \left[(1-u^\rho)^\alpha - (u^\rho)^\alpha\right] u^{\rho-1}\mathcal{F}'(u^\rho h_1^\rho + (1-u^\rho)h_2^\rho)du\right| \\ &\le \frac{h_2^\rho-h_1^\rho}{2}\int_0^1 u^{\rho-1}|(1-u^\rho)^\alpha - (u^\rho)^\alpha||\mathcal{F}'(u^\rho h_1^\rho + (1-u^\rho)h_2^\rho)|du \\ &\le \frac{h_2^\rho-h_1^\rho}{2}\int_0^1 u^{\rho-1}|(1-u^\rho)^\alpha-(u^\rho)^\alpha|\left[u^\rho\left|\frac{\mathcal{F}'(h_1^\rho)}{e^{\beta h_1^\rho}}\right| + (1-u^\rho)\left|\frac{\mathcal{F}'(h_2^\rho)}{e^{\beta h_2^\rho}}\right|\right] \\ &= \frac{h_2^\rho-h_1^\rho}{2}\left\{\int_0^{1/\sqrt[\rho]{2}} u^{\rho-1}[(1-u^\rho)^\alpha - u^{\rho\alpha}]\left[u^\rho\left|\frac{\mathcal{F}'(h_1^\rho)}{e^{\beta h_1^\rho}}\right| + (1-u^\rho)\left|\frac{\mathcal{F}'(h_2^\rho)}{e^{\beta h_2^\rho}}\right|\right]\right. \\ &\quad \left. + \int_{1/\sqrt[\rho]{2}}^1 u^{\rho-1}[u^{\rho\alpha} - (1-u^\rho)^\alpha]\left[u^\rho\left|\frac{\mathcal{F}'(h_1^\rho)}{e^{\beta h_1^\rho}}\right| + (1-u^\rho)\left|\frac{\mathcal{F}'(h_2^\rho)}{e^{\beta h_2^\rho}}\right|\right]\right\}.\end{aligned}$$

Since

$$
\begin{aligned}
&\int_0^{1/\sqrt[\rho]{2}} u^{\rho-1}u^{\rho}(1-u^{\rho})^{\alpha}du - \int_0^{1/\sqrt[\rho]{2}} u^{\rho-1}u^{\rho(\alpha+1)}du \\
&= \int_{1/\sqrt[\rho]{2}}^{1} u^{\rho-1}u^{\rho\alpha}(1-u^{\rho})du - \int_{1/\sqrt[\rho]{2}}^{1} u^{\rho-1}(1-u^{\rho})^{\alpha+1}du \qquad (21)\\
&= \frac{1}{\rho}\left[\frac{1}{(\alpha+1)(\alpha+2)} - \frac{\frac{1}{2^{\alpha+1}}}{(\alpha+1)}\right],
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^{1/\sqrt[\rho]{2}} u^{\rho-1}(1-u^{\rho})^{\alpha+1}du - \int_0^{1/\sqrt[\rho]{2}} u^{\rho-1}u^{\rho\alpha}(1-u^{\rho})du \\
&= \int_{1/\sqrt[\rho]{2}}^{1} u^{\rho-1}u^{\rho(\alpha+1)}du - \int_{1/\sqrt[\rho]{2}}^{1} u^{\rho-1}u^{\rho}(1-u^{\rho})^{\alpha}du \qquad (22)\\
&= \frac{1}{\rho}\left[\frac{1}{(\alpha+2)} - \frac{\frac{1}{2^{\alpha+1}}}{(\alpha+1)}\right].
\end{aligned}
$$

Thus by using (21) and (22) in (20), we get (19). □

Remark 3.2. *In Theorem* 3.1. *By letting* $\beta = 0$*, then the inequality* (19) *becomes the inequality* (17) *of Theorem* 2.5 *in* [2].

Theorem 3.3. *Let* $\alpha > 0, \rho > 0$*. function* $\mathcal{F} : [h_1^{\rho}, h_2^{\rho}] \subseteq (0, \infty) \to \mathbb{R}$ *on* (h_1^{ρ}, h_2^{ρ}) *with* $h_1^{\rho} < h_2^{\rho}$ *and* $\mathcal{F}' \in L_1[h_1^{\rho}, h_2^{\rho}]$*. If* $|\mathcal{F}'|$ *is exponentially* s*-convex in second sense on* $[h_1^{\rho}, h_2^{\rho}]$*, then the following inequality holds:*

(23)

$$
\begin{aligned}
&\left|\frac{\mathcal{F}(h_1^{\rho}) + \mathcal{F}(h_2^{\rho})}{2} - \frac{\alpha\rho^{\alpha}\Gamma(\alpha+1)}{2(h_2^{\rho}-h_1^{\rho})^{\alpha}}\left[\ {}^{\rho}I^{\alpha}_{h_1+}\mathcal{F}(h_2^{\rho}) + \ {}^{\rho}I^{\alpha}_{h_2-}\mathcal{F}(h_1^{\rho})\right]\right| \\
&\le \frac{h_2^{\rho}-h_1^{\rho}}{2\rho(\alpha+1)}\left[\left|\frac{\mathcal{F}'(h_1^{\rho})}{e^{\beta h_1^{\rho}}}\right| + \left|\frac{\mathcal{F}'(h_2^{\rho})}{e^{\beta h_2^{\rho}}}\right|\right] \\
&\quad \times \left\{\mathcal{B}(\frac{1}{2}; s+1, \alpha+1) - \mathcal{B}(\frac{1}{2}; \alpha+1, s+1) + \frac{2^{\alpha+s}-1}{2^{\alpha+s}(\alpha+s+1)}\right\}.
\end{aligned}
$$

Where $\mathcal{B}_v$ *is an incomplete beta function defined by* $\mathcal{B}_v(h_1, h_2) = \int_0^v t^{h_1-1}(1-t)^{h_2-1}dt$, $v \in (0, 1)$.

Proof. Using Lemma 1.7 and exponential s-convexity of $|\mathcal{F}'|$, we get

(24)
$$\begin{aligned}
&\left|\frac{\mathcal{F}(h_1^\rho)+\mathcal{F}(h_2^\rho)}{2}-\frac{\alpha\rho^\alpha\Gamma(\alpha+1)}{2(h_2^\rho-h_1^\rho)^\alpha}\left[{}^\rho I^\alpha_{h_1+}\mathcal{F}(h_2^\rho)+{}^\rho I^\alpha_{h_2-}\mathcal{F}(h_1^\rho)\right]\right|\\
&=\left|\frac{h_2^\rho-h_1^\rho}{2}\int_0^1\left[(1-u^\rho)^\alpha-(u^\rho)^\alpha\right]u^{\rho-1}\mathcal{F}'(u^\rho h_1^\rho+(1-u^\rho)h_2^\rho)du\right|\\
&\le\frac{h_2^\rho-h_1^\rho}{2}\int_0^1 u^{\rho-1}|(1-u^\rho)^\alpha-(u^\rho)^\alpha||\mathcal{F}'(u^\rho h_1^\rho+(1-u^\rho)h_2^\rho)|du\\
&\le\frac{h_2^\rho-h_1^\rho}{2}\int_0^1 u^{\rho-1}|(1-u^\rho)^\alpha-(u^\rho)^\alpha|\left[u^{\rho s}\left|\frac{\mathcal{F}'(h_1^\rho)}{e^{\beta h_1^\rho}}\right|+(1-u^\rho)^s\left|\frac{\mathcal{F}'(h_2^\rho)}{e^{\beta h_2^\rho}}\right|\right]\\
&=\frac{h_2^\rho-h_1^\rho}{2}\left\{\int_0^{1/\sqrt[\rho]{2}}u^{\rho-1}[(1-u^\rho)^\alpha-u^{\rho\alpha}]\left[u^{\rho s}\left|\frac{\mathcal{F}'(h_1^\rho)}{e^{\beta h_1^\rho}}\right|+(1-u^\rho)^s\left|\frac{\mathcal{F}'(h_2^\rho)}{e^{\beta h_2^\rho}}\right|\right]\right.\\
&\quad\left.+\int_{1/\sqrt[\rho]{2}}^1 u^{\rho-1}[u^{\rho\alpha}-(1-u^\rho)^\alpha]\left[u^{\rho s}\left|\frac{\mathcal{F}'(h_1^\rho)}{e^{\beta h_1^\rho}}\right|+(1-u^\rho)^s\left|\frac{\mathcal{F}'(h_2^\rho)}{e^{\beta h_2^\rho}}\right|\right]\right\}.
\end{aligned}$$

Since

$$\begin{aligned}
\int_0^{1/\sqrt[\rho]{2}}u^{\rho-1}u^{\rho s}(1-u^\rho)^\alpha d\tau&=\int_{1/\sqrt[\rho]{2}}^1 u^{\rho-1}u^{\rho\alpha}(1-u^\rho)^s du\\
&=\frac{1}{\rho}\mathcal{B}(\frac{1}{2};s+1,\alpha+1),
\end{aligned}\tag{25}$$

$$\begin{aligned}
\int_0^{1/\sqrt[\rho]{2}}u^{\rho-1}u^{\rho\alpha}(1-u^\rho)^s d\tau&=\int_{1/\sqrt[\rho]{2}}^1 u^{\rho-1}u^{\rho s}(1-u^\rho)^\alpha du\\
&=\frac{1}{\rho}\mathcal{B}(\frac{1}{2};\alpha+1,s+1),
\end{aligned}\tag{26}$$

$$\begin{aligned}
\int_0^{1/\sqrt[\rho]{2}}u^{\rho-1}u^{\rho(\alpha+s)}du&=\int_{1/\sqrt[\rho]{2}}^1 u^{\rho-1}(1-u^\rho)^{\alpha+s}du\\
&=\frac{1}{\rho}\frac{1}{2^{s+\alpha+1}(s+\alpha+1)},
\end{aligned}\tag{27}$$

and

(28)

$$\int_0^{1/\sqrt[\rho]{2}} u^{\rho-1}(1-u^\rho)^{\alpha+s}du = \int_{1/\sqrt[\rho]{2}}^{1} u^{\rho-1}u^{\rho(\alpha+s)}du$$
$$= \frac{1}{\rho}\left[\frac{1}{s+\alpha+1} - \frac{1}{2^{s+\alpha+1}(s+\alpha+1)}\right].$$

Thus by using (25)~(28) in (24), we get (23). □

Corollary 3.4. *In Theorem* 3.3. *By letting* $\beta = 0$, *we get*

(29)

$$\left|\frac{\mathcal{F}(h_1^\rho)+\mathcal{F}(h_2^\rho)}{2} - \frac{\alpha\rho^\alpha\Gamma(\alpha+1)}{2(h_2^\rho-h_1^\rho)^\alpha}\left[\ {}^\rho I^\alpha_{h_1+}\mathcal{F}(h_2^\rho) +\ {}^\rho I^\alpha_{h_2-}\mathcal{F}(h_1^\rho)\right]\right|$$
$$\leq \frac{h_2^\rho - h_1^\rho}{2\rho(\alpha+1)}\left[\left|\mathcal{F}'(h_1^\rho)\right| + \left|\mathcal{F}'(h_2^\rho)\right|\right]$$
$$\times\left\{\mathcal{B}(\frac{1}{2}; s+1,\alpha+1) - \mathcal{B}(\frac{1}{2};\alpha+1,s+1) + \frac{2^{\alpha+s}-1}{2^{\alpha+s}(\alpha+s+1)}\right\}.$$

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