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RIGIDITY CHARACTERIZATION OF COMPACT RICCI SOLITONS

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Abstract. In this paper, we firstly define the Ricci mean value along the gradient vector field of the Ricci potential function and show that it is non-negative on a compact Ricci soliton. Furthermore a Ricci soliton is Einstein if and only if its Ricci mean value is vanishing. Finally, we obtain a compact Ricci soliton $(M^n, g)(n \geq 3)$ is Einstein if its Weyl curvature tensor and the Kulkarni-Nomizu product of Ricci curvature are orthogonal.

1. Introduction

The concept of Ricci solitons was introduced by Hamilton in mid 80's. They are natural generalizations of Einstein metrics. Ricci solitons also correspond to self-similar solutions of Hamilton's Ricci flow and often arise as limits of dilations of singularities in the Ricci flow. They can be viewed as fixed points of the Ricci flow, as a dynamical system, on the space of Riemannian metrics modulo diffeomorphisms and scalings. Ricci solitons are of interests to physicists as well and are called quasi-Einstein metrics in physics literature.

A complete n-dimensional Riemannian manifold (M^n, g) is called a Ricci soliton if there exists a smooth vector field X such that the Ricci tensor satisfies the following equation

$$Ric + \frac{1}{2}\mathcal{L}_X g = \mu g \tag{1}$$

for some constant μ, where Ric is the Ricci tensor of M and $\mathcal{L}_X$ denotes the Lie derivative operator along the vector field X. The Ricci soliton is said to be shrinking, steady, and expanding accordingly as μ is positive, zero, and

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negative, respectively. If X is a gradient vector field, then we have a gradient Ricci soliton, satisfying the equation

$$Ric + Hess(f) = \mu g, \tag{2}$$

where $Hess(f)$ denote the Hessian of f. The function f is called a potential function of the gradient steady soliton. The case $X = 0$ (i.e., f being a constant function) is an Einstein metric or is said to be trivial, and vice versa.

Many works have been directed at classifying the Ricci soliton, as it is an important problem in the theory of the Ricci flow. In [23], Perelman have proved that every compact Ricci soliton is a gradient Ricci soliton. The classification of gradient Ricci solitons has been a very interesting problem. For compact expanding and steady gradient Ricci solitons, it is well-known that they must be Einstein (see [8], [23]). The shrinking case is a little bit more complicated. When $n = 2, 3$, it is known that compact shrinking Ricci solitons are Einstein (see [15] and [18]).

In dimension 2 Hamilton [16] proved that the shrinking gradient Ricci solitons with bounded curvature are $\mathbb{S}^2$, $\mathbb{RP}^2$, and $\mathbb{R}^2$ with constant curvature [14]. The two-dimensional case is special: Every shrinking compact two-dimensional Ricci soliton is $\mathbb{S}^2$ or $\mathbb{RP}^2$ with the standard metric. This result was first proved by Hamilton [16] with an argument using the Uniformization theorem which can be strongly simplified by means of Kazdan-Warner identity (see [9]). In [6], Chen, Lu and Tian found a simple proof independent by uniformization of surfaces.

Ivey proved the first classification result in dimension 3 showing that compact shrinking gradient solitons have constant positive curvature [18]. In the noncompact case Perelman [23] has shown that the 3-dimensional shrinking gradient Ricci solitons with bounded nonnegative sectional curvature are $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$, and $\mathbb{R}^3$ or quotients. Ni and Wallach [21] have given an alternative approach to prove the classification of 3-dimensional shrinkers which extends to higher dimensional manifolds with zero Weyl tensor, while every 3-manifold has zero Weyl tensor. Their argument also requires non-negative Ricci curvature. Also see Naber's paper [20] for a different argument in the 3-dimensional case. By using a different set of formulas they remove the non-negative curvature assumption.

When $n \geq 4$, there exist nontrivial compact gradient shrinking solitons. Also, there exist complete noncompact Ricci solitons (steady, shrinking and expanding) that are not Einstein (cf. [2] and [19]).

To characterize Ricci solitons, various rigidity results under appropriate curvature pinching assumptions were proved. In [1], the result implies that gradient Ricci solitons with positive curvature operator must be of constant curvature. Specially, for the case of vanishing Weyl conformal curvature tensor, there are many interesting results (cf. [3–5, 11] and [24]).

In [3], they proved that a complete noncompact non-flat conformally flat gradient steady Ricci soliton is the Bryant soliton up to scaling. In [21], they proved a classification result on gradient shrinking solitons with vanishing Weyl curvature tensor which includes the rotationally symmetric ones, in high dimension. In [4], the author proved several identities on compact gradient Ricci solitons. As an application of these identities, later, in [5] they obtain that a compact gradient shrinking Ricci soliton, which is locally conformally flat, must be Einstein. In [11], they obtain the same result.

One naturally asks how to know a compact gradient soliton is Einstein? In this paper, a quantity will be defined to judge it, and the motivation comes from the fact that Z. Guo [12] defined the Ricci mean value of a hypersurface and obtained the gap theorem for the scalar curvature recently. Similarly, we define the Ricci mean value of a gradient Ricci soliton as follows:

$$\delta = \frac{1}{nV}\int_M Ric(\nabla f, \nabla f) dM, \tag{3}$$

where V is the volume of M and $Ric(\nabla f, \nabla f)$ denotes the Ricci curvature along the gradient vector ∇f. We obtain that δ is non-negative on a compact gradient Ricci soliton. More importantly, a compact gradient Ricci soliton is Einstein if and only if its Ricci mean value is vanishing. Explicitly, we prove following results:

Theorem 1.1. *Let $(M^n, g)(n \geq 2)$ be a compact Ricci soliton. Then*

$$\delta \geq 0. \tag{4}$$

Moreover, the equality sign holds if and only if M is trivial.

We recall that for two symmetric $(0,2)$-tensors A and B defined the Kulkarni-Nomizu product as the $(0,4)$-tensor

$$\begin{aligned} A \circ B(X,Y,Z,W) = & \frac{1}{2}[A(X,Z)B(Y,W) + A(Y,W)B(X,Z)] \\ & - \frac{1}{2}[A(X,W)B(Y,Z) + A(Y,Z)B(X,W)], \end{aligned} \tag{5}$$

where X, Y, Z, W are smooth vector fields on M.

In this paper, we study compact Ricci solitons that its Weyl conformal curvature tensor W and the Kulkarni-Nomizu product of Ricci curvature $Ric \circ Ric$ are orthogonal, i.e.,

$$\langle W, Ric \circ Ric \rangle = 0. \tag{6}$$

Firstly, we compute the Laplacian of $|Ric|^2/R^2$, where $|Ric|^2$ is the squared length of the Ricci tensor of M, and then derive an algebraic lemma (Lemma 4.2) to estimate the formula. Finally applying with the strong maximum principle (Lemma 2.4), we obtain the following theorem:

Theorem 1.2. *Let $(M^n, g)(n \geq 3)$ be a compact Ricci soliton. If its Weyl conformal curvature tensor W and $Ric \circ Ric$ are orthogonal, M must be Einstein.*

Particularly, when the Weyl conformal curvature tensor vanishes, (6) holds autometically. Hence we also have the result:

Corollary 1.3. *Let $(M^n, g)(n \geq 3)$ be a compact Ricci soliton with vanishing Weyl conformal curvature tensor, and then M is Einstein.*

Note that the Weyl conformal curvature tensor is identically zero for every 3-manifold, while the metric is called locally conformally flat when its Weyl conformal curvature tensor vanishes in dimension $n > 3$. Hence, every compact 3-dimensional Ricci soliton is Einstein. When $n > 3$, every compact conformally flat Ricci soliton must be also Einstein.

We organize the paper as follows. In Section 2, we give some formulas and notations for a Riemannian manifold and some fundamental formulas of Ricci solitons by using the method of moving frames. In Section 3, we give the properties of the Ricci mean value for a gradient Ricci soliton and complete the proof of Theorem 1.1. In Section 4, we obtain some properties of Ricci solitons and an algebraic lemma and prove Theorem 1.2.

2. Preliminaries

In this section, we first recall the some formulas and notations for a Riemannian manifold by using the method of moving frames. Then we give some fundamental formulas of Ricci solitons and an algebraic lemma to complete the proof of the main theorem.

Let $M^n(n \geq 3)$ be an n-dimensional Riemannian manifold, $e_1, \ldots, e_n$ be a local orthonormal frame fields on M^n, and $\omega_1, \ldots, \omega_n$ be their dual 1-forms. In this paper we make the following conventions on the range of indices:

$$1 \leq i, j, k, \ldots \leq n,$$

and agree that repeated indices are summed over the respective ranges. Then we can write the structure equations of M^n as follows:

$$d\omega_i = \omega_j \wedge \omega_{ji}, \ \omega_{ij} + \omega_{ji} = 0, \tag{7}$$

$$d\omega_{ij} = \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} R_{ijkl}\omega_k \wedge \omega_l, \ R_{ijkl} = -R_{jikl}, \tag{8}$$

where d is external differential operator on M, ω_{ij} is the Levi-Civita connection form and R_{ijkl} is the Riemannian curvature tensor of M. It is known that Riemannian curvature tensor satisfies the following identities:

$$R_{ijkl} = -R_{ijlk}, \ R_{ijkl} = R_{klij}, \ R_{ijkl} + R_{iklj} + R_{iljk} = 0. \tag{9}$$

Ricci tensor R_{ij} and scalar curvature R are defined respectively by

$$R_{ij} := \sum_k R_{ikjk}, \ R = \sum_i R_{ii}. \tag{10}$$

Let f be a smooth function on M^n, we define the covariant derivatives f_i, $f_{i,j}$, $f_{i,jk}$ as follows:

$$f_i\omega_i := df, \ f_{i,j}\omega_j := df_i + f_j\omega_{ji}, \tag{11}$$

$$f_{i,jk}\omega_k := df_{i,j} + f_{k,j}\omega_{ki} + f_{i,k}\omega_{kj}. \tag{12}$$

We know that

$$f_{i,j} = f_{j,i},\ f_{i,jk} - f_{i,kj} = f_l R_{lijk}. \tag{13}$$

Its gradient, Hessian and Laplacian are defined by the following formulas:

$$\nabla f := f_i e_i,\ Hess(f) := f_{i,j}\omega_i \otimes \omega_j,\ \Delta f := \sum_i f_{i,i}. \tag{14}$$

The covariant derivatives of tensors R_{ij} and R_{ijkl} are defined by the following formulas:

$$R_{ij,k}\omega_k := dR_{ij} + R_{kj}\omega_{ki} + R_{ik}\omega_{kj}, \tag{15}$$

$$R_{ij,kl}\omega_l := dR_{ij,k} + R_{lj,k}\omega_{li} + R_{il,k}\omega_{lj} + R_{ij,l}\omega_{lk}, \tag{16}$$

$$R_{ijkl,m}\omega_m := dR_{ijkl} + R_{mjkl}\omega_{mi} + R_{imkl}\omega_{mj} + R_{ijml}\omega_{mk} + R_{ijkm}\omega_{ml}. \tag{17}$$

By exterior differentiation of (8), one can get the second Bianchi identity

$$R_{ijkl,m} + R_{ijlm,k} + R_{ijmk,l} = 0. \tag{18}$$

From (10), (15) and (18), we have

$$R_{ij,k} = R_{ik,j} = -\sum_l R_{lijk,l}, \tag{19}$$

and so

$$\sum_j R_{ji,j} = \frac{1}{2}R_i. \tag{20}$$

We define the Schouten tensor $S = S_{ij}\omega_i \otimes \omega_j$, where

$$S_{ij} := R_{ij} - \frac{1}{2(n-1)}R\delta_{ij}, \tag{21}$$

then $S_{ij} = S_{ji}$. The tensor

$$C_{ijkl} := R_{ijkl} - \frac{1}{n-2}(S_{ik}\delta_{jl} + S_{jl}\delta_{ik} - S_{il}\delta_{jk} - S_{jk}\delta_{il}) \tag{22}$$

is called Weyl conformal curvature tensor which does not change under conformal transformation of the metric. The Weyl conformal curvature tensor is identically zero for every 3-dimensional manifold. In dimension $n \geq 4$, when its Weyl conformal curvature tensor vanishes, the metric called conformally flat is locally conformally equivalent to a flat metric.

Let

$$\phi_{ij} := R_{ij} - \frac{1}{2}R\delta_{ij}, \tag{23}$$

and we get $\phi = \phi_{ij}\omega_i \otimes \omega_j$ is a symmetric tensor defined on M. We introduce an operator $\Box$ associated to ϕ acting on any function $f \in C^2(M)$ by

$$\Box f = \phi_{ij} f_{i,j}. \tag{24}$$

Then, in [7] Cheng and Yau prove that $\Box$ is self-adjoint relative to the L^2 inner product of M, that is

$$\int_M (h\Box f)dM = \int_M (f\Box h)dM \tag{25}$$

for any C^2-functions f and h.

Now, let (M^n, g) be a gradient compact Ricci soliton and the soliton equation(2) can be written as

$$R_{ij} + f_{i,j} = \mu\delta_{ij}. \tag{26}$$

We will give some well-known facts of gradient Ricci solitons.

Lemma 2.1 ([11])**.** *Suppose that* (M^n, g) *is a gradient Ricci soliton satisfying* (26)*. Then the following formulas hold,*

$$R + \Delta f = n\mu,\ R_{ij,k} + f_{i,jk} = 0,\ R_i + (\Delta f)_i = 0, \tag{27}$$

$$\Delta R_{ij} = R_{ij,k}f_k + 2\mu R_{ij} - 2R_{kl}R_{ikjl}, \tag{28}$$

$$\Delta R = \langle \nabla R, \nabla f\rangle + 2\mu R - 2|Ric|^2, \tag{29}$$

where $\Delta R_{ij} = \sum_k R_{ij,kk}$, $|Ric|^2 = \sum_{i,j} R_{ij}^2$.

Lemma 2.2 ([11])**.** *Let* (M^n, g) *be a compact Ricci soliton. If its scalar curvature* R *is nonconstant, then it must be positive everywhere.*

In this paper, we also need the well-known algebraic lemma which was first used by Okumura and the strong maximum principle.

Lemma 2.3 ([22])**.** *Let* $\alpha_i, i = 1, \ldots, n$*, be real numbers such that* $\sum_i \alpha_i = 0$, $\sum_i \alpha_i^2 = constant \geq 0$*. Then*

$$|\sum_i \alpha_i^3| \leq \frac{n-2}{\sqrt{n(n-1)}}(\sum_i \alpha_i^2)^{\frac{3}{2}}. \tag{30}$$

Moreover, the equality holds in (30) *if and only if* $(n-1)$ *of the* α_i *are equal or* $\alpha_i = 0$ *for all* i*.*

Lemma 2.4 ([10], [13], [17])**.** *A nonconstant* C^∞ *function* f *on a Riemannian manifold* (M, g) *without boundary such that* $\Delta f \geq V(f)$ *for some* C^∞ *vector field* V *cannot assume a maximum value anywhere in* M*.*

3. The Ricci mean value of a gradient Ricci soliton

In this section, we mainly investigate the Ricci mean value along the gradient vector field for a compact gradient Ricci soliton which is defined in (2).

Lemma 3.1. *Suppose that* (M^n, g) *is a compact Ricci soliton. Then*

$$\int_M \Big((1-\frac{n}{2})n\mu^2 + \frac{1}{2}R^2 - |Ric|^2\Big)dM = 0. \tag{31}$$

Proof. Taking the function $h = 1$ in (25), then

$$\int_M (R_{ij} - \frac{1}{2}R\delta_{i,j})f_{i,j}dM = 0 \tag{32}$$

From (26), we have

$$(R_{ij} - \frac{1}{2}R\delta_{ij})f_{i,j} = (1 - \frac{n}{2})\mu R + \frac{1}{2}R^2 - |Ric|^2. \tag{33}$$

We can easily obtain (31) by putting the first equation of (27) and (33) into (32) because M is compact. □

Proof of Theorem 1.1. Calculating the Laplacian of $\Delta(|\nabla f|^2)$ and using the Ricci identities, it follows that

$$\frac{1}{2}\Delta(|\nabla f|^2) = \sum_{i,j} f_{i,j}^2 + f_i(\Delta f)_i + R_{ij}f_if_j. \tag{34}$$

From (20) and (27), we have

$$\frac{1}{2}(\Delta f)_i + R_{ij}f_j = 0. \tag{35}$$

Using (26) and (27),

$$\sum_{i,j} f_{i,j}^2 = |Ric|^2 - n\mu^2 + 2\mu\Delta f. \tag{36}$$

Plugging (35) and (36) into (34) implies that

$$\frac{1}{2}\Delta(|\nabla f|^2) = |Ric|^2 - n\mu^2 + 2\mu\Delta f - R_{ij}f_if_j. \tag{37}$$

Since M is compact, noting (3), we obtain

$$\delta = \frac{1}{nV}\int_M [|Ric|^2 - n\mu^2]dM. \tag{38}$$

Combining with (27) and (31), then

$$\delta = \frac{1}{2nV}\int_M (R - n\mu)^2 dM. \tag{39}$$

It follows that δ is non-negative, and the equality sign holds if and only if $R = n\mu$ which implies M is trivial. In fact, from the first equation of (27), M is Einstein if the scalar curvature R is constant. This is the proof of Theorem 1.1. □

Remark 3.2. For a given compact Riemannian manifold $(M^n, g)(n \geq 3)$ and a real number μ, if it is a gradient Ricci soliton, we have a non-empty set

$$RS = \{f \in C^2(M) : Ric + Hess(f) = \mu g\}. \tag{40}$$

Functional $\delta : RS \to \mathbb{R}$ which is defined by (3) needs to satisfy

$$\frac{1}{n}\min |Ric(g)|^2 - \mu^2 \leq \delta(f) \leq \frac{1}{n}\max |Ric(g)|^2 - \mu^2 \tag{41}$$

from (38) and Theorem 1.1, where $f \in RS$. Hence we can define two numbers $\alpha(g)$ and $\beta(g)$ as follows:

$$\alpha(g) := \inf\{\delta(f) : f \in RS\},\ \beta(g) := \sup\{\delta(f) : f \in RS\}. \tag{42}$$

Our inequality can be written as follows:

$$\frac{1}{n}\min|Ric(g)|^2 - \mu^2 \le \alpha(g) \le \beta(g) \le \frac{1}{n}\max|Ric(g)|^2 - \mu^2. \tag{43}$$

4. Compact Ricci solitons under the Weyl curvature tensor condition

In this section, we will give some properties and complete the proof of Theorem 1.2. Let M be a compact Ricci soliton. Since every compact Ricci soliton is a gradient Ricci soliton by means of Perelman work, there exists a potential function f such that the soliton equation (26) holds. Hence we next only need to deal with the case of shrinking ($\mu > 0$) gradient Ricci solitons, because compact expanding and steady gradient Ricci solitons must be Einstein. M is trivial if the scalar curvature R is constant. So we only need to consider the case which R is nonconstant. In this case, it must be positive everywhere from lemma 2.2. Hence we can consider the function $|Ric|^2/R^2$.

Lemma 4.1. *Suppose that (M^n, g) be a gradient Ricci soliton. Then*

$$\begin{aligned}\Delta\frac{|Ric|^2}{R^2} &= \langle\nabla\frac{|Ric|^2}{R^2}, \nabla f + 2\frac{1}{R}\nabla R - \frac{2}{R^2}\nabla R^2\rangle \\ &+ \frac{4}{R^3}\Big(|Ric|^4 - \frac{2n-1}{(n-1)(n-2)}R^2|Ric|^2 \\ &+ \frac{R^4}{(n-1)(n-2)} + \frac{2R}{n-2}tr(Ric)^3\Big) \\ &+ 2\sum_{i,j,k}\Big|\frac{1}{R}R_{ij,k} - \frac{1}{R^2}R_{ij}R_k\Big|^2 - \frac{4}{R^2}R_{ij}R_{kl}C_{ikjl}.\end{aligned} \tag{44}$$

Proof. By the definition of Δ, we compute that

$$\Delta\frac{|Ric|^2}{R^2} = \frac{1}{R^4}\Big(R^2\Delta|Ric|^2 - |Ric|^2\Delta R^2\Big) + \langle\nabla\frac{|Ric|^2}{R^2}, -\frac{2}{R^2}\nabla R^2\rangle. \tag{45}$$

Next, we deal with the first item of the right side of (45). Calculating the Laplacian of R^2 and $|Ric|^2$ respectively, using (28) and (29), we know

$$\Delta R^2 = 2|\nabla R|^2 + 2R\langle\nabla R, \nabla f\rangle + 4\mu R^2 - 4R|Ric|^2, \tag{46}$$

$$\Delta|Ric|^2 = 2\sum_{i,j,k}R_{ij,k}^2 + 2R_{ij}R_{ij,k}f_k + 4\mu|Ric|^2 - 4R_{ij}R_{kl}R_{ijkl}. \tag{47}$$

Substituting (46) and (47) into the first item of the right side of (45), then

$$\begin{aligned}&\frac{1}{R^4}\Big(R^2\Delta|Ric|^2-|Ric|^2\Delta R^2\Big)\\ &=\frac{2}{R^2}\sum_{i,j,k}R_{ij,k}^2-\frac{2}{R^4}|Ric|^2|\nabla R|^2+\frac{2}{R^2}R_{ij}R_{ij,k}f_k\\ &\quad-\frac{2}{R^3}|Ric|^2\langle\nabla R,\nabla f\rangle+\frac{4}{R^3}\Big(|Ric|^4-RR_{ij}R_{kl}R_{ikjl}\Big).\end{aligned}\tag{48}$$

Noticing

$$\Big(\frac{|Ric|^2}{R^2}\Big)_k=\frac{2}{R^2}R_{ij}R_{ij,k}-\frac{2}{R^3}|Ric|^2R_k,\tag{49}$$

we have

$$\begin{aligned}\Delta\frac{|Ric|^2}{R^2}=&\frac{2}{R^2}\sum_{i,j,k}R_{ij,k}^2-\frac{2}{R^4}|Ric|^2|\nabla R|^2\\ &+\frac{4}{R^3}\Big(|Ric|^4-RR_{ij}R_{kl}R_{ikjl}\Big)\\ &+\langle\nabla\frac{|Ric|^2}{R^2},\nabla f-\frac{2}{R^2}\nabla R^2\rangle.\end{aligned}\tag{50}$$

On the other hand, one can work out the following equation

$$\begin{aligned}&\sum_{i,j,k}\Big|\frac{1}{R}R_{ij,k}-\frac{1}{R^2}R_{ij}R_k\Big|^2\\ &=\frac{1}{R^2}\sum_{i,j,k}R_{ij,k}^2-\frac{1}{R^4}|Ric|^2|\nabla R|^2-\frac{1}{R}\langle\nabla\frac{|Ric|^2}{R^2},\nabla R\rangle.\end{aligned}\tag{51}$$

By the use of (22), we compute that

$$\begin{aligned}R_{ij}R_{kl}R_{ikjl}=&-\frac{2n-1}{(n-1)(n-2)}R|Ric|^2+\frac{R^3}{(n-1)(n-2)}\\ &+\frac{2}{n-2}tr(Ric)^3+R_{ij}R_{kl}C_{ikjl}.\end{aligned}\tag{52}$$

Therefore, we can easily obtain (44) by putting (51) and (52) into (50). □

In this paper, we also need the following algebraic lemma:

Lemma 4.2. *Let $A=(a_{ij})$ be a symmetric $(n\times n)$-matrix. When $n\geq 3$, then*

$$|A|^4-\frac{2n-1}{(n-1)(n-2)}(trA)^2|A|^2+\frac{1}{(n-1)(n-2)}(trA)^4+\frac{2}{n-2}(trA)(trA^3)\geq 0,\tag{53}$$

where $|A|^2=\sum_{i,j}a_{ij}^2$, $trA=\sum_i a_{ii}$. Moreover the equality holds for the matrix A if and only if A can be transformed simultaneously by an orthogonal matrix *into scalar multiples of the unit matrix I_n or $\tilde{A}$, where*

$$\tilde{A} = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-1} \end{pmatrix},$$

and I_k is the $(k \times k)$ unit matrix.

Proof. We may assume that A is diagonal and denote by $a_1, \dots, a_n$ the diagonal entries in A. By a simple calculation we obtain that the left side of (53) is written by $F(a_1, \dots, a_n)$, where

$$\begin{aligned} F(a_1, \dots, a_n) = &\Big(\sum_i a_i^2\Big)^2 - \frac{2n-1}{(n-1)(n-2)}\Big(\sum_i a_i^2\Big)\Big(\sum_i a_i\Big)^2 \\ &+ \frac{1}{(n-1)(n-2)}\Big(\sum_i a_i\Big)^4 + \frac{2}{n-2}\Big(\sum_i a_i\Big)\Big(\sum_i a_i^3\Big). \end{aligned} \tag{54}$$

There are $(n+2)$ real numbers $s, t, \alpha_1, \dots, \alpha_n$, such that

$$a_i = t + s\alpha_i,\ i = 1, , \dots, n, \tag{55}$$

where $\alpha_1, \dots, \alpha_n$ satisfy the following equations,

$$\sum_i \alpha_i = 0,\ \sum_i \alpha_i^2 = 1. \tag{56}$$

When $t = 0$, then $a_i = s\alpha_i$ and $F(a_1, \dots, a_n) = s^4 F(\alpha_1, \dots, \alpha_n)$. Noting (56), it is straightforward to get $F(\alpha_1, \dots, \alpha_n) = 1$. Consequently, in this case, $F(a_1, \dots, a_n) = s^4 \geq 0$, and the equality holds if and only if $A = 0_{n \times n}$.

On the other hand, if $t \neq 0$, let

$$a_i = t(1 + x\alpha_i),\ i = 1, \dots, n, \tag{57}$$

where $x = \frac{s}{t}$, and then

$$F(a_1, \dots, a_n) = t^4 F(1 + x\alpha_1, \dots, 1 + x\alpha_n) \tag{58}$$

from (54) and (55). Noticing (53), from (51) and (54), we obtain

$$F(1 + x\alpha_1, \dots, 1 + x\alpha_n) = x^2\Big[x^2 + \frac{2n}{n-2}\Big(\sum_i \alpha_i^3\Big)x + \frac{n}{n-1}\Big]. \tag{59}$$

Let

$$G(x) = x^2 + \frac{2n}{n-2}\Big(\sum_i \alpha_i^3\Big)x + \frac{n}{n-1} \tag{60}$$

and it can be seen as a quadratic function about x. Therefore from (56) and Lemma 2.3, we can get its discriminant

$$\Delta = \frac{4n^2}{(n-2)^2}\Big(\sum_i \alpha_i^3\Big)^2 - \frac{4n}{n-1} \leq 0. \tag{61}$$

So $G(x)$ is non-negative. Moreover, it vanishes if and only if $(n-1)$ of the α_i are equal since $\sum_i \alpha_i^2 = 1$, that is

$$\alpha_1 = \mp\sqrt{\frac{n-1}{n}},\ \alpha_2 = \cdots = \alpha_n = \pm\sqrt{\frac{1}{n(n-1)}}. \tag{62}$$

Therefore we can easily obtain $F(a_1, \ldots, a_n) \geq 0$ by using (58), (59) and (60), and the equality holds if and only if $a_1 = \cdots = a_n = t$ or

$$a_1 = 0,\ a_2 = \cdots = a_n = t\frac{n}{n-1}. \tag{63}$$

Hence, in this case, the equality of (53) holds if and only if A can be transformed simultaneously by an orthogonal matrix into scalar multiples of I_n or $\tilde{A}$. □

Proof of Theorem 1.2. Now, let $(M^n, g)(n \geq 3)$ be a compact Ricci soliton whose Weyl conformal curvature tensor W and $Ric \circ Ric$ are orthogonal. By (5) the definition of $Ric \circ Ric$ and the properties of Weyl curvature tensor W, we can get that the two tensors W and $Ric \circ Ric$ are orthogonal if and only if

$$\sum_{ijkl} R_{ik}R_{jl}C_{ijkl} = 0. \tag{64}$$

Combining Lemmas 4.1 and 4.2, then

$$\Delta\frac{|Ric|^2}{R^2} \geq \langle\nabla\frac{|Ric|^2}{R^2}, \nabla f + 2\frac{1}{R}\nabla R - \frac{2}{R^2}\nabla R^2\rangle. \tag{65}$$

By making use of the strong maximum principle for the above formula, then

$$\frac{|Ric|^2}{R^2} = c, \tag{66}$$

where c is constant. Moreover, from (66) and (53) we get

$$|Ric|^4 - \frac{2n-1}{(n-1)(n-2)}R^2|Ric|^2 + \frac{R^4}{(n-1)(n-2)} + \frac{2R}{n-2}tr(Ric)^3 = 0, \tag{67}$$

$$\sum_{i,j,k}\Big|\frac{1}{R}R_{ij,k} - \frac{1}{R^2}R_{ij}R_k\Big|^2 = 0. \tag{68}$$

The equality (67) implies that (R_{ij}) can be transformed simultaneously by an orthogonal matrix into scalar multiples of I_n or $\tilde{A}$ from Lemma 4.2. Therefore, we consider the following two cases separately. Now, we can choose a local orthonormal frames field on M^n such that

$$R_{ij} = \lambda_i\delta_{ij}, \tag{69}$$

where i is not summing index.

Case 1: $\lambda_1 = \cdots = \lambda_n$, and then M is Einstein.

Case 2:

$$\lambda_1 = 0,\ \lambda_2 = \cdots = \lambda_n = \lambda \neq 0, \tag{70}$$

and then

$$R = (n-1)\lambda, \ |Ric|^2 = (n-1)\lambda^2. \tag{71}$$

Putting (71) into (66), we get

$$c = \frac{1}{n-1}. \tag{72}$$

When $n > 3$, (68) yields

$$R_{ij,k} = \frac{1}{R} R_{ij} R_k \tag{73}$$

for any i, j, k. From (20) and (73), then

$$\frac{1}{2} R_j = \frac{\lambda_j}{(n-1)\lambda} R_j. \tag{74}$$

When $j = 1$ in the above equation, as $\lambda_1 = 0$, we can have $R_1 = 0$. When $j \geq 2$, then $\lambda_j = \lambda$ and $R_j = 0$. Therefore the scalar curvature R is constant.

When $n = 3$, $c = \frac{1}{2}$ in M since c is constant. From (66) and Lemma 3.1, we can obtain

$$(1 - \frac{n}{2}) n\mu^2 V = (c - \frac{1}{2}) \int_M R^2 dM, \tag{75}$$

where V is the volume of M. Hence, the left side of (75) is negative and its right side is vanishing for $n = 3$, which is a contradiction.

Consequently, we complete the proof of Theorem 1.2. □

Remark 4.3. Let $(M^n, g)(n \geq 3)$ be a Riemannian manifold, and R_{ij} and R are denoted the Ricci tensor and scalar curvature respectively. If $R \neq 0$ everywhere in M, let

$$T_{ij} := \frac{1}{R} R_{ij}, \tag{76}$$

and then $T = T_{ij} \omega_i \otimes \omega_j$ is a symmetric tensor defined on M. The equation (68) is equivalent that the tensor T is parallel. In Lemma 3.1 of [10], Derdzinski pointed out that if the tensor T is parallel, then R is constant. In fact this result is not exact. For example, let M^2 be a 2-dimension Riemannian manifold with nonconstant curvature, and Γ be a 1-dimension manifold. Let M^3 is the Riemannian product manifold $\Gamma \times M^2$, and then the tensor T of M^3 is parallel, but its the scalar curvature is not constant.

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