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ADAPTATION OF THE MINORANT FUNCTION FOR LINEAR PROGRAMMING

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Abstract. In this study, we propose a new logarithmic barrier approach to solve linear programming problem using the projective method of Karmarkar. We are interested in computation of the direction by Newton's method and of the step-size using minorant functions instead of line search methods in order to reduce the computation cost. Our new approach is even more beneficial than classical line search methods. We reinforce our purpose by many interesting numerical simulations proved the effectiveness of the algorithm developed in this work.

1. Introduction

Interior-point methods are one of the efficient methods developed to solve linear and non linear programming problems.

Several algorithms have been proposed to solve the linear programming problem, where, we distinguish three fundamental classes of interior point methods namely: projective interior point methods and their alternatives, central trajectory methods, barrier/penalty methods [2]. Our work is based on the latter type of interior point methods for solving linear programming problems.

In this paper, we propose a logarithmic barrier interior-point method for solving linear programming problems (LP). In fact, the main difficulty to be anticipated in establishing an iteration in such a method will come from the determination and computation of the step-size. Various approaches are developed to overcome this difficulty. It is known in [2, 6] that the computation of the step-size is expensive specically when line search methods are used. Leulmi and all. [5] proposed efficient and less expensive procedures in semidefinite programming not only to avoid line search methods, but also to accelerate the algorithm's convergence. The purpose of this paper is to exploit this idea for LP problems.

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We consider the following linear programming problem

$$(D)\begin{cases} \min\limits_{x} f\left(x\right) \\ g_i(x)\leq 0,\ \ i=1,...,n \\ x\in\mathbb{R}^n, \end{cases}$$

where $f,\ g_i$ étant convexes et différentiables sur un ouvert convexe contenant

$$X=\left\{x\in\mathbb{R}^m:g_i(x)\leq 0,\ \ i=1,...,n\right\}.$$

In all which follows, we denot by

(1) $X=\{x\in\mathbb{R}^m:g_i(x)\leq 0,\ \ i=1,...,n\}$, the set of feasible solutions of (D).
(2) $\widehat{X}=\{x\in\mathbb{R}^m:g_i(x)<0,\ \ i=1,...,n\}$, the set of strictly feasible solutions of (D).

Let $u,v\in\mathbb{R}^n$, their scalar product is defined by

$$\langle u,v\rangle=u^Tv=\sum_{i=1}^{n}u_iv_i.$$

We suppose that the set $\widehat{X}$ (coincides with $int(X)$) is not empty.

The problem (D) is approximated by the following perturbed problem (D_η)

$$(D_\eta)\begin{cases} \min\ \ f_\eta(x) \\ x\in\mathbb{R}^n, \end{cases} \tag{1}$$

with the penalty parameter $\eta>0$, and f_η is the barrier function defined by

$$f_\eta(x)=\begin{cases} f\left(x\right)+n\eta\ln\eta-\eta\sum\limits_{i=1}^{n}\ln\left(-g_i\left(x\right)\right) & \text{if} \quad x\in X \\ +\infty & \text{if not.} \end{cases}$$

We are interested then in solving the problem (D_η).

The idea of this new approach consists to introduce one original process to calculate the step-size based on minorant functions.

The main advantage of (D_η) resides in the strict convexity of its objective function and its feasible domain. Consequently, the conditions of optimality are necessary and sufficient. This fosters theoretical and numerical studies of the problem.

One of the advantages of the problem (D) with respect to its prturbed problem (D_η) is that variable of the objective function is a vector instead of a matrix in the type problem (D_η).

We study in the next section, the existence and uniqueness of optimal solution of the problem (D_η), and we show its convergence to problem (D), in particular the behavior of its optimal value and its optimal solutions when $\eta\to 0$, then $\lim\limits_{\eta\to 0}x_\eta=x^*$ is an optimal solution of (D).

In Section 3, we propose an interior point algorithm based on the Newton's approach which allows us to solve the nonlinear system resulting from the optimality conditions. The iteration of this algorithm is of descent type, defined by $x_{k+1} = x_k + \alpha_k d_k$, where d_k is the descent direction and α_k is the step-size. Also, we present different steps-size by minimizing a minorant functions which approximate the unidimensional function $\theta(\alpha_k) = \min_{\alpha>0} f(x + \alpha d)$. The last section, is dedicated to the presentation of comparative numerical tests to illustrate the effectiveness of our approaches and to determine the most efficient algorithm.

The main advantage of (D_η) resides in the strict convexity of its objective function and its feasible domain. Consequently, the conditions of optimality are necessary and sufficient. This, fosters theoretical and numerical studies of the problem.

Before this, it is necessary to show that (D_η) has at least an optimal solution.

2. Existence and uniqueness of optimal solution of perturbed problem and its convergence to problem (D)

2.1. Existence and uniqueness of optimal solution of perturbed problem

Firstly, we give the following definition

Definition 2.1. *Let f be a function defined from $\mathbb{R}^m$ to $\mathbb{R}\cup\{\infty\}$, f is called inf-compact if for all $\eta > 0$, the set $X_\eta(f) = \{x \in \mathbb{R}^m : f(x) \leq \eta\}$ is compact, which comes in particular to say that its cone of recession is reduced to zero.*

To prove that (D_η) has an optimal solution, we show that f_η is inf-compact. For that, it is enough to prove that the cone of recession

$$\widehat{X}\left((f_\eta)_\infty\right) = \left\{d \in \mathbb{R}^n, (f_\eta)_\infty(d) \leq 0\right\},$$

is reduced to the origin, i.e.,

$$\left((f_\eta)_\infty(d) \leq 0\right) \Rightarrow (d = 0),$$

where $(f_\eta)_\infty$ is defined by

$$(f_\eta)_\infty(d) = \lim_{\alpha\to+\infty} \frac{f_\eta(x+\alpha d) - f_\eta(x)}{\alpha} = b^T d.$$

This needs to prove the following proposition.

Proposition 2.1. *$d = 0$ whenever $b^T d \leq 0$ and $A^T d \in \widehat{X}$.*

Proof. Assume that $d \neq 0$, $b^t d \leq 0$ and $U = \sum_{i=1}^m x_i A_i \in S_n^+$. As the set $\widehat{F}$ is nonempty, then there exists $x > 0$, such that $Ax = b$.
We have

$$\langle b,\ d\rangle = \langle Ax, d\rangle = \left\langle x, A^T d\right\rangle > 0 \Rightarrow b^T d > 0,$$

contradiction.
Hence, f_η is inf-compact, therefore we conclude that the problem (D_η) admits

at least an optimal solution.
The proposition is proved. □

Then, The problem (D_η) has an optimal solution.

We know that the Hessian matrix $H = \nabla^2 f_\eta(x)$ is positive definite, then the problem (D_η) is strictly convex and if it has an optimal solution then it is unique.

We have

$$f_\eta(x) = f(x) + n\eta \ln \eta - \eta \sum_{i=1}^{n} \ln\left(-g_i(x)\right),$$

As f_η is inf-compact and strictly convex, therefore the problem (D_η) admits a unique optimal solution.

We denote by $x(\eta)$ or x_η the unique optimal solution of (D_η).

2.2. Convergence of the perturbed problem to the problem (D)

For $x \in \widehat{X}$, let's introduce the symmetrical definite positive matrix B_i of rank m, $i = 1, ..., n$ and the lower triangular matrix L, such that $B_i = Ae_i \left(Ae_i\right)^T = LL^T$, which implies that H is a positive definite matrix.

In what follows, we will be interested by the behavior of the optimal value and the optimal solution $x(\eta)$ of the problem (D_η). For that, let us introduce the function θ defined by

$$\theta(x,\eta) = f_\eta(x) = \begin{cases} f_\eta(x) & \text{if} \qquad x \in \widehat{X}, \\ +\infty & \text{if not.} \end{cases}$$

Proposition 2.2. *For $\eta > 0$, let x_η an optimal solution of the problem (D_η), then there exists $x \in X$ an optimal solution of (D), such that, $\lim\limits_{\eta \longrightarrow 0} x_\eta = x$.*

Proof. The function θ is differentiable at the optimal point $(x(\eta), \eta)$ and checks

$$\nabla_x \theta\left(x(\eta), \eta\right) = \nabla f_\eta\left(x_\eta\right) = 0.$$

So, for all $x \in \widehat{X}$ we have

$$f(x) = \theta(x, 0) \geq \theta\left(x(\eta), \eta\right) + \langle x - x(\eta), \nabla_x \theta\left(x(\eta), \eta\right)\rangle + (0 - \eta)\nabla_\eta \theta\left(x(\eta), \eta\right),$$

This gives

$$f(x) \geq f(x_\eta) - n\eta, \ \forall x \in \widehat{X}.$$

Hence

$$\min_{x \in X} f(x) \geq f\left(x_\eta\right) - n\eta,$$

then

$$f\left(x_\eta\right) \geq \min_{x \in X} f(x) \geq f\left(x_\eta\right) - n\eta.$$

When η tends towards zero, we obtain

$$\min_{x \in X} f(x) = \lim_{\eta \longrightarrow 0} f(x_\eta).$$

Finally, if x_η is an optimal solution of the problem (D_η) then there exists $x = \lim_{\eta \longrightarrow 0} x_\eta$ an optimal solution of the problem (D). The proposition is proved. □

In all that follows, we adopt the following conventions : The vector $e \in \mathbb{R}^n$ is the vector whose all the components are equal to 1, given a vector $x \in \mathbb{R}^n$, Y is the diagonal matrix whose diagonal elements are the components of x (**i.e.**, $Y = diag\{x\}$). The convergence of the algorithm is based on the following function, called "multiplicative potential function", defined for all $x \in X,\ x > 0$, by

$$f(x) = \frac{\langle b, x \rangle^n}{\prod_{i=1}^{n} x_i},$$

that we extend by semi-continuity on Y.

We can also consider the function "logarithmic potential function", defined by

$$q(x) = \ln f(x) = n \ln(\langle b, x \rangle) - \sum_{i=1}^{n} \ln(x_i),$$

where f verifies the following properties

1) $0 < f(x) < +\infty$ if $x > 0$ and $Ax = 0$.

2) $f(x) = +\infty$ if x belongs to the relative boundary X without solution of (D).

3) $f(x) = 0$ if x is solution of (D) or if $x = 0$.

4) $f(kx) = f(x)$ for all $x \in X$ and all $k > 0$.

Thus, the problem (D) is to find the optimal solutions of the problem

$$\begin{cases} \min f(x) = 0 \\ \quad Ax = 0 \\ x \geq 0,\ x \neq 0. \end{cases}$$

Starting from the known $x \in X$ point, the Karmarkar algorithm is a descent method that generates, because of the barrier character of the objective function f, a sequence of points all contained in the relative interior of X, of where the denomination of method of interior points. We will describe the transition from the initial iterated x to the iterated $\tilde{x}$. We assume that iterated $\tilde{x}$ satisfies $\tilde{x} > 0$ and $A\tilde{x} = 0$.

Remark 1. Normalization we normalize x by the relation $x = \sqrt{\frac{n}{\langle x, x \rangle}}x$,in order to have $\langle x, x \rangle = n$.

3. Computation of the Newton descent direction

In this part, we are interested in the numerical solution of the problem (D_η). Interior point methods of types logarithmic barrier are conceived for solving this problem type while being based on the optimality conditions which are necessary and sufficient. x_η is an optimal solution of (D_η) if it satisfies the following condition

$$\nabla f_\eta (x_\eta) = 0. \tag{2}$$

To solve (1), we use the Newton's approach which means to find at each iteration a vector $x_{\eta k} + d_k$ checking the following linear system

$$H_k d_k = -\nabla f_\eta (x_{\eta k}). \tag{3}$$

As $H_k = \nabla^2 f_\eta (x_{\eta k})$ is a symmetric positive definite matrix, the Cholesky methods and the conjugate gradient methods are the best convenient for solving the system (2). It's easy to see that we have

$$\frac{f(\tilde{x})}{f(x)} = g(z),$$

with

$$z = X^{-1}\tilde{x},\ g(z) = \frac{\langle b, z\rangle^n}{\prod\limits_{i=1}^{n} z_i},$$

and $b = \frac{1}{\langle c,x\rangle} Xc$. The conditions $A\tilde{x} = 0, \tilde{x} \geq 0$ and $\tilde{x} \neq 0$ are transposed into

$$AXz = 0, z \geq 0 \text{ and } z \neq 0.$$

Let's put $B = AX$. Problem (D) is equivalent to the problem

$$(D) \begin{cases} \min g(z) = 0 \\ Bz = 0 \\ z \geq 0,\ z \neq 0, \end{cases}$$

e is feasible solution of this problem and we have $g(e) = \langle b, e\rangle = 1$.

Since we have $g(kz) = g(z)$ for all $z \geq 0$ and all $k > 0$, we will work on the following standard problem

$$(D_N) \begin{cases} \min g(z) = 0 \\ Bz = 0 \\ \langle e, z\rangle = n,\ z \geq 0. \end{cases}$$

It is easy to see that the matrix (A^t, x) is of rank $m+1$, it is the same of the matrix (B^t, e). Obtaining a Newton descent direction at point e for the problem (D_N) is obtained by solving the quadratic problem

$$(D_Q) \begin{cases} \min \frac{1}{2} \langle \nabla^2 g(e) d, d\rangle + \langle \nabla g(e), d\rangle \\ Bd = 0,\ \langle e, d\rangle = 0. \end{cases}$$

For that, let's introduce the matrix

$$P = I - (B^t, e) \left[(B^t, e)^t (B^t, e) \right]^{-1} (B^t, e)^t,$$

which corresponds to the projection on the linear subspace

$$E = \{d : Bd = 0, \langle e, d \rangle = 0\},$$

we have $P^2 = P = P^t, PB^t = 0$ and $Pe = 0$. It's easy to see that we have

$$P\nabla g(e) = Pb \text{ and } P\nabla^2 g(e)P = I + n(n-1)Pbb^tP,$$

the quadratic problem is equivalent to the problem

$$\begin{cases} \min \frac{1}{2} \left\langle \nabla^2 g(e) d, d \right\rangle + \langle \nabla g(e), d \rangle \\ Pd = d, \end{cases}$$

whose optimal solution is collinear to $d = -Pb = -P\nabla g(e)$.

The direction d coincides with the direction given by the projected gradient. We are now interested in some properties of d. First, we have $\langle d, e \rangle = -\langle Pb, e \rangle = -\langle b, Pe \rangle = 0$, we then observe that we have on the one hand

$$\left\{ \begin{array}{c} z : \langle e, z \rangle = n \\ \|z - e\|^2 \leq \frac{n}{n-1} \end{array} \right\} \subset \left\{ \begin{array}{c} z : \langle e, z \rangle = n \\ z \geq 0 \end{array} \right\} \subset \left\{ \begin{array}{c} z : \langle e, z \rangle = n \\ \|z - e\|^2 \leq n(n-1) \end{array} \right\},$$

on the other hand

$$\begin{cases} \min \langle b, z - e \rangle = -1 \\ Bz = 0 \\ \langle e, z \rangle = n, \ z \geq 0, \end{cases}$$

since $P(e - \bar{z}) = e - \bar{z}$, then $\langle b, z - e \rangle = \langle Pb, z - e \rangle$, we obtain

$$\|Pb\| \sqrt{\frac{n}{n-1}} \leq 1 \leq \|Pb\| \sqrt{n(n-1)},$$

In the following, we denote by $\bar{d}$ and σ the mean and the standard deviation of $(d_1, d_2, ..., d_n)$. we have

$$\bar{d} = \frac{1}{n} \sum_{i=1}^{n} d_i = 0 \text{ and } \sigma^2 = \frac{\|d\|^2}{n} - \bar{d}^2.$$

Note that all optimal solutions of this problem is a convex polyhedron contained in the relative boundary of the set of feasible points. To ensure the convergence of the algorithm towards an optimal solution x^* of (D_η), it should be made sure that all the iterate $x_{\eta k} + d_k$ remains strictly feasible.

In the next section, we calculate the step-size for our new approach.

4. Computation of the step-size with the minorant functions

In the descent methods, the line search methods are known to compute the optimal step-size α_k. It suffice to minimize the unidimensional function

$$\theta(\alpha_k) = \min_{\alpha > 0} f_\eta(x + \alpha d).$$

The most used methods of the type line search are those of Goldstein–Armijo, Fibonacci, etc. Unfortunately, these methods are expensive in computational volume, and even inapplicable to semidefinite problems. To avoid this difficulty, we exploit the idea suggested by J.P. Crouzeix and B. Merikhi [2] which approaches the function

$$\varphi(\alpha) = \frac{1}{\eta}[f_\eta(x+\alpha d) - f_\eta(x)], \tag{4}$$

by the simple minorant function giving at each iteration k, a step-size α_k in an easy way, simple and much less expensive than line search methods.

But, we propose a new idea, we suggest the simple minorant functions, we approaches the function (4) .

Remark 2. To keep the function $\varphi(\alpha)$ well defined, it is necessary that for all $x \in \widehat{X}$, $(x+\alpha d)$ still in $\widehat{X}$. Which returns to find $\widehat{\alpha} > 0$ such that for any $\alpha \in [0, \widehat{\alpha}]$, $x + \alpha d \in \widehat{X}$.

Proposition 4.1. *Let* $\widehat{\alpha} = \sup\{\alpha,\ 1 + d_i\alpha\}$ *with* $d_i = -P\nabla g(e)$, $\forall i = 1, ..., n$. *Far all* $\alpha \in [0, \widehat{\alpha}]$, *the following function* $\varphi(\alpha)$ *is well defined*

$$\varphi(\alpha) = n\ln\left(1 - \|d\|^2\alpha\right) - \sum_{i=1}^{n} \ln(1 + d_i\alpha),\ \ \alpha \in [0, \widehat{\alpha}].$$

4.1. Some useful inequalities

Before determining these functions, we need the following results

The following result is caused by H. Wolkowicz and al. [9], see also J. P. Crouzeix and al. [3] for additional results.

Proposition 4.2. [9]

$$\begin{aligned} \bar{x} - \sigma_x\sqrt{n-1} &\leq \min_i x_i \leq \bar{x} - \frac{\sigma_x}{\sqrt{n-1}}, \\ \bar{x} + \frac{\sigma_x}{\sqrt{n-1}} &\leq \max_i x_i \leq \bar{x} + \sigma_x\sqrt{n-1}. \end{aligned}$$

Let's recall that, B. Merikhi and al. (2008) [2] proposed some useful inequalities related to the maximum and to the minimum of $x_i > 0$ for any $i = 1, ..., n$

$$n\ln(\bar{x} - \sigma_x\sqrt{n-1}) \leq A \leq \sum_{i=1}^{n} \ln(x_i) \leq B \leq n\ln(\bar{x}), \tag{7}$$

with

$$\begin{aligned} A &= (n-1)\ln(\bar{x} + \frac{\sigma_x}{\sqrt{n-1}}) + \ln(\bar{x} - \sigma_x\sqrt{n-1}), \\ B &= \ln(\bar{x} + \sigma_x\sqrt{n-1}) + (n-1)\ln(\bar{x} - \frac{\sigma_x}{\sqrt{n-1}}). \end{aligned}$$

Such that $\bar{x}$ and σ_x are respectively, the mean and the standard deviation of a statistical series $\{x_1, x_2, ..., x_n\}$ of n real numbers. These quantities are defined as follows

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{and} \quad \sigma_x^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2.$$

Based on this results, we give in the following, new notions of the non expensive minorant functions for φ, that offers some variable steps-size to every iteration with a simple technique.

Thanks to definite positivity results in linear algebra, we propose three different alternatives that offers some variable steps-size α to every iteration.

The efficient one to the other can be translated by numerical tests that we will present at the end of this work.

4.2. The minorant functions

We seek a minorant function $\widehat{\varphi}$ of φ on $[0, \widehat{\alpha}_i]$, $i = 1, 2$ and 3, such that

$$\|d\|^2 = n(\bar{d}^2 + \sigma^2) = \widehat{\varphi}''(0) = -\widehat{\varphi}'(0),\ \widehat{\varphi}\,(0) = 0.$$

In the following, we take $x_i = 1 + \alpha d_i$, $x = 1 + \alpha d$ and $\sigma_x = \alpha\sigma$.

4.2.1. *First minorant function.* This strategy consists to minimize the minorant approximations $\widetilde{\varphi}$ of φ over $[0, \widehat{\alpha}[$. To be efficient, this minorant approximation needs to be simple and sufficiently near φ. In our case, it requires

$$0 = \widetilde{\varphi}(0),\ \|d\|^2 = \widetilde{\varphi}''(0) = -\widetilde{\varphi}'(0).$$

By applying inequalities (7) , we give

$$\sum_{i=1}^{n} \ln(x_i) \leq B.$$

Where

$$n \ln\left(1 - n\alpha\sigma^2\right) - \sum_{i=1}^{n} \ln(1 + d_i\alpha) \geq n \ln\left(1 - n\alpha\sigma^2\right) - B.$$

Thus the first minorant function can be defined as follows

$$\begin{aligned} \varphi_1(\alpha) &= n \ln\left(1 - n\alpha\sigma^2\right) - B \\ \varphi_1(\alpha) &= n \ln\left(1 - n\alpha\sigma^2\right) - (n-1)\ln\left(1 - \frac{\sigma\alpha}{\sqrt{n-1}}\right) - \ln\left(1 + \sigma\alpha\sqrt{n-1}\right). \end{aligned}$$

The minorant function φ_1 is definite and convex on $[0, \widehat{\alpha}_1]$ and we have $\varphi_1(\alpha) \leq \varphi(\alpha)$,
with $\varphi_1(0) = 0$ and $\varphi_1''(0) = -\varphi_1'(0) = \|d\|^2$.

This minimum is obtained in $\bar{\alpha}_1 = \alpha_{opt}$, such that, $\varphi_1'(\alpha) = 0$. We have

$$\varphi_1'(\alpha) = -\frac{n^2\sigma^2}{1 - n\alpha\sigma^2} + \frac{(n-1)\,\sigma}{\sqrt{n-1} - \sigma\alpha} - \frac{\sigma\sqrt{n-1}}{1 + \sigma\alpha\sqrt{n-1}}.$$

Thus, we deduce that the function φ_1 reaches its minimum at a unique point $\bar{\alpha}_1$ which is the root of $\varphi_1'(\alpha) = 0$.

$$\bar{\alpha}_1 = \frac{n\sqrt{n-1}}{\sqrt{n-1} - n\left(n-2\right)\sigma}.$$

We take for new iterated

$$\tilde{x} = X\left(e + \bar{\alpha}_1 d\right) = x + \bar{\alpha}_1 X d.$$

By construction we have $\tilde{x} > 0$ and $A\tilde{x} = 0$. By replacing $\bar{\alpha}_1$ by its value we obtain

En remplaçant $\bar{\alpha}$ par sa valeur on obtient

$$\varphi_1\left(\bar{\alpha}_1\right) = \left(n-1\right)\ln\left(1 + \frac{n\sigma}{\sqrt{n-1}}\right) + \ln\left(1 - n\sigma\sqrt{n-1}\right).$$

The quantity $\varphi_1\left(\bar{\alpha}_1\right) - \varphi_1\left(0\right) = \varphi_1\left(\bar{\alpha}_1\right)$ depends on σ. We pose $u = n\sigma\sqrt{n-1}$, we obtain : $1 \leq u \leq n-1$ and

$$\begin{aligned} \varphi_1\left(\bar{\alpha}_1\right) &= \left(n-1\right)\ln\left(1 + \frac{n\sigma}{\sqrt{n-1}}\right) + \ln\left(1 - n\sigma\sqrt{n-1}\right) \\ &= \left(n-1\right)\ln\left(1 + \frac{u}{n-1}\right) + \ln\left(1-u\right) = \xi\left(u\right). \end{aligned}$$

We deduce

$$\varphi_1\left(\bar{\alpha}_1\right) = \xi\left(u\right) > \ln\left(2\right) - 1 + \frac{n}{2\left(n-1\right)}\left(u+1\right) \geq \ln\left(\frac{2}{e}\right) \simeq -0.307.$$

4.2.2. *Second minorant function.* One can also thought of simpler functions than φ_1 (involving only one logarithm) to extract from the known inequality

$$n\ln\left(1 - n\alpha\sigma^2\right) - \sum_{i=1}^{n}\ln(1 + d_i\alpha) - \ln\left(1 + \|d\|\alpha\right) \leqslant 0,$$

where

$$\varphi_2\left(\alpha\right) = n\ln\left(1 - n\alpha\sigma^2\right) - \ln\left(1 + \beta_2\alpha\right), \ \alpha \in \left[0, \widehat{\alpha}_2\right],$$

with $\widehat{\alpha}_2 = \frac{-1}{\beta_2}$ and $\beta_2 = \|d\|$.

The minorant function φ_2 is definite and convex on $[0, \widehat{\alpha}_2]$ and we have $\varphi\left(\alpha\right) \geq \varphi_2\left(\alpha\right)$,

with $\varphi_2\left(0\right) = 0$ and $\varphi_2''\left(0\right) = -\varphi_2'\left(0\right) = \|d\|^2$.

4.2.3. *Third minorant function.* Another minorant function simpler than φ_1 involving also only one logarithm.

We consider functions of the following type

$$\breve{\varphi}(\alpha) = n\ln\left(1 - n\alpha\sigma^2\right) - \breve{\gamma}\ln(1 + \breve{\beta}\alpha), \ \alpha \in [0, \breve{\alpha}[,$$

where in order to fulfill the requirements

$$\breve{\alpha} = \sup[\alpha : 1 + \alpha\breve{\beta} > 0]. \tag{5}$$

We can also think of another minorant function φ_3 better approximating φ_1 than φ_2, i.e.,

$$\varphi_2(\alpha) \leq \varphi_3(\alpha) \leq \varphi_1(\alpha),$$

such that $\beta_3 = \bar{d} - \frac{\sigma}{\sqrt{n-1}}$, and we are looking for $\gamma_3 = \frac{\|d\|^2}{\beta_3^2}$ which checks (5), which gives

$$\varphi_3(\alpha) = n \ln\left(1 - n\alpha\sigma^2\right) - \gamma_3 \ln(1 + \beta_3 \alpha),$$

Proposition 4.3. *φ_i, $i = 1, 2, 3$, is strictly convex over $[0, \mathring{\alpha}[$, with $\mathring{\alpha} = \min(\widehat{\alpha}, \widehat{\alpha}_1, \widehat{\alpha}_2)$, $\varphi(\alpha) \to +\infty$ when $\alpha \to \widehat{\alpha}$. So we have*

$$\varphi_2(\alpha) \leq \varphi_3(\alpha) \leq \varphi_1(\alpha) \leq \varphi(\alpha),\ \forall \alpha \in [0, \mathring{\alpha}[\,.$$

Proof. The first inequality is obvious. The inequality $\varphi(\alpha) \geq \varphi_1(\alpha)$ is a direct consequence of (7). Let's consider

$$g(\alpha) = \varphi_3(\alpha) - \varphi_1(\alpha).$$

Since $\beta_1 = \beta_2$ and $\beta_1 \leq \gamma_1$, we have for any $\alpha \in [0, \mathring{\alpha}[$

$$g''(\alpha) = \frac{\gamma_2\beta_2^2 - (n-1)\beta_1^2}{(1+\beta_1\alpha)^2} - \frac{\gamma_1^2}{(1+\gamma_1\alpha)^2} \leq \frac{\gamma_1^2}{(1+\beta_1\alpha)^2} - \frac{\gamma_1^2}{(1+\gamma_1\alpha)^2} \leq 0,$$

and since $g(0) = g'(0) = 0$ and $g''(\alpha) \leq 0$, it becomes $g(\alpha) \leq 0$ for any $\alpha \in [0, \mathring{\alpha}[$. Then, let's put

$$h(\alpha) = \varphi_2(\alpha) - \varphi_3(\alpha),$$

so

$$h(0) = h'(0) = 0 \text{ and } h''(\alpha) = \frac{\beta_2^2}{(1+\beta_2\alpha)^2} - \frac{\gamma_3\beta_3^2}{(1+\beta_3\alpha)^2}.$$

Since $\|d\|^2 = \gamma_2\beta_2^2$ and so $\beta_3 = \|d\|$

$$h''(\alpha) = \|d\|^2 \left(\frac{1}{(1+\beta_2\alpha)^2} - \frac{1}{(1+\beta_3\alpha)^2} \right) \leq 0.$$

because $\beta_3 \leq \beta_2$. Therefore $h(\alpha) \leq 0$ for any $\alpha \in [0, \mathring{\alpha}[$. $\square$

Thus, we deduce that the function φ_i reaches its minimum at a unique point $\bar{\alpha}_i$ which is the root of $\varphi_i'(\alpha) = 0$. Thus, the three values $\bar{\alpha}_i$, $i = 1, 2, 3$ are explicitly computed, then, we take $\bar{\alpha}_1$, $\bar{\alpha}_2$ and $\bar{\alpha}_3$ are belongs to the interval $(0, \widehat{\alpha} - \varepsilon)$ and $\varphi'(\alpha) < 0$, with $\varepsilon > 0$, is a fixed precision.

Remark 3. The calculation of $\bar{\alpha}$ is performed by a dichotomous procedure, in the cases where $\bar{\alpha}_i \notin (0, \widehat{\alpha} - \varepsilon)$, and $\varphi'(\alpha) > 0$, as follows
Put $a = 0$ and $b = \widehat{\alpha} - \varepsilon$.
while $|b - a| > 10^{-4}$
If $\varphi(\frac{a+b}{2}) < 0$ then $b = \frac{a+b}{2}$
else $a = \frac{a+b}{2}$, so $\bar{\alpha} = b$.
This calculation guarantees a better approximation of the minimizer of $\varphi'(\alpha)$ while remaining in the domain of φ.

Proposition 4.4. *Let x_{k+1} and x_k two strictly feasible solutions of (D_η), obtained respectively at the iteration $k+1$ and k, so we have*

$$f_\eta(x_{k+1}) < f_\eta(x_k).$$

Proof. We have

$$f_\eta(x_{k+1}) \simeq f_\eta(x_k) + \langle \nabla f_\eta(x_k), x_{k+1} - x_k \rangle,$$

and $x_{k+1} = x_k + \alpha_k d_k$ thus

$$\begin{aligned} f_\eta(x_{k+1}) - f_\eta(x_k) &\simeq \langle \nabla f_\eta(x_k), \alpha_k d_k \rangle = \alpha_k \left\langle -\nabla^2 f_\eta(x_k) d_k, d_k \right\rangle \\ &\simeq -\alpha_k \left\langle \nabla^2 f_\eta(x_k) d_k, d_k \right\rangle < 0. \end{aligned}$$

Therefore $f_\eta(x_{k+1}) < f_\eta(x_k)$.

□

5. The algorithm

In this section, we present the algorithm of our approach to obtain an optimal solution $\bar{x}$ to the problem (D).

For simplicity, we consider x_k instead of $x_{\eta k}$ and x instead of x_η.

Begin algorithm

Initialization

$x_0 > 0$ is a strictly feasible solution of (D), $d_0 \in \mathbb{R}^n$, $\varepsilon > 0$ is a given precision.

Iteration

- **While** $\left|b^T d^k\right| > \varepsilon$ do
  1. Normalization : $x = \sqrt{\frac{n}{\langle x,x \rangle}} x$
  2. Take $b = \frac{1}{\langle c,x \rangle} Xc, B = AX$.
  3. Solve $\{d : B_k d = 0, \langle e, d \rangle = 0\}$, $\delta = Xd$.
  4. Compute the step-size using the strategies $S_i,\ i = 1, 2, 3$.
  5. Take the new iterate $x^{k+1} = x^k + \alpha_k d_k$.
  6. Take $k = k + 1$.
- **End while**

End algorithm

This approach reaches to reduce the number of the iteration and the time of calculation. In the following Section, we present some examples.

6. Numerical tests

The following examples are taken from the literature see for instance [[1],[4],[8]] and implemented in MATLAB R2013a on Pentium(R) Dual Core CPU T4400 (2.20 GHz) with 3.00 Go RAM. We have taken $\varepsilon = 1.0e-006$. In the table of results, (size) represents the size of the example, (itrat) represents the number of iterations necessary to obtain an optimal solution, (time) represents the time of computation in seconds (s) and (st) represents the strategy. We note that the matrices used in the numerical tests are full matrices.

6.1. Examples of fixed sizes

Example 01:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \; b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } c = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^t$$

Example 02:

$$A = \begin{bmatrix} 2 & 3 & 1 & 2 \\ 3 & 0 & -2 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \text{ and } c = \begin{bmatrix} 4 & 1 & 2 & 0 \end{bmatrix}^T.$$

Example 03:

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & 1 & -1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}, \; b = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \text{ and } c = \begin{bmatrix} 3 & 2 & 1 & 3 \end{bmatrix}^T.$$

Example 04:

$$A = \begin{bmatrix} 2 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \; b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } c = \begin{bmatrix} 3 & -1 & 1 & 0 & 0 & 0 \end{bmatrix}^T.$$

Example 05:

$$A = \begin{bmatrix} -1 & 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & -3 & 2 & 0 & 1 & 0 \\ -3 & 2 & 1 & 0 & 0 & 0 & 1 \\ 3 & 5 & 4 & 0.5 & 0 & 0 & 0 \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix} \text{ and } c = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & -2 \end{bmatrix}^T.$$

Example 06:

$$A = \begin{bmatrix} 0 & 1 & 2 & -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & -2 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & -1 & -2 & 0 & 0 & 0 & 1 \\ 1 & 3 & 4 & 2 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{bmatrix} \text{ and } c = \left[\begin{matrix} 1 & 0 & -2 & 1 & 1 & 0 & 0 \end{matrix} \right.$$

Example 07:

$$A = \begin{bmatrix} 1 & 0 & -4 & 3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 5 & 3 & 1 & 0 & -1 & 3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 4 & 5 & -3 & 3 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & 1 & -5 & 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & -3 & 2 & -1 & 4 & 5 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 4 \\ 4 \\ 5 \\ 7 \\ 5 \end{bmatrix}$$

$$\text{and } c = \begin{bmatrix} -4 & -5 & -1 & -3 & 5 & -8 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

Example 08:

The matrix A is

$$\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

The vector c and b are

$$c = \begin{bmatrix} 2 & -1 & -3 & 5 & -2 & 0 & 4 & 1 & 2 & -1 & 1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$b = \begin{bmatrix} 8 & 4 & 6 & 2 & 5 & 1 & 2 & 6 & 3 & 9 & 4 \end{bmatrix}^T$$

The last examples can be given in the following table

size	st_1		st_3		st_2	
	itrat	time	itrat	time	itrat	time
2×3	2	0.0016	2	0.0016	4	0.0025
2×4	3	0.012	5	0.032	7	0.045
3×4	1	0.001	1	0.0019	4	0.0034
3×6	4	0.024	6	0.042	9	0.077
4×7	6	0.032	9	0.045	10	0.069
5×9	5	0.049	8	0.079	12	0.089
6×12	7	0.055	9	0.082	13	0.091
11×25	9	0.058	13	0.086	15	0.098

6.2. Example cube

$n = 2m,\ n = 2m,\ A[i,j] = 0$ if $i \neq j$ or $(i+1) \neq j$.
$A[i,j] = A[i, i+m] = 1,\ b[i] = 2$, for $i, j = 1, ..., m$.
The following table resumes the obtained results

size	st_1		st_2		st_3	
	itrat	time	itrat	time	itrat	time
50×100	1	0.014	12	0.099	3	0.35
100×200	1	0.023	17	0.55	6	0.51
200×400	2	0.051	18	0.59	7	0.66
450×900	3	0.069	21	0.98	9	0.85

Commentary

These tests show, clearly, the impact of our three strategies offer an optimal solution of (D) in a reasonable time and with a small number of iterations.

We notice that the 1^{st} strategy is the best. The obtained comparative numerical results favor this last strategy moreover, it requires a computing time largely low vis-a-vis the other two strategies. This seems to be quite expected, because theoretically the strategy st_1 uses the function φ_1 that is the closest (best approximation) of the function φ.

7. Conclusion

In spite of the mathematical development in the domain of the linear programming, a lot of problems remain to develop. For it, in our survey, we treated a theoretical and numerical survey of our new approach, based on the notion of minorant functions. This allows us to determine the step-size by a simple and easy manner. As expected, the technique of minorant functions for the computation of the step-size proves its efficiency, and this by reducing the computational cost in the projective algorithm of Karmarkar compared to the line search method.

The numerical simulations confirm the effectiveness of our approaches. Our algorithm converges to the same optimal solution, using any strategy among the three proposed strategies. The first strategy is the best approach versus computing time and number of iterations. Thus, the numerical tests prove that our approache was reducing the cost of iteration for the linear programming. Our survey, opens interesting perspective for the non linear programming (NLP).

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