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EVOLUTION AND MONOTONICITY FOR A CLASS OF QUANTITIES ALONG THE RICCI-BOURGUIGNON FLOW

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Abstract. In this paper we consider the monotonicity of the lowest constant $\lambda_a^b(g)$ under the Ricci-Bourguignon flow and the normalized Ricci-Bourguignon flow such that the equation

$$-\Delta u + au\log u + bRu = \lambda_a^b(g)u$$

with $\int_M u^2\,\mathrm{dV} = 1$, has positive solutions, where a and b are two real constants. We also construct various monotonic quantities under the Ricci-Bourguignon flow and the normalized Ricci-Bourguignon flow. Moreover, we prove that a compact steady breather which evolves under the Ricci-Bourguignon flow should be Ricci-flat.

1. Introduction

Geometric flows are essential tools in the study of Riemannian manifolds and play important role in mathematics and physics. On the other hand, eigenvalues are also very important tools in the study of geometry and topology of Riemannian manifolds. That is why one can find so many mathematicians who have worked on the eigenvalues of geometric operators along geometric flows. In this paper we apply the Ricci-Bourguignon flow which is a generalization of the Ricci flow. To be familiar with this flow, suppose $(M, g(t))$ is a smooth closed Riemannian manifold of dimension $n \geq 2$ which is evolving under the following second order quasilinear parabolic PDE

$$\frac{\partial g(t)}{\partial t} = -2Ric_{g(t)} + 2\rho R_{g(t)}g(t), \tag{1}$$

where $\mathrm{Ric}_{g(t)}$ is the Ricci tensor, $R_{g(t)}$ is the scalar curvature of the manifold and ρ is a real constant. Depending on the choice of ρ, the Ricci-Bourguignon flow may turn to certain celebrated geometric flows. Namely, for $\rho = \frac{1}{2}$ this flow will turn to the Einstein flow, for $\rho = \frac{1}{2(n-1)}$ it will turn to the Schouten flow and for $\rho = 0$ it will turn to the famous Ricci flow. If $\rho < \frac{1}{2(n-1)}$, then every

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initial compact Riemannian manifold (M, g_0) has a unique smooth solution $g(t)$ solving the flow equation (1) with $g(0) = g_0$ (see [5, Theorem 3.1.2]).

Perelman [9] introduced the functional

$$\mathcal{F}(g, f) = \int_M (|\nabla f|^2 + R)e^{-f}\, \mathrm{dV},$$

and showed the $\mathcal{F}$-functional is nondecreasing under the Ricci flow coupled to a backward heat-type equation. Since the functional $\mathcal{F}$ is nondecreasing, it turns out that the lowest eigenvalue of the operator $-4\Delta + R$ is nondecreasing along the Ricci flow. Cao [1] showed that on manifolds with nonnegative curvature operator, the eigenvalues of the operator $-\Delta + \frac{R}{2}$ are nondecreasing along the Ricci flow. Afterwards, Li [8] obtained the same monotonicity for the first eigenvalue of the operator $-\Delta + \frac{R}{2}$ without any curvature assumption. Later, Cao [2] considered a general operator $-\Delta + bR$, where $b \geq \frac{1}{4}$, and proved that the first eigenvalue of this operator is nondecreasing along the Ricci flow on manifolds without curvature assumption.

Recently Chen [3] *et al.* did analogous work to what aforementioned. They studied the monotonicity of eigenvalues of operator $-\Delta + bR$, where b is a constant, along the Ricci-Bourguignon flow and derived the monotonicity of the lowest eigenvalue of $-\Delta + bR$. Wang [11] demonstrated the monotonicity of the lowest eigenvalue of the Schrödinger operator

$$\frac{(1-(n-1)\rho)^2}{1-2(n-1)\rho} R - 4\Delta,$$

where ρ is the same as in (1), along the Ricci-Bourguignon flow and ruled out the nontrivial steady breathers. Huang and Li [7] investigated the monotonicity of the lowest constant $\lambda_a^b(g)$ along the Ricci flow such that the following nonlinear equation has positive solutions:

$$-\Delta u + au \log u + bRu = \lambda_a^b(g)u \tag{2}$$

with

$$\int_M u^2\, \mathrm{dV} = 1, \tag{3}$$

where a and b are real constants. In fact, they extended the equation

$$-\Delta u + bRu = \lambda^b(g)u \tag{4}$$

with $\int_M u^2\, \mathrm{dV} = 1$, to the equation (2) and generalized some results of Cao [2] and Li [8] as well. Motivated by them, in this paper we will study the monotonicity of the lowest constant $\lambda_a^b(g)$ along the Ricci-Bourguignon flow such that the nonlinear equation (2) with (3) has positive solutions. In the following, we write RB-flow rather Ricci-Bourguignon flow for abbreviation.

The following theorem will be proved in Section 2:

Theorem 1.1. *Let $g(t)$, $t \in [0,T)$, be a solution to the RB-flow on a closed manifold M^n and let $\lambda_a^b(g)$ be the lowest constant such that the nonlinear equation* (2) *with* (3) *has positive solutions. Under assumptions $\rho \le 0$, $b \ge \frac{1}{4}$, $R(0) \ge na$ and $|\nabla f|^2 \ge \Delta f$, the quantity*

$$\lambda_a^b(t) + \frac{na^2(1-n\rho)^2}{8}t$$

is strictly increasing along the RB-flow in $[0,T)$. Here $f = -2\log\phi$.

In the case of evolving an initially locally symmetric manifold, we will prove the following:

Theorem 1.2. *Suppose $g(t)$, $t \in [0,T)$, is a solution to the RB-flow on a closed manifold M^n. Moreover suppose (M, g_0) is locally symmetric. Let $\lambda_a^b(g)$ be the lowest constant such that the nonlinear equation* (2) *with* (3) *has positive solutions. Then under assumptions $\rho \le 0$, $b \ge \frac{1}{4}$ and $R(0) \ge na$ the quantity*

$$\lambda_a^b(t) + \frac{na^2(1-n\rho)^2}{8}t$$

is strictly increasing along the RB-flow in $[0,T)$.

Define the symmetric tensor $\mathsf{S} = \mathrm{Ric} - \rho R g$ and let $S = \mathrm{tr}\,\mathsf{S} = (1-n\rho)R$ be its trace. The normalized RB-flow is

$$\frac{\partial g(t)}{\partial t} = -2\mathsf{S} + \frac{2}{n}sg(t), \tag{5}$$

where

$$s = \frac{\int_M S\,\mathrm{dV}}{\int_M \mathrm{dV}} = (1-n\rho)\frac{\int_M R\,\mathrm{dV}}{\int_M \mathrm{dV}} = (1-n\rho)r \tag{6}$$

and r is the average scalar curvature. In Section 3, we focus on the normalized RB-flow and we will prove the following theorem:

Theorem 1.3. *Let $g(t)$, $t \in [0,T)$, be a solution to the normalized RB-flow on a compact surface M^2 and let $\lambda_a^b(g)$ be the lowest constant such that the nonlinear equation* (2) *with* (3) *has positive solutions. Under assumptions $\rho \le 0$, $b = \frac{1}{2}$ and $R(0) \ge 2a$, the quantity*

$$\lambda_a^b(t) + \frac{a^2(1-2\rho)^2}{4}t + \frac{sa}{2}t + s\int_0^t \lambda^b(s)\mathrm{ds}$$

is strictly increasing along the normalized RB-flow in $[0,T)$. Here λ^b is the lowest eigenvalue of (4).

We also prove the following result when an initial homogeneous manifold is evolving along the RB-flow:

Theorem 1.4. *Suppose $g(t)$, $t \in [0,T)$, is a solution to the normalized RB-flow on a closed manifold M^n. Moreover suppose (M, g_0) is homogeneous. Let $\lambda_a^b(g)$ be the lowest constant such that the nonlinear equation* (2) *with* (3) *has* *positive solutions. Then under assumptions $\rho \leq 0$, $b \geq \frac{1}{4}$ and $R(0) \geq na$, the quantity*

$$\lambda_a^b(t) + \frac{na^2(1-n\rho)^2}{8}t + \frac{sa}{2}t + \frac{2s}{n}\int_0^t \lambda^b(s)\mathrm{ds}$$

is strictly increasing along the normalized RB-flow in $[0, T)$, where λ^b is the lowest eigenvalue of (4).

2. Monotonicity of $\lambda_a^b(g)$ along the RB-flow

We define the lowest constant $\lambda_a^b(g)$ which satisfies (2) with (3) as

$$\lambda_a^b(g) = \inf\left\{\mathcal{G}_a^b(g,u) : \int_M u^2\mathrm{dV} = 1,\ u > 0,\ u \in C^\infty(M)\right\},$$

in which

$$\mathcal{G}_a^b(g,u) = \int_M (|\nabla u|^2 + au^2\log u + bRu^2)\mathrm{dV}.$$

Generally it is not clear for us that the constant $\lambda_a^b(g)$ and the corresponding function $u(x,t)$ are differentiable in t along the RB-flow. So we can not use the differentiability property for $\lambda_a^b(g)$ and $u(x,t)$. But we can use a common trick to resolve this problem. Namely, we may proceed similar to Chen *et al.*'s work [3] to bypass the differentiability of $\lambda_a^b(g)$. For this purpose, we show that on a compact manifold M, the constant $\lambda_a^b(g)$ exists and it is a continuous function along the Ricci-Bourguignon flow on $[0, T)$.

Now we first show that for any compact Riemanniam manifold (M, g), the constant $\lambda_a^b(g)$ exists. To see that, we should prove the set

$$\left\{\mathcal{G}_a^b(g,u) : \int_M u^2\mathrm{dV} = 1,\ u > 0,\ u \in C^\infty(M)\right\} \tag{7}$$

is bounded from below. We may write

$$\begin{aligned}
&\left\{\mathcal{G}_a^b(g,u) : \int_M u^2\mathrm{dV} = 1,\ u > 0,\ u \in C^\infty(M)\right\}\\
\subseteq\ &\left\{\int_M (|\nabla u|^2 + bRu^2)\mathrm{dV} : \int_M u^2\mathrm{dV} = 1,\ u > 0,\ u \in C^\infty(M)\right\}\\
&+\left\{\int_M au^2\log u\ \mathrm{dV} : \int_M u^2\mathrm{dV} = 1,\ u > 0,\ u \in C^\infty(M)\right\}.
\end{aligned}$$

By [3] we know that when M is compact, the set

$$\left\{\int_M (|\nabla u|^2 + bRu^2)\mathrm{dV} : \int_M u^2\mathrm{dV} = 1,\ u > 0,\ u \in C^\infty(M)\right\} \tag{8}$$

takes its infimum and hence is bounded from below. On the other hand, since $at^2\log t$ takes its minimum at $t = \frac{1}{\sqrt{e}}$, where e is the Euler's number, one can easily get the lower bound of the set

$$\left\{\int_M au^2\log u\,\mathrm{dV} : \int_M u^2\mathrm{dV}=1,\ u>0,\ u\in C^\infty(M)\right\} \tag{9}$$

for $a>0$.

Since the sets (8) and (9) are bounded from below, we can conclude that the set (7) is bounded from below and consequently $\lambda_a^b(g)$ exists.

Following the techniques of [3], we will show that $\lambda := \lambda_a^b$ is a continuous function along the Ricci-Bourguignon flow.

Lemma 2.1. *If g_1 and g_2 are two metrics on M satisfying*

$$(1+\epsilon)^{-1}g_1 \le g_2 \le (1+\epsilon)g_1 \quad \text{and} \quad R(g_1)-\epsilon \le R(g_2) \le R(g_1)+\epsilon,$$

then

$$\begin{aligned}&\lambda(g_2)-\lambda(g_1)\\ \le\ &\left((1+\epsilon)^{\frac{n}{2}+1}-(1+\epsilon)^{-\frac{n}{2}}\right)(1+\epsilon)^{\frac{n}{2}}\left(\lambda(g_1)-\min(bR_{g_1}+a\log w)\right)\\ &+\left((1+\delta)|b|\max|R_{g_2}-R_{g_1}|+2\delta\max|bR_{g_1}+a\log w|\right)(1+\epsilon)^{\frac{n}{2}},\end{aligned}$$

where $\delta\to 0$ as $\epsilon\to 0$. In particular, $\lambda=\lambda_a^b$ is a continuous function with respect to the C^2-topology.

Proof. One can prove this using arguments similar to Lemma 5.24 in [4]. □

All over this paper we denote

$$\frac{\partial g_{ij}}{\partial t}=h_{ij},$$

where h is a symmetric 2-tensor. The following lemma is required throughout the paper.

Lemma 2.2 (L. Cremaschi [5])**.** *Suppose $g(t)$ is a solution to the RB-flow. Let R and $\mathrm{d}V$ denote the scalar curvature and volume element of metric $g(t)$ respectively. Then the following evolution equations hold*:

$$\frac{\partial R}{\partial t}=(1-2\,(n-1)\,\rho)\,\Delta R+2|Ric|^2-2\rho R^2,$$

$$\frac{\partial dV}{\partial t}=(n\rho-1)R\,\mathrm{dV}.$$

Again, suppose that $\lambda_a^b(g)$ is the lowest constant such that (2) with (3) has positive solution $u(x,t)$. Now we are going to apply a trick in order to bypass time derivatives of the constant $\lambda_a^b(g)$ and the corresponding function $u(x,t)$. According to [6, Theorem 7.2], for any $t_0\in[0,T)$, there exists a smooth function $\phi(t)>0$ satisfying

$$\int_M \phi(t)^2\mathrm{dV}=1$$

and $\phi(t_0) = u(t_0)$. Let

$$\mu(t) = \int_M \left(-\phi(t)\Delta\phi(t) + a\phi(t)^2 \log\phi(t) + bR\phi(t)^2\right) \text{ dV}.$$

Then $\mu(t)$ is a smooth function by definition. And at time t_0, we conclude that $\lambda_a^b(t_0) = \mu(t_0)$.

Here we state an essential lemma:

Lemma 2.3. *Let $g(t)$, $t \in [0,T)$, be a solution to the RB-flow on a closed manifold M^n and let $\lambda_a^b(g)$ be the lowest constant such that the nonlinear equation (2) with (3) has positive solutions. Suppose that $u(t_0)$ is the corresponding solution to $\lambda_a^b(t_0)$. Then we have*

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\left(\mu(t)+\frac{na^2(1-n\rho)^2}{8}t\right)\Big|_{t=t_0} = {} & \frac{1}{2}\int_M \left|R_{ij} + \nabla_i\nabla_j f + \frac{a}{2}(1-n\rho)g_{ij}\right|^2 e^{-f} \text{ dV} \\
& + \left(2b - \frac{1}{2}\right)\int_M |R_{ij}|^2 e^{-f} \text{ dV} \\
& - 2\rho b \int_M R^2 e^{-f} \text{ dV} \\
& + \frac{\rho}{2}\int_M |\nabla f|^2 (na - R) e^{-f} \text{ dV} \\
& + \left(\frac{n}{2} - 2(n-1)b\right)\rho \int_M \Delta R e^{-f} \text{ dV},
\end{aligned} \tag{10}$$

where $f = -2\log\phi$.

Proof. By definition we know

$$\mu(t) = \int_M \left(|\nabla\phi|^2 + a\phi^2\log\phi + bR\phi^2\right) \text{dV}. \tag{11}$$

Using Lemma 2.2 we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\mu(t)\Big|_{t=t_0} = {} & \int_M \big((2R_{ij} - 2\rho R g_{ij})\left(\nabla^i\phi\nabla^j\phi\right) + 2g_{ij}\nabla^i\phi_t\nabla^j\phi + 2a\phi\phi_t\log\phi \\
& + a\phi\phi_t + bR_t\phi^2 + 2bR\phi\phi_t\big)\text{dV} \\
& + \int_M \left(|\nabla\phi|^2 + a\phi^2\log\phi + bR\phi^2\right)(n\rho - 1)\, R\,\text{dV}.
\end{aligned} \tag{12}$$

Applying

$$2\int_M R_{ij}\nabla^i\phi\nabla^j\phi\,\text{dV} = \int_M \left((-\nabla_i R\,\nabla^i\phi)\phi - 2R_{ij}(\nabla^i\nabla^j\phi)\phi\right) \text{ dV}$$

and

$$-\int_M |\nabla\phi|^2 R\,\text{dV} = \int_M \left(R\Delta\phi + \nabla_i R\nabla^i\phi\right)\phi\,\text{dV}$$

into (12), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\mu(t)\big|_{t=t_0} = & \int_M \Big[-2R_{ij}(\nabla^i\nabla^j\phi)\phi - 2\rho R|\nabla\phi|^2 + bR_t\phi^2 + a\phi\phi_t \\ & + 2\phi_t\left(-\Delta\phi + a\phi\log\phi + bR\phi\right) \\ & + R\phi\left(-\Delta\phi + a\phi\log\phi + bR\phi\right)(n\rho-1) - n\rho(\nabla_i R\nabla^i\phi)\phi\Big]\,\mathrm{dV} \\ = & \int_M \Big[-2R_{ij}(\nabla^i\nabla^j\phi)\phi - 2\rho R|\nabla\phi|^2 + bR_t\phi^2 + \frac{a}{2}(\phi^2)_t\Big]\,\mathrm{dV} \\ & + \mu(t_0)\left(\int_M \phi^2\,\mathrm{dV}\right)_t\Big|_{t=t_0} - \int_M n\rho(\nabla_i R\nabla^i\phi)\phi\,\mathrm{dV} \\ = & \int_M \Big[-2R_{ij}(\nabla^i\nabla^j\phi)\phi - 2\rho R|\nabla\phi|^2 + bR_t\phi^2 + \frac{a}{2}(\phi^2)_t\Big]\,\mathrm{dV} \\ & - \int_M n\rho(\nabla_i R\nabla^i\phi)\phi\,\mathrm{dV}, \end{aligned} \tag{13}$$

where the last equality is obtained by

$$\int_M (\phi^2)_t + (n\rho-1)R\phi^2\,\mathrm{dV} = 0$$

from (3). According to Lemma 2.2, we have $\frac{\partial R}{\partial t} = (1-2(n-1)\rho)\Delta R + 2|Ric|^2 - 2\rho R^2$. So, (13) leads to the following

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\mu(t)\big|_{t=t_0} = & \int_M \Big[-2R_{ij}(\nabla^i\nabla^j\phi)\phi - 2\rho R|\nabla\phi|^2 \\ & + b\phi^2\left((1-2(n-1)\rho)\Delta R + 2|R_{ij}|^2 - 2\rho R^2\right) \\ & + \frac{a}{2}(1-n\rho)R\phi^2\Big]\,\mathrm{dV} - \int_M n\rho(\nabla_i R\nabla^i\phi)\phi\,\mathrm{dV} \\ = & \int_M \Big[-2R_{ij}(\nabla^i\nabla^j\phi)\phi - 2\rho R|\nabla\phi|^2 \\ & + \big((1-2(n-1)\rho)bR\Delta\phi^2 + 2b\phi^2|R_{ij}|^2 \\ & - 2\rho b\phi^2R^2\big) + \frac{a}{2}(1-n\rho)R\phi^2\Big]\,\mathrm{dV} - \int_M n\rho(\nabla_i R\nabla^i\phi)\phi\,\mathrm{dV}. \end{aligned} \tag{14}$$

Under a transformation $f = -2\log\phi$ which is equivalent to $\phi^2 = e^{-f}$, we get

$$\nabla^i\nabla^j\phi = \left(-\frac{1}{2}\nabla^i\nabla^j f + \frac{1}{4}\nabla^i f\nabla^j f\right)e^{-\frac{f}{2}} \tag{15}$$

and

$$\int_M (\nabla_i R\nabla^i\phi)\phi\,\mathrm{dV} = -\frac{1}{2}\int_M R\Delta e^{-f}\,\mathrm{dV}. \tag{16}$$

Hence, (14) can be written as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\mu(t)\big|_{t=t_0} =& \int_M \Big[R_{ij}(\nabla^i\nabla^j f) - \frac{1}{2}R_{ij}\nabla^i f\nabla^j f - \frac{\rho}{2}R|\nabla f|^2 \\ & - (1-2\,(n-1)\,\rho)\,bR\Delta f + (1-2\,(n-1)\,\rho)\,bR|\nabla f|^2 \\ & + 2b|R_{ij}|^2 - 2\rho bR^2 + \frac{a}{2}(1-n\rho)R\Big] e^{-f}\,\mathrm{dV} \\ & + \frac{n\rho}{2}\int_M R\Delta e^{-f}\,\mathrm{dV}. \end{aligned} \tag{17}$$

Applying the second Bianchi identity $2\nabla_l R^l_j = \nabla_j R$, we attain

$$\begin{aligned} & -b\,(1-2\,(n-1)\,\rho)\int_M R\Delta f e^{-f}\,\mathrm{dV} \\ =\; & (1-2\,(n-1)\,\rho)\int_M (b\nabla_i R\nabla^i f - bR|\nabla f|^2)e^{-f}\,\mathrm{dV} \\ =\; & (1-2\,(n-1)\,\rho)\int_M \big(-2bR_{ij}\nabla^i\nabla^j f + 2bR_{ij}\nabla^i f\nabla^j f - bR|\nabla f|^2\big)e^{-f}\,\mathrm{dV}. \end{aligned} \tag{18}$$

Thus, inserting (18) into (17) yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\mu(t)\big|_{t=t_0} =\; & (1-2b\,(1-2\,(n-1)\,\rho))\int_M R_{ij}(\nabla^i\nabla^j f)e^{-f}\,\mathrm{dV} \\ & + \left(2b\,(1-2\,(n-1)\,\rho) - \frac{1}{2}\right)\int_M R_{ij}(\nabla^i f\nabla^j f)e^{-f}\,\mathrm{dV} \\ & + 2b\int_M |R_{ij}|^2 e^{-f}\,\mathrm{dV} + \int_M \left(-2\rho bR^2 + \frac{a}{2}(1-n\rho)R\right)e^{-f}\,\mathrm{dV} \\ & + \frac{n\rho}{2}\int_M R\Delta e^{-f}\,\mathrm{dV} - \frac{\rho}{2}\int_M R|\nabla f|^2 e^{-f}\,\mathrm{dV}. \end{aligned} \tag{19}$$

Integrating by parts, we achieve

$$\int_M R_{ij}(\nabla^i\nabla^j f)e^{-f}\,\mathrm{dV} = \int_M R_{ij}(\nabla^i f\nabla^j f)e^{-f}\,\mathrm{dV} - \frac{1}{2}\int_M R\Delta e^{-f}\,\mathrm{dV} \tag{20}$$

and

$$\begin{aligned} & \int_M R_{ij}(\nabla^i\nabla^j f)e^{-f}\,\mathrm{dV} + \int_M |\nabla_i\nabla_j f|^2 e^{-f}\,\mathrm{dV} \\ =\; & \frac{1}{2}\int_M \Delta|\nabla f|^2 e^{-f}\,\mathrm{dV} - \int_M (\nabla_i\Delta f)(\nabla^i f)e^{-f}\,\mathrm{dV} - \frac{1}{2}\int_M R\Delta e^{-f}\,\mathrm{dV} \\ =\; & -\int_M \left[\Delta f - \frac{1}{2}|\nabla f|^2 + \frac{1}{2}R\right]\Delta e^{-f}\,\mathrm{dV} \\ =\; & (2b - \frac{1}{2})\int_M R\Delta e^{-f}\,\mathrm{dV} - a\int_M |\nabla f|^2 e^{-f}\,\mathrm{dV}, \end{aligned} \tag{21}$$

where the last equality deduced from the fact that

$$2\mu(t_0) = \left(\Delta f - \frac{1}{2}|\nabla f|^2 - af + 2bR\right)(t_0). \tag{22}$$

According to (20) and (21), we get

$$\begin{aligned}\int_M |\nabla_i\nabla_j f|^2 e^{-f}\,\mathrm{dV} = 2b\int_M R\Delta e^{-f}\,\mathrm{dV} - \int_M R_{ij}(\nabla^i f\nabla^j f)e^{-f}\,\mathrm{dV}\\ - a\int_M |\nabla f|^2 e^{-f}\,\mathrm{dV}.\end{aligned} \tag{23}$$

Using (20) and (23), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\mu(t)\Big|_{t=t_0} =\ & (1-2b\,(1-2\,(n-1)\,\rho))\int_M R_{ij}(\nabla^i\nabla^j f)e^{-f}\,\mathrm{dV}\\ & + \left(2b\,(1-2\,(n-1)\,\rho) - \frac{1}{2}\right)\int_M R_{ij}(\nabla^i f\nabla^j f)e^{-f}\,\mathrm{dV}\\ & + 2b\int_M |R_{ij}|^2 e^{-f}\,\mathrm{dV} + \int_M \left(-2\rho bR^2 + \frac{a}{2}(1-n\rho)R\right)e^{-f}\,\mathrm{dV}\\ & + \frac{n\rho}{2}\int_M R\Delta e^{-f}\,\mathrm{dV} - \frac{\rho}{2}\int_M R|\nabla f|^2 e^{-f}\,\mathrm{dV}\\ =\ & \int_M R_{ij}(\nabla^i\nabla^j f)e^{-f}\,\mathrm{dV} - \frac{1}{2}\int_M R_{ij}(\nabla^i f\nabla^j f)e^{-f}\,\mathrm{dV}\\ & + b\,(1-2\,(n-1)\,\rho)\int_M R\Delta e^{-f}\,\mathrm{dV} + 2b\int_M |R_{ij}|^2 e^{-f}\,\mathrm{dV}\\ & + \int_M \left(-2\rho bR^2 + \frac{a}{2}(1-n\rho)R\right)e^{-f}\,\mathrm{dV} + \frac{n\rho}{2}\int_M R\Delta e^{-f}\,\mathrm{dV}\\ & - \frac{\rho}{2}\int_M R|\nabla f|^2 e^{-f}\,\mathrm{dV}\\ =\ & \int_M R_{ij}(\nabla^i\nabla^j f)e^{-f}\,\mathrm{dV} + 2b\int_M |R_{ij}|^2 e^{-f}\,\mathrm{dV}\\ & + \int_M \left(-2\rho bR^2 + \frac{a}{2}(1-n\rho)R\right)e^{-f}\,\mathrm{dV} + \frac{1}{2}\int_M |\nabla_i\nabla_j f|^2 e^{-f}\,\mathrm{dV}\\ & + \frac{a}{2}\int_M (\Delta f)e^{-f}\,\mathrm{dV} + \left(\frac{n}{2} - 2(n-1)b\right)\rho\int_M R\Delta e^{-f}\,\mathrm{dV}\\ & - \frac{\rho}{2}\int_M R|\nabla f|^2 e^{-f}\,\mathrm{dV}\\ =\ & \int_M R_{ij}(\nabla^i\nabla^j f)e^{-f}\,\mathrm{dV} + 2b\int_M |R_{ij}|^2 e^{-f}\,\mathrm{dV} - 2\rho b\int_M R^2 e^{-f}\,\mathrm{dV}\\ & + \frac{a}{2}(1-n\rho)\int_M Re^{-f}\,\mathrm{dV} + \frac{1}{2}\int_M |\nabla_i\nabla_j f|^2 e^{-f}\,\mathrm{dV}\\ & + \frac{a}{2}(1-n\rho)\int_M (\Delta f)e^{-f}\,\mathrm{dV} + \frac{a}{2}n\rho\int_M (\Delta f)e^{-f}\,\mathrm{dV}\end{aligned} \tag{24}$$

$$
\begin{aligned}
&+\frac{n\rho}{2}\int_M R\Delta e^{-f}\,\mathrm{dV}-2(n-1)b\rho\int_M R\Delta e^{-f}\,\mathrm{dV}\\
&-\frac{\rho}{2}\int_M R|\nabla f|^2 e^{-f}\,\mathrm{dV}\\
=&\ \frac{1}{2}\int_M\left|R_{ij}+\nabla_i\nabla_j f+\frac{a}{2}(1-n\rho)g_{ij}\right|^2 e^{-f}\,\mathrm{dV}\\
&+\left(2b-\frac{1}{2}\right)\int_M|R_{ij}|^2e^{-f}\,\mathrm{dV}-2\rho b\int_M R^2e^{-f}\,\mathrm{dV}\\
&+\frac{\rho}{2}\int_M|\nabla f|^2(na-R)e^{-f}\,\mathrm{dV}\\
&+\left(\frac{n}{2}-2(n-1)b\right)\rho\int_M\Delta Re^{-f}\,\mathrm{dV}-\frac{na^2(1-n\rho)^2}{8}.
\end{aligned}
$$

□

Theorem 2.4. *Let $g(t)$, $t\in[0,T)$, be a solution to the RB-flow on a closed manifold M^n and let $\lambda_a^b(g)$ be the lowest constant such that the nonlinear equation* (2) *with* (3) *has positive solutions. Under assumptions $\rho\le 0$, $b\ge\frac{1}{4}$, $R(0)\ge na$ and $|\nabla f|^2\ge\Delta f$, the quantity*

$$\lambda_a^b(t)+\frac{na^2(1-n\rho)^2}{8}t$$

is strictly increasing along the RB-flow in $[0,T)$. Here $f=-2\log\phi$, where the function ϕ has been introduced before.

Proof. Using divergence theorem, we have

$$0=\int_M\Delta e^{-f}\,\mathrm{dV}=\int_M(|\nabla f|^2-\Delta f)e^{-f}\,\mathrm{dV}. \tag{25}$$

Since by hypothesis $|\nabla f|^2-\Delta f\ge 0$, we obtain from (25)

$$|\nabla f|^2=\Delta f. \tag{26}$$

Integrating by parts and using (26), yields

$$\int_M\Delta Re^{-f}\,\mathrm{dV}=\int_M R\Delta e^{-f}\,\mathrm{dV}=\int_M R(|\nabla f|^2-\Delta f)e^{-f}\,\mathrm{dV}=0.$$

On the other hand, by assumption we have $R(0)\ge na$. Thus, according to Lemma 2.3 and Lemma 2.6 in [3], we conclude that either $\max R(t)>na$ or $g(t)=g(0)$ for every $t\in(0,T)$. Suppose that $\max R(t)>na$, because otherwise the proof is trivial. Therefore, considering $\rho\le 0$, $b\ge\frac{1}{4}$ and $|\nabla f|^2\ge\Delta f$, Lemma 2.3 yields

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\mu(t)+\frac{na^2(1-n\rho)^2}{8}t\right)\Big|_{t=t_0}>0.$$

By definition, $\mu(t)$ is a smooth function in variable t. Therefore there exists a sufficiently small $\delta > 0$ such that in the interval $(t_0 - \delta, t_0 + \delta)$,

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\mu(t) + \frac{na^2(1-n\rho)^2}{8}t\right) > 0.$$

Thus

$$\mu(t_0) + \frac{na^2(1-n\rho)^2}{8}t_0 > \mu(t_1) + \frac{na^2(1-n\rho)^2}{8}t_1$$

for any $t_1 \in (t_0 - \delta, t_0 + \delta)$ and $t_1 < t_0$.

We note that

$$\mu(t_0) = \lambda_a^b(t_0) \quad \text{and} \quad \mu(t_1) \geq \lambda_a^b(t_1).$$

This implies that

$$\lambda_a^b(t_0) + \frac{na^2(1-n\rho)^2}{8}t_0 > \lambda_a^b(t_1) + \frac{na^2(1-n\rho)^2}{8}t_1$$

for any $t_0 > t_1$. Since according to Lemma 2.1 the quantity $\lambda_a^b(t) + \frac{na^2(1-n\rho)^2}{8}t$ is continuous and since $t_0 \in [0, T)$ is arbitrary, we can conclude that $\lambda_a^b(t) + \frac{na^2(1-n\rho)^2}{8}t$ is strictly increasing in $[0, T)$. □

Corollary 2.5. *Suppose $g(t)$, $t \in [0, T)$, is a solution to the RB-flow on a closed manifold M^n. Under assumptions $\rho \leq 0$, $b = \frac{n}{4(n-1)}$ and $R(0) \geq na$, the quantity*

$$\lambda_a^b + \frac{na^2(1-n\rho)^2}{8}t$$

is strictly increasing along the RB-flow.

Now, we claim that a locally symmetric manifold remains the same along the RB-flow. To prove this, we first need the following.

Proposition 2.6 (L. Cremaschi [5])**.** *During the Ricci-Bourguignon flow of a Riemannian manifold $(M^n, g(t))$, the Riemann tensor satisfies the following evolution equation.*

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijkl} = {} & \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) \\ & - g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + R_{ijkp}R_{ql}) \\ & - \rho(\nabla_i\nabla_k R g_{jl} - \nabla_i\nabla_l R g_{jk} - \nabla_j\nabla_k R g_{il} + \nabla_j\nabla_l R g_{ik}) \\ & + 2\rho R R_{ijkl},\end{aligned}$$

where the tensor B is defined as $B_{ijkl} = g^{pq}g^{rs}R_{ipjr}R_{kqls}$.

Remark 2.7. Recall that for a tensor field A of arbitrary type, we have the following formula for commuting the covariant derivative and the Laplacian:

$$\nabla(\Delta A) - \Delta(\nabla A) = \nabla \mathrm{Rm} * A + \mathrm{Rm} * \nabla A, \tag{27}$$

and for a t-dependent tensor field $A = A(t)$ we have

$$\frac{\partial}{\partial t}\nabla A - \nabla \frac{\partial}{\partial t} A = A * \nabla h. \tag{28}$$

Substituting A with Riemann curvature tensor Rm in (27) and (28), along with the fact that $h = 2(Ric - \rho R g)$, we get

$$\nabla(\Delta \mathrm{Rm}) = \Delta(\nabla \mathrm{Rm}) + \mathrm{Rm} * \nabla \mathrm{Rm} \tag{29}$$

and

$$\nabla \frac{\partial}{\partial t}\mathrm{Rm} = \frac{\partial}{\partial t}\nabla \mathrm{Rm} + \mathrm{Rm} * \nabla \mathrm{Rm} + \rho g * \mathrm{Rm} * \nabla \mathrm{Rm}. \tag{30}$$

Writing Proposition 2.6 in $*$-notation, yields

$$\frac{\partial}{\partial t}\mathrm{Rm} = \Delta \mathrm{Rm} + \mathrm{Rm} * \mathrm{Rm} + \rho g * \nabla^2 \mathrm{Rm}. \tag{31}$$

Finally, taking covariant derivative of (31) and using (29) together with (30), we have

$$\frac{\partial}{\partial t}\nabla \mathrm{Rm} = \Delta \nabla \mathrm{Rm} + \mathrm{Rm} * \nabla \mathrm{Rm} + \rho g * \nabla^3 \mathrm{Rm} + \rho g * \mathrm{Rm} * \nabla \mathrm{Rm}. \tag{32}$$

Now we are able to prove our claim.

Proposition 2.8. *Suppose $g(t)$, $t \in [0, T)$, is a solution to the RB-flow on a closed manifold M^n. If $(M, g(t))$ is locally symmetric at the initial time $t = 0$, then it remains locally symmetric for all times $t \in [0, T)$.*

Proof. Consider the original ODE which according to (32), $\nabla \mathrm{Rm}_{g(t)}$ is its answer. On the other hand, zero is an answer for this ODE too. Therefore, by the uniqueness of the solution of an ODE, we conclude that $\nabla \mathrm{Rm}_{g(t)} \equiv 0$ for all $t \in [0, T)$. □

Lemma 2.3 along with the above fact, gives:

Corollary 2.9. *Suppose $g(t)$, $t \in [0, T)$, is a solution to the RB-flow on a closed manifold M^n with $g(0) = g_0$. Moreover suppose (M, g_0) is locally symmetric. Then under assumptions $\rho \leq 0$, $b \geq \frac{1}{4}$ and $R(0) \geq na$, the quantity*

$$\lambda_a^b + \frac{na^2(1 - n\rho)^2}{8} t$$

is strictly increasing along the RB-flow.

Proof. By hypothesis, (M, g_0) is locally symmetric. Hence due to Proposition 2.8, we get

$$\nabla \mathrm{Rm(t)} \equiv 0$$

for $t \in [0, T)$. Accordingly,

$$\nabla R(t) = \mathrm{tr} \nabla \mathrm{Rm}_{g(t)} = 0. \tag{33}$$

Integrating by parts and using (33), we obtain

$$\int_M \Delta R e^{-f}\,\mathrm{dV} = \int_M R\Delta e^{-f}\,\mathrm{dV} = -\int_M \langle \nabla R\,, \nabla e^{-f}\rangle\,\mathrm{dV} = 0. \tag{34}$$

Employing Lemma 2.3 along with the expression (34) and proceeding similar to the proof of Theorem 2.4, we can conclude the proof of the corollary. □

Our next aim is to show that isometries are preserved along the RB-flow. By Theorem 3.1.2 in [5] we know that for $\rho < \frac{1}{2(n-1)}$ any initial compact Riemannian manifold (Mg_0) has a unique smooth solution $(M, g(t))$ solving the flow equation (1). Thus we have the following proposition.

Proposition 2.10. *Suppose $(M, g(t))$, $t \in [0, T)$, is a compact solution to the RB-flow with $g(0) = g_0$ and $\rho < \frac{1}{2(n-1)}$. If $\phi : (M, g_0) \longrightarrow (M, g_0)$ is an isometry, then it will remain an isometry along the RB-flow for all $t \in [0, T)$.*

Proof. Since $g(t)$, $t \in [0, T)$, is a solution to the RB-flow, we have

$$\frac{\partial g(t)}{\partial t} = -2Ric_{g(t)} + 2\rho R_{g(t)} g(t).$$

Accordingly, for $t \in [0, T)$ we attain

$$\frac{\partial}{\partial t}\phi^* g(t) = -2Ric_{\phi^* g(t)} + 2\rho R_{\phi^* g(t)} \phi^* g(t),$$

which shows that $\phi^* g(t)$ is a solution to the RB-flow with $\phi^* g(0) = g_0$ as well. Hence, by uniqueness of the solutions to the RB-flow, we get $\phi^* g(t) = g(t)$, for all $t \in [0, T)$. □

Hereunder, using the above fact, we show that the homogeneity is preserved along the RB-flow.

Proposition 2.11. *Suppose $g(t)$, $t \in [0, T)$, is a solution to the RB-flow on a closed manifold M^n with $\rho < \frac{1}{2(n-1)}$. If $(M, g(t))$ is homogeneous at the initial time $t = 0$, then it remains homogeneous for all times $t \in [0, T)$.*

Proof. Since (M, g_0) is homogeneous, its isometry group $\mathrm{Iso}(M, g_0)$, acts transitively. Now let $\phi : (M, g_0) \longrightarrow (M, g_0)$ be an isometry. Proposition 2.10 shows that $\phi : (M, g(t)) \longrightarrow (M, g(t))$, $t \in [0, T)$, is an isometry as well. Consequently $\phi \in \mathrm{Iso}(M, g(t))$ and thus $\mathrm{Iso}(M, g_0) \subset \mathrm{Iso}(M, g(t))$ for $t \in [0, T)$. So, if $x, y \in M$ are arbitrary, then there exists a member of $\mathrm{Iso}(M, g_0)$ and hence a member of $\mathrm{Iso}(M, g(t))$ such that $\phi(x) = y$. Therefore $\mathrm{Iso}(M, g(t))$, $t \in [0, T)$, also acts transitively and thus $(M, g(t))$, $t \in [0, T)$, is homogeneous too. □

Corollary 2.12. *Suppose $g(t)$, $t \in [0, T)$, is a solution to the RB-flow on a closed manifold M^n. Moreover suppose (M, g_0) is homogeneous. Then under assumptions $\rho \le 0$, $b \ge \frac{1}{4}$ and $R(0) \ge na$, the quantity*

$$\lambda_a^b(g) + \frac{na^2(1-n\rho)^2}{8}t$$

is strictly increasing along the RB-flow.

Proof. Since (M, g_0) is homogeneous, thus by virtue of Proposition 2.11, $(M, g(t))$ must be homogeneous along the RB-flow. On the other hand, we know that a homogeneous manifold has constant scalar curvature. Therefore,

$$\Delta R_{g(t)} \equiv 0$$

for $t \in [0, T)$. Hence, by Lemma 2.3 we can conclude the proof. □

We now recall the definitions of a breather.

Definition. A metric $g(t)$ evolving under the RB-flow flow is called an breather, if there exist some $t_1 < t_2$ and $\alpha > 0$ such that $\alpha g(t_1)$ and $g(t_2)$ differ only by a diffeomorphism. A breather is called steady, shrinking or expanding if $\alpha = 1$, $\alpha < 1$ or $\alpha > 1$, respectively.

Theorem 2.13. *On a closed manifold M^n, a steady breather $g(t)$ evolving by the RB-flow is Ricci-flat when $\rho \leq 0$.*

Proof. We proceed as in [11] to prove the theorem. Let

$$\mathcal{F}^b(g, f) = \int_M (|\nabla f|^2 + bR)e^{-f} \text{ dV}$$

and

$$\mathcal{F}^b_a(g, f) = \int_M (|\nabla f|^2 - a\frac{f}{2} + bR)e^{-f} \text{ dV}.$$

So, we can write

$$\mathcal{F}^b_a(g, f) = \mathcal{F}^b(g, f) - \int_M a\frac{f}{2}e^{-f} \text{ dV}. \tag{35}$$

We define

$$\tilde{\lambda}^b_a(g) = \inf\left\{\mathcal{F}^b_a(g, f) : \int_M e^{-f}\text{dV} = 1,\ f \in C^\infty(M)\right\}.$$

By [11, Theorem 3.1] and (35) we get

$$\begin{aligned}
&\frac{\text{d}}{\text{dt}}\mathcal{F}^b_a(g, f)\\
&= 2b\int_M |Ric + \frac{1-(n-1)\rho}{b}Hess(f)|^2 e^{-f} \text{ dV}\\
&\quad - \frac{2\rho b}{n}\int_M |Rg + \frac{n}{2b}Hess(f)|^2 e^{-f} \text{ dV}\\
&\quad + \left[2(1-2(n-1)\rho) - \frac{2(1-(n-1)\rho)^2}{b} + \frac{n\rho}{2b}\right]\int_M |Hess(f)|^2 e^{-f} \text{ dV}\\
&\quad - \frac{a}{2}\int_M \big[(n\rho-1)R - (1-2(n-1)\rho)\\
&\quad + f(1-2(n-1)\rho)\big](\Delta f - |\nabla f|^2)e^{-f} \text{ dV},
\end{aligned} \tag{36}$$

where f evolves by

$$\frac{\partial f}{\partial t} = (n\rho - 1)R - (1 - 2(n-1)\rho)(\Delta f - |\nabla f|^2).$$

If we assume that $|\nabla f|^2 \geq \Delta f$, then using divergence theorem we have

$$0 = \int_M \Delta e^{-f}\ \mathrm{dV} = \int_M (|\nabla f|^2 - \Delta f)e^{-f}\ \mathrm{dV}.$$

Thus we obtain

$$|\nabla f|^2 = \Delta f. \tag{37}$$

So, considering $|\nabla f|^2 \geq \Delta f$, (36) yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{F}_a^b(g, f) \\
&= 2b\int_M |Ric + \frac{1-(n-1)\rho}{b}Hess(f)|^2 e^{-f}\ \mathrm{dV} \\
&\quad - \frac{2\rho b}{n}\int_M |Rg + \frac{n}{2b}Hess(f)|^2 e^{-f}\ \mathrm{dV} \\
&\quad + \left[2(1-2(n-1)\rho) - \frac{2(1-(n-1)\rho)^2}{b} + \frac{n\rho}{b}\right]\int_M |Hess(f)|^2 e^{-f}\ \mathrm{dV} \\
&= \frac{\mathrm{d}}{\mathrm{dt}}\mathcal{F}^b(g, f).
\end{aligned} \tag{38}$$

Under assumptions $\rho \leq 0$, $b \geq \frac{4(1-(n-1)\rho)^2 - n\rho}{4(1-2(n-1)\rho)} > 0$ and using Corollary 3.3 in [11], one can conclude that $\mathcal{F}^b(g, f)$ and so by (38), the functional $\mathcal{F}_a^b(g, f)$ is nondecreasing with respect to variable t. Moreover, by (38) along with Corollary 3.3 in [11], we conclude that $\mathcal{F}_a^b(g, f)$ is strictly monotone unless the metric is Ricci-flat. Now suppose that $g(t)$ is a breather. Therefore, by definition of a breather, there exist some $t_1 < t_2$ such that $g(t_1)$ and $g(t_2)$ differ only by a diffeomorphism. Thus $\tilde{\lambda}_a^b(t_1) = \tilde{\lambda}_a^b(t_2)$. Let $\tilde{f}(t_2)$ be the corresponding function to $\tilde{\lambda}_a^b(t_2)$. we have

$$\begin{aligned}
\tilde{\lambda}_a^b(t_2) &= \mathcal{F}_a^b(g(t_2), \tilde{f}(t_2)) \\
&\geq \mathcal{F}_a^b(g(t_1), \tilde{f}(t_1)) \\
&\geq \inf \mathcal{F}_a^b(g(t_1), f) \\
&= \tilde{\lambda}_a^b(t_1).
\end{aligned}$$

As mentioned before, $\mathcal{F}^b(g, f)$ is strictly monotone unless the metric is Ricci-flat. So the equality $\tilde{\lambda}_a^b(t_1) = \tilde{\lambda}_a^b(t_2)$ shows that $g(t)$ is Ricci-flat. □

Remark 2.14. A similar result was obtained by Wang (see [11, Corollary 2.8]).

3. Monotonicity of $\lambda_a^b(g)$ along the normalized RB-flow

In this section we first state the following lemma which provides the evolution formula for the scalar curvature and volume element along the normalized RB-flow.

Lemma 3.1. *Suppose $g(t)$ is a solution to the normalized RB-flow. Let R and* dV *denote the scalar curvature and volume element of metric $g(t)$ respectively. Then the following evolution equations hold*:

$$\frac{\partial R}{\partial t} = 2|Ric|^2 + (1+2\rho\,(1-n))\,\Delta R - 2\rho R^2 - \frac{2}{n}sR,$$

$$\frac{\partial dV}{\partial t} = -(1-n\rho)(R-r)\,\mathrm{dV},$$

where s is as introduced in Section 1.

Proof. Due to Proposition 2.3.12 in [10], the volume element evolves as $\frac{\partial \mathrm{dV}}{\partial t} = \frac{1}{2}\mathrm{tr}h\,\mathrm{dV}$. Since along the normalized RB-flow, $h = -2Ric + 2\rho Rg + \frac{2}{n}sg$, we obtain

$$\frac{\partial \mathrm{dV}}{\partial t} = \frac{1}{2}\mathrm{tr}\left(-2Ric + 2\rho Rg + \frac{2}{n}sg\right)\,\mathrm{dV} = -\,(1-n\rho)\,(R-r)\,\mathrm{dV}.$$

In order to find the evolution equation for the scalar curvature R along the normalized RB-flow, we note that by Proposition 2.3.9 in [10], the scalar curvature R evolves as $\frac{\partial R}{\partial t} = -\langle Ric\,, h\rangle + \delta^2 h - \Delta(\mathrm{tr}h)$, Thus

$$\begin{aligned}\frac{\partial R}{\partial t} &= -\left\langle Ric\,, -2Ric + 2\rho Rg + \frac{2}{n}sg\right\rangle \\ &\quad + \delta^2\left(-2Ric + 2\rho Rg + \frac{2}{n}sg\right) + 2\,(1-n\rho)\,\Delta(R-r),\end{aligned} \tag{39}$$

where for a tensor T, $\delta(\mathsf{T}) = -\mathrm{tr}_{12}\nabla\mathsf{T}$ is the divergence operator. On the one hand, by the contracted second Bianchi identity we have $\delta^2(Ric) = \frac{1}{2}\Delta R$. On the other hand, a direct computation shows that $\delta^2(Rg) = \Delta R$. Hence, by (39) we attain

$$\frac{\partial R}{\partial t} = 2|Ric|^2 + (1+2\rho\,(1-n))\,\Delta R - 2\rho R^2 - \frac{2}{n}sR. \qquad \square$$

We will need to the following lemma:

Lemma 3.2. *Let $g(t)$, $t \in [0,T)$, be a solution to the normalized RB-flow on a closed manifold M^n and let $\lambda_a^b(g)$ be the lowest constant such that the nonlinear equation* (2) *with* (3) *has positive solutions. Suppose that $u(t_0)$ is the corresponding solution to $\lambda_a^b(t_0)$. Then we have*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}\left(\mu(t) + \frac{na^2(1-n\rho)^2}{8}t + \frac{sa}{2}t + \frac{2s}{n}\int_0^t \lambda^b(s)\,\mathrm{ds}\right)\Big|_{t=t_0} \\ &= \frac{1}{2}\int_M \left|R_{ij} + \nabla_i\nabla_j f + \frac{a}{2}(1-n\rho)g_{ij}\right|^2 e^{-f}\,\mathrm{dV}\end{aligned} \tag{40}$$

$$
\begin{aligned}
&+\left(2b-\frac{1}{2}\right)\int_M |R_{ij}|^2 e^{-f}\,\mathrm{dV}\\
&-2\rho b\int_M R^2 e^{-f}\,\mathrm{dV}+\frac{\rho}{2}\int_M |\nabla f|^2(na-R)e^{-f}\,\mathrm{dV}\\
&+\left(\frac{n}{2}-2(n-1)b\right)\rho\int_M \Delta R e^{-f}\,\mathrm{dV}.
\end{aligned}
$$

Here $f=-2\log\phi$, where the function ϕ has been introduced before. And λ^b is the lowest eigenvalue of (4).

Proof. Let the metric $g(t)$ evolve under the normalized RB-flow. According to the Lemma 3.1 we have $\frac{\partial dV}{\partial t}=-(1-n\rho)(R-r)\,\mathrm{dV}$. By a direct computation we also have,

$$
\frac{\partial}{\partial t}|\nabla\phi|^2=\left(2R^{ij}-2\rho Rg^{ij}-\frac{2}{n}sg^{ij}\right)(\nabla_i\phi\nabla_j\phi)+2g^{ij}\nabla_i\phi_t\nabla_j\phi.
$$

Thus, by virtue of (11) we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\mu(t)\big|_{t=t_0}=&\int_M\Big[2R_{ij}(\nabla^i\phi\nabla^j\phi)-2\rho R|\nabla\phi|^2-\frac{2}{n}s|\nabla\phi|^2+2\nabla^i\phi_t\nabla_i\phi\\
&+2a\phi\phi_t\log\phi+a\phi\phi_t+bR_t\phi^2+2bR\phi\phi_t\Big]\,\mathrm{dV}\\
&-\int_M\left(|\nabla\phi|^2+a\phi^2\log\phi+bR\phi^2\right)((1-n\rho)(R-r))\,\mathrm{dV}.
\end{aligned}\tag{41}
$$

Applying

$$
2\int_M R_{ij}\nabla^i\phi\nabla^j\phi\,\mathrm{dV}=\int_M\left((-\nabla_iR\,\nabla^i\phi)\phi-2R_{ij}(\nabla^i\nabla^j\phi)\phi\right)\,\mathrm{dV}
$$

and

$$
-\int_M|\nabla\phi|^2\,(R-r)\;\mathrm{dV}=\int_M\left[(R-r)\,\Delta\phi+\nabla_iR\nabla^i\phi\right]\phi\,\mathrm{dV}
$$

into (41), we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\mu(t)\big|_{t=t_0}=&\int_M\Big[-2R_{ij}(\nabla^i\nabla^j\phi)\phi-\nabla_iR(\nabla^i\phi)\phi-2\rho R|\nabla\phi|^2-\frac{2}{n}s|\nabla\phi|^2\\
&+bR_t\phi^2+a\phi\phi_t+2\phi_t\left(-\Delta\phi+a\phi\log\phi+bR\phi\right)\\
&-(1-n\rho)(R-r)\phi\left(-\Delta\phi+a\phi\log\phi+bR\phi\right)\\
&+(1-n\rho)(\nabla_iR\nabla^i\phi)\phi\Big]\,\mathrm{dV}\\
=&\int_M\Big[-2R_{ij}(\nabla^i\nabla^j\phi)\phi-2\rho R|\nabla\phi|^2-\frac{2}{n}s|\nabla\phi|^2+bR_t\phi^2\\
&+\frac{a}{2}(\phi^2)_t\Big]\,\mathrm{dV}+\mu(t_0)\left(\int_M\phi^2\,\mathrm{dV}\right)_t\Big|_{t=t_0}\\
&-n\rho\int_M\nabla_iR(\nabla^i\phi)\phi\,\mathrm{dV}
\end{aligned}\tag{42}
$$

$$= \int_M \Big[-2R_{ij}(\nabla^i\nabla^j\phi)\phi - 2\rho R|\nabla\phi|^2 - \frac{2}{n}s|\nabla\phi|^2 + bR_t\phi^2 + \frac{a}{2}(1-n\rho)(R-r)\phi^2\Big]\,\mathrm{dV} - n\rho\int_M \nabla_i R(\nabla^i\phi)\phi\,\mathrm{dV},$$

where the last equality is obtained by

$$\int_M \left((\phi^2)_t - (1-n\rho)(R-r)\phi^2\right)\,\mathrm{dV} = 0$$

from (3). By Lemma 3.1 we have, $\frac{\partial R}{\partial t} = 2|Ric|^2 + (1+2\rho\,(1-n))\,\Delta R - 2\rho R^2 - \frac{2}{n}sR$. Thus from (42), we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\mu(t)\big|_{t=t_0} &= \int_M \Big[-2R_{ij}(\nabla^i\nabla^j\phi)\phi - 2\rho R|\nabla\phi|^2 - \frac{2s}{n}|\nabla\phi|^2 + bR_t\phi^2 \\
&\quad + \frac{a}{2}(1-n\rho)(R-r)\phi^2\Big]\,\mathrm{dV} - n\rho\int_M \nabla_i R(\nabla^i\phi)\phi\,\mathrm{dV} \\
&= \int_M \Big[-2R_{ij}(\nabla^i\nabla^j\phi)\phi - 2\rho R|\nabla\phi|^2 - \frac{2s}{n}|\nabla\phi|^2 \\
&\quad + b\phi^2\left((1+2\rho\,(1-n))\,\Delta R + 2|R_{ij}|^2 - 2\rho R^2 - \frac{2}{n}sR\right) \\
&\quad + \frac{a}{2}(1-n\rho)(R-r)\phi^2\Big]\,\mathrm{dV} - n\rho\int_M \nabla_i R(\nabla^i\phi)\phi\,\mathrm{dV} \\
&= \int_M \Big[-2R_{ij}(\nabla^i\nabla^j\phi)\phi - 2\rho R|\nabla\phi|^2 + (1-2(n-1)\rho)\,bR\Delta\phi^2 \\
&\quad + 2b\phi^2|R_{ij}|^2 - 2\rho b\phi^2R^2 + \frac{a}{2}(1-n\rho)(R-r)\phi^2\Big]\,\mathrm{dV} \\
&\quad - \frac{2s}{n}\int_M \left(|\nabla\phi|^2 + bR\phi^2\right)\,\mathrm{dV} - n\rho\int_M \nabla_i R(\nabla^i\phi)\phi\,\mathrm{dV}.
\end{aligned} \tag{43}$$

By hypothesis we know that $f = -2\log\phi$ which is equivalent to $\phi^2 = e^{-f}$. Keeping this in mind, according to (15) and (16), we can write (43) as follows:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\mu(t)\big|_{t=t_0} &= \int_M \Big[R_{ij}(\nabla^i\nabla^j f) - \frac{1}{2}R_{ij}\nabla^i f\nabla^j f - \frac{\rho}{2}R|\nabla f|^2 \\
&\quad + (1-2(n-1)\rho)\,bR|\nabla f|^2 - (1-2(n-1)\rho)\,bR\Delta f + 2b|R_{ij}|^2 \\
&\quad - 2\rho bR^2 + \frac{a}{2}(1-n\rho)(R-r)\Big]e^{-f}\,\mathrm{dV} \\
&\quad - \frac{2s}{n}\int_M \left(\frac{1}{4}|\nabla f|^2 + bR\right)e^{-f}\,\mathrm{dV} + \frac{n\rho}{2}\int_M R\Delta e^{-f}\,\mathrm{dV}.
\end{aligned} \tag{44}$$

Doing a similar computation such as Lemma 2.3, we achieve

$$\frac{\mathrm{d}}{\mathrm{dt}}\mu(t)\big|_{t=t_0} = \frac{1}{2}\int_M \left|R_{ij} + \nabla_i\nabla_j f + \frac{a}{2}(1-n\rho)g_{ij}\right|^2 e^{-f}\,\mathrm{dV}$$

$$
\begin{aligned}
&+\left(2b-\frac{1}{2}\right)\int_M |R_{ij}|^2 e^{-f}\,\mathrm{dV}-\frac{na^2(1-n\rho)^2}{8}\\
&-\frac{2s}{n}\int_M\left(\frac{1}{4}\Delta f+bR\right)e^{-f}\,\mathrm{dV}-2\rho b\int_M R^2e^{-f}\,\mathrm{dV}\\
&-\frac{a}{2}(1-n\rho)r+\frac{\rho}{2}\int_M|\nabla f|^2(na-R)e^{-f}\,\mathrm{dV}\\
&+\left(\frac{n}{2}-2(n-1)b\right)\rho\int_M R\Delta e^{-f}\,\mathrm{dV},
\end{aligned}
$$

which according to (4) and (6) gives

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\left(\mu(t)+\frac{na^2(1-n\rho)^2}{8}t+\frac{sa}{2}t+\frac{2s}{n}\int_0^t\lambda^b(s)\,\mathrm{ds}\right)\Big|_{t=t_0}\\
=&\ \frac{1}{2}\int_M\left|R_{ij}+\nabla_i\nabla_j f+\frac{a}{2}(1-n\rho)g_{ij}\right|^2e^{-f}\,\mathrm{dV}\\
&+\left(2b-\frac{1}{2}\right)\int_M|R_{ij}|^2e^{-f}\,\mathrm{dV}-2\rho b\int_M R^2e^{-f}\,\mathrm{dV}\\
&+\frac{\rho}{2}\int_M|\nabla f|^2(na-R)e^{-f}\,\mathrm{dV}\\
&+\left(\frac{n}{2}-2(n-1)b\right)\rho\int_M R\Delta e^{-f}\,\mathrm{dV}.
\end{aligned}
$$

Therefore, the proof is complete. □

Remark 3.3. In dimension $n = 2$, the scalar curvature determines the full curvature tensor. In fact, in two dimension the scalar curvature is twice the Gaussian curvature. By Gauss-Bonnet theorem, for a closed Manifold M^2, we have

$$\int_M K\,\mathrm{dV}=2\pi\chi(M),$$

where K is the Gaussian curvature and $\chi(M)$ is the Euler characteristic of M. According to the above argument we get

$$r=\frac{\int_M 2K\,\mathrm{dV}}{\int_M \mathrm{dV}}=\frac{4\pi\chi(M)}{A},$$

where A is the area of M^2. Thus r and consequently $s=(1-n\rho)r$ are constant. Then due to Lemma 3.2, we achieve the following monotonicity theorem.

Theorem 3.4. *Let $g(t)$, $t\in[0,T)$, be a solution to the normalized RB-flow on a compact surface M^2 and let $\lambda_a^b(g)$ be the lowest constant such that the nonlinear equation* (2) *with* (3) *has positive solutions. Under assumptions $\rho\le 0$, $b=\frac{1}{2}$ and $R(0)\ge 2a$, the quantity*

$$\lambda_a^b(t)+\frac{a^2(1-2\rho)^2}{4}t+\frac{sa}{2}t+s\int_0^t\lambda^b(s)\mathrm{ds}$$

is strictly increasing along the normalized RB-flow in $[0, T)$. *Here* λ^b *is the lowest eigenvalue of* (4).

Remark 3.5. The normalized RB-flow and unnormalized RB-flow are essentially the same flows. They only differ by a reparametrization of time and scaling factor in space. So it is reasonable to anticipate that some features of the RB-flow, such isometry preserving or homogeneity preserving hold for the normalized RB-flow as well. Thus we are able to state and prove the analogous of Proposition 2.10 and Proposition 2.11 along the normalized RB-flow. We do not prove them here, because of the similarity of the proofs to the case of unnormalized RB-flow.

Proposition 3.6. *Suppose* $(M, g(t))$, $t \in [0, T)$, *is a compact solution to the normalized RB-flow with* $g(0) = g_0$ *and* $\rho < \frac{1}{2(n-1)}$. *If* $\phi : (M, g_0) \longrightarrow (M, g_0)$ *is an isometry, then it will remain an isometry along the normalized RB-flow for all* $t \in [0, T)$.

The subsequent fact shows that the homogeneity is preserved along the RB-flow.

Proposition 3.7. *Suppose* $g(t)$, $t \in [0, T)$, *is a solution to the normalized RB-flow on a closed manifold* M^n *with* $\rho < \frac{1}{2(n-1)}$. *If* $(M, g(t))$ *is homogeneous at initial time* $t = 0$, *then it remains homogeneous for all times* $t \in [0, T)$.

The following fact is a direct consequence of Lemma 3.2 and Proposition 3.7.

Theorem 3.8. *Suppose* $g(t)$, $t \in [0, T)$, *is a solution to the normalized RB-flow on a closed manifold* M^n. *Moreover suppose* (M, g_0) *is homogeneous. Let* $\lambda_a^b(g)$ *be the lowest constant such that the nonlinear equation* (2) *with* (3) *has positive solutions. Then under assumptions* $\rho \leq 0$, $b \geq \frac{1}{4}$ *and* $R(0) \geq na$, *the quantity*

$$\lambda_a^b(t) + \frac{na^2(1-n\rho)^2}{8}t + \frac{sa}{2}t + \frac{2s}{n}\int_0^t \lambda^b(s)\mathrm{ds}$$

is strictly increasing along the normalized RB-flow in $[0, T)$, *where* λ^b *is the lowest eigenvalue of* (4).

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