Korean J. Math. **27** (2019), No. 3, pp. 645–655
https://doi.org/10.11568/kjm.2019.27.3.645

EXTENDING AND LIFTING OPERATORS ON BANACH SPACES

JeongHeung Kang

Abstract. In this article, we show that the nuclear operator defined on Banach space has an extending and lifting operator. Also we give new proofs of the well known facts which were given Pelczýnski theorem for complemented subspaces of ℓ_1 and Lewis and Stegall's theorem for complemented subspaces of $L_1(\mu)$.

1. Introduction

In this article, we first show that the nuclear operator on Banach space has an extension property and a lifting property such as an operator defined on ℓ_∞-space and as an operator on ℓ_1-space which may not be norm preserving. Also we give a new proof of the well known facts which were given by Pelczýnski for complemented subspaces of ℓ_1 in [17] and [2, p.114] and by Lewis and Stegall's theorem for a complemented subspace of $L_1(\mu)$ in [2] and [10].

We start our discussion with well known results concerning with geometric structures of ℓ_∞ and ℓ_1 spaces and with bounded linear operators between these Banach spaces. We first recall the well known generalization of the Hahn-Banach extension theorem due to Nachbin [16].

Received April 17, 2019. Revised July 20, 2019. Accepted July 22, 2019.
2010 Mathematics Subject Classification: 46B03.
Key words and phrases: Extension Property, lifting property, absolutely p-summing operator, nuclear operator.
This paper is partially supported by the Hwa-Rang Dae Research Institute in 2018.

THEOREM 1.1. [16, 17]. *Let (Ω, Σ, μ) be a measure space and let Y be a closed subspace of a Banach space X. Then every bounded linear operator $T : Y \to L_\infty(\mu)$ extends to an operator $\tilde{T} : X \to L_\infty(\mu)$ such that $\tilde{T}|_Y = T$, $\|\tilde{T}\| = \|T\|$ and they satisfy the following diagram.*

$$\begin{array}{ccc} Y & \xrightarrow{T} & L_\infty(\mu) \\ i\downarrow & \nearrow \tilde{T} & \\ X & & \end{array} \tag{1.1}$$

We can readily show that with $\ell_\infty(\Gamma)$ in place of $L_\infty(\mu)$ this result is a consequence of the Hahn-Banach extension theorem.

In this direction of research, we can give one of motivating fact that the absolutely 2-summing operators between Banach spaces have the extension property due to the particular form of their factorization through as $L_\infty(\mu)$ space where μ is a suitable probability measure. Now for this we will give the definition of $p-$summing operators on a Banach space which is given several books such as in [1], [2] and [18].

DEFINITION 1.2. [1] Suppose that $1 \leq p < \infty$ and $T : X \to Y$ is a bounded linear operator between Banach spaces. We say that T is *absolutely p-summing operator* if there is a constant $C > 0$ such that for any finite natural number n and for any choice of $x_1, x_2, \cdots, x_n$ in X, we have

$$(\sum_{i=1}^{n} \|Tx_i\|^p)^{1/p} \leq C \sup\{(\sum_{i=1}^{n} | < x^*, x_i > |^p)^{1/p} : x^* \in B_{X^*}\}. \tag{1.2}$$

The least C for which inequality as given above (1.2) always holds is denoted by $\pi_p(T)$. Moreover, we will denote by $\Pi_p(X, Y)$ the set of all $p-$summing operators $T : X \to Y$. Then we know that $\Pi_p(X, Y)$ is a linear subspaces of $L(X, Y)$ the space of all bounded linear operators from X into Y and that for all $T \in \Pi_p(X, Y)$,

$$\|T\| \leq \pi_p(T).$$

Also, it is known that $1 \leq p < \infty$, $\pi_p(T)$ is a norm on $\Pi_p(X, Y)$, with which this space is a Banach space. Also the space $\Pi_p(X, Y)$ satisfies the following (so-called) "ideal property". If $T : X \to Y$ is a $p-$summing, and if $R : W \to X$ and $S : Y \to Z$ are bounded operators between Banach spaces, then the composition STR is a $p-$summing, and we have $\pi_p(STR) \leq \|S\|\pi_p(T)\|R\|$. For more facts for absolutely $p-$summing operator refers to books [1] and [2].

Now let's go back to the extension property of linear operators. As a well known fact, for any probability measure μ, $L_\infty(\mu)$ is an injective Banach space given in Theorem 1.1. The following theorem is an easy consequence of Pietch factorization theorem for 2-summing operators in [1, pp.86].

THEOREM 1.3. [1] [π_2-extension theorem]. *Let X, Y, and Z be Banach spaces with X a subspace of Z. Then every 2-summing operator $T : X \to Y$ admits a 2-summing extension $\tilde{T} : Z \to Y$ with $\tilde{T}|_X = T$ and $\pi_2(\tilde{T}) = \pi_2(T)$.*

DEFINITION 1.4. [2] A bounded linear operator $T : X \to Y$ is called a *nuclear operator* if there exist sequences (x_n^*) in X^* and (y_n) in Y such that

$$\sum_{n=1}^{\infty} \|x_n^*\| \|y_n\| < \infty$$

and such that for all $x \in X$,

$$T(x) = \sum_{n=1}^{\infty} x^*(x) y_n.$$

Also the nuclear norm of T is defined by

$$\|T\|_{nuc} = inf\{\sum_{n=1}^{\infty} \|x_n^*\| \|y_n\| : T(x) = \sum_{n=1}^{\infty} x^*(x) y_n\} \tag{1.3}$$

where the infimum is taken over all sequences (x_n^*) in X^* and (y_n) in Y.

For later references, we now give one more definition concerning the ℓ_1-space which has a property dual to the extension property saying the "lifting property".

DEFINITION 1.5. We say that a Banach space X has the *lifting property* if for every bounded linear operator q from a Banach space Z onto a Banach space Y and for every $T \in L(X, Y)$, there exists a $\tilde{T} \in L(X, Z)$ such that $T = q \circ \tilde{T}$, $\|\tilde{T}\| \leq \lambda \|T\|$, for some $\lambda > 0$ and the following diagram commutes;

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \tilde{T}\downarrow & \nearrow q & \\ Z & & \end{array} \tag{1.4}$$

Also for the later reference, we gave the well known theorem which is given several authors such as in [9] and in [17].

Theorem 1.6. ([13], pp.107) *Let X and Y be Banach spaces such that there is an operator q from Y onto X. Then for every $T \in L(\ell_1, X)$ there is a $\tilde{T} \in L(\ell_1, Y)$ for which $q \circ \tilde{T} = T$ and $\|\tilde{T}\| \leq \lambda \|T\|$, for some $\lambda > 0$.*

Moreover if q is a quotient map then for every $\epsilon > 0$, $\tilde{T}$ may be chosen so that $\|\tilde{T}\| \leq (1+\epsilon)\|T\|$ and the following diagram commutes;

$$\begin{array}{ccc} \ell_1 & \xrightarrow{T} & X \\ \tilde{T}\downarrow & \nearrow q & \\ Y & & \end{array} \tag{1.5}$$

For the converse of above theorem, Köthe [9] also proved as following theorem in [13]. From this theorem we can see that the space having the lifting property is characterized in the nonseparable case also.

Theorem 1.7. [9] *Every Banach space X with the lifting property is isomorphic to $\ell_1(\Gamma)$, for some index set Γ.*

2. Main results

In this article, first one of main question is which bounded linear operator has extending on its upper spaces, as like Hahn-Banach extension theorem. There are a lot of such extending operators as you can see several theorems(Theorem 4.1 and Theorem 4.2, p.1723-1726) given by M. Zippin in [4]. Also we can see the π_2-extension Theorem 1.3. Second, we are concerned with the lifting property, that is, which bounded linear operators on X into Y can have the lifting property? The space with the lifting property is completely characterized in separable and nonseparable cases as given in Theorem 1.7. Now we will give another fact for extension property for a nuclear operator on Banach spaces.

Theorem 2.1. *Let X, Y and Z be Banach spaces with X a subspace of Z. Then if a bounded linear operator $T : X \to Y$ is a nuclear operator, then there exists a nuclear extended operator $\tilde{T} : Z \longrightarrow Y$ such that $\tilde{T}|_X = T$ and $\|\tilde{T}\|_{nuc} \leq (1+\epsilon)\|T\|_{nuc}$, for any $\epsilon > 0$.*

Proof. Suppose $T : X \to Y$ is a nuclear operator. Then for $\epsilon > 0$, by the definition of nuclear operator norm we can find sequences (x_n^*) in X^* and (y_n) in Y such that

$$\sum_{n=1}^{\infty} \|x_n^*\|\|y_n\| \leq (1+\epsilon)\|T\|_{nuc}. \tag{2.1}$$

Now define $S : X \to \ell_\infty$ by $S(x) = (\frac{x_n^*(x)}{\|x_n^*\|})$ and $R : \ell_\infty \to \ell_1$ by $R((a_n)) = ((a_n\|x_n^*\|\|y_n\|))$. Then we have

$$\|S(x)\| = \|(\frac{x_n^*(x)}{\|x_n^*\|})\|_\infty \leq \|x\|.$$

Hence $\|S\| \leq 1$. Also, for $(a_n) \in \ell_\infty$ with $\|(a_n)\|_\infty \leq 1$,

$$\begin{aligned} \|R((a_n))\| &= \sum_{n=1}^{\infty} |a_n|\|x_n^*\|\|y_n\| \\ &\leq \sum_{n=1}^{\infty} \|x_n^*\|\|y_n\| \qquad \text{by (2.1)} \\ &\leq (1+\epsilon)\|T\|_{nuc}. \qquad (2.2) \end{aligned}$$

Therefore we also have $\|R\|_{nuc} \leq (1+\epsilon)\|T\|_{nuc}$.

Finally, define $U : \ell_1 \to Y$ by $U(b_n) = \sum_{n=1}^{\infty} \frac{b_n y_n}{\|y_n\|}$. Then we have

$$\begin{aligned} \|U(b_n)\| &\leq \sum_{n=1}^{\infty} |b_n|\|y_n\|/\|y_n\| \\ &\leq \sum_{n=1}^{\infty} |b_n| \\ &= \|b_n\|_1. \qquad (2.3) \end{aligned}$$

Hence we have $\|U\| \leq 1$. For all $x \in X$,

$$\begin{aligned} URS(x) &= UR(\frac{x_n^*(x)}{\|x_n^*\|}) \\ &= U(x_n^*(x)\|y_n\|) \\ &= \sum_{n=1}^{\infty} x_n^*(x) y_n = T(x). \qquad (2.4) \end{aligned}$$

From the above argument, we have the following diagram;

$$
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
{\scriptstyle S}\downarrow & & \uparrow{\scriptstyle U} \\
\ell_\infty & \xrightarrow{R} & \ell_1
\end{array}
\tag{2.5}
$$

Now by extension property of ℓ_∞, we can define $\tilde{S} : Z \to \ell_\infty$ such that $\tilde{S}|_X = S$ and $\|\tilde{S}\| = \|S\|$. Then we can consider the following diagram.

$$
\begin{array}{ccc}
X & \xrightarrow{S} & \ell_\infty \\
\downarrow & \nearrow \tilde{S} & \\
Z & &
\end{array}
\tag{2.6}
$$

Then we define $\tilde{T} = U \circ R \circ \tilde{S}$ on Z. Hence by the ideal property of a nuclear operator R and above argument, we have

$$
\begin{aligned}
\|\tilde{T}\|_{nuc} &= \|U \circ R \circ \tilde{S}\|_{nuc} \\
&\leq \|U\|\|R\|_{nuc}\|\tilde{S}\| \\
&\leq \|R\|_{nuc}, \qquad \text{by (2,2)} \\
&\leq (1+\epsilon)\|T\|_{nuc}.
\end{aligned}
\tag{2.7}
$$

Hence $\tilde{T}$ is an extension of T with $\|\tilde{T}\|_{nuc} \leq (1+\epsilon)\|T\|_{nuc}$ and $\tilde{T}|_X = T$. This proves the theorem. □

We now are going back to the work of the ℓ_1-space which has a property dual to the extension property saying the “lifting property”. For this time, we can show that a nuclear operator on Banach space X can have the liftable operator.

Theorem 2.2. *Let X, Y and Z be Banach spaces and $T : X \to Y$ be a nuclear operator. If for every surjective linear map q from a Banach space Z onto a Banach space Y, there exists a nuclear operator $\tilde{T} : X \to Z$ such that $T = q \circ \tilde{T}$ and $\|\tilde{T}\|_{nuc} \leq (1+\epsilon)\|T\|_{nuc}$, for $\epsilon > 0$ and the following diagram commutes;*

$$
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\tilde{T}\downarrow & \nearrow q & \\
Z & &
\end{array}
\tag{2.8}
$$

Proof. Suppose $T : X \to Y$ is a nuclear operator. Then for $\epsilon > 0$, by the definition of nuclear operator norm, we can find sequences (x_n^*) in X^* and (y_n) in Y such that

$$\sum_{n=1}^{\infty} \|x_n^*\|\|y_n\| \leq (1+\epsilon)\|T\|_{nuc}.$$

Now define $S : X \to \ell_\infty$ by $S(x) = (\frac{x_n^*(x)}{\|x_n^*\|})$ and $R : \ell_\infty \to \ell_1$ by $R((a_n)) = ((a_n\|x_n^*\|\|y_n\|))$ as in Theorem 2.1. Then applying the same argument as in Theorem 2.1, we can have $\|S\| \leq 1$ in (2.2) and $\|R\|_{nuc} \leq (1+\epsilon)\|T\|_{nuc}$ in (2.3). Finally, define $U : \ell_1 \to Y$ by $U(b_n) = \sum_{n=1}^{\infty} \frac{b_n y_n}{\|y_n\|}$. Then we also have $\|U\| \leq 1$ in (2.3) as in Theorem 2.1. For all $x \in X$, we also have in (2.4)

$$URS(x) = \sum_{n=1}^{\infty} x_n^*(x) y_n = T(x)$$

From the same argument in theorem 2.1, we have the following diagram as in (2.5) ;

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ S\big\downarrow & & \big\uparrow U \\ \ell_\infty & \xrightarrow{R} & \ell_1 \end{array} \tag{2.9}$$

Since $U : \ell_1 \to Y$ is a bounded linear operator with $\|U\| \leq 1$ and by the lifting property of ℓ_1, we can find a lifting $\tilde{U} : \ell_1 \to Z$ such that $q \circ \tilde{U} = U$ and $\|\tilde{U}\| = \|U\|$. Then define a lifting of T by $\tilde{T} = \tilde{U} \circ R \circ S$ where R is a nuclear operator on ℓ_∞. Now for all $x \in X$,

$$\begin{aligned} q \circ \tilde{T}(x) &= q \circ \tilde{U} \circ R \circ S(x) \\ &= U \circ R \circ S(x) \\ &= T(x). \end{aligned} \tag{2.10}$$

Hence $q \circ \tilde{T} = T$ and

$$\begin{aligned} \|\tilde{T}\|_{nuc} &\leq \|\tilde{U} \circ R \circ S\|_{nuc} \\ &\leq \|\tilde{U}\|\|R\|_{nuc}\|S\| \\ &\leq (1+\epsilon)\|T\|_{nuc}. \end{aligned} \tag{2.11}$$

This proves the theorem. □

Now by using Theorem 1.7, we can prove the following Pełczyński's theorem in [17] and ([2], pp.114) which is complemented subspaces of ℓ_1.

THEOREM 2.3. *Every infinite dimensional complemented subspaces of ℓ_1 is isomorphic to ℓ_1.*

Proof. Let X be an infinite dimensional complemented subspace of ℓ_1. Now let $P : \ell_1 \to X$ be the projection on ℓ_1 into X with norm $\|P\| = 1$. Suppose Y and Z are Banach spaces and $T : X \to Y$ is a any bounded linear operator. And let q be a surjective linear operator Z onto Y, then we have the following diagram;

$$\begin{tikzcd} \ell_1 \arrow[r, shift left, "P"] \arrow[dr] & X \arrow[l, shift left] \arrow[r, "T"] \arrow[d] & Y \\ & Z \arrow[ur, "q"'] & \end{tikzcd} \tag{2.12}$$

Then by the lifting property of ℓ_1, $T \circ P : \ell_1 \to Y$ has a lifting operator $\widehat{T \circ P} : \ell_1 \to Z$ with $q \circ \widehat{T \circ P} = T \circ P$ and

$$\|\widehat{T \circ P}\| = \|T \circ P\| \leq \|T\|\|P\| = \|T\|.$$

Let $J : X \to \ell_1$ be just an inclusion map with norm $\|J\| = 1$. Finally define a lifting of T by $\tilde{T} = \widehat{T \circ P} \circ J : X \to Z$. Then for all $x \in X$,

$$\begin{aligned} q \circ \tilde{T}(x) &= q \circ \widehat{T \circ P} \circ J(x) \\ &= T \circ P \circ J(x) \qquad \text{since} \quad P \circ J = Id_X \\ &= T(x). \end{aligned} \tag{2.13}$$

Moreover, we have

$$\begin{aligned} \|\tilde{T}\| &= \|\widehat{T \circ P} \circ J\| \\ &\leq \|\widehat{T \circ P}\|\|J\| \\ &\leq \|T\|. \end{aligned} \tag{2.14}$$

Hence this shows X has the lifting property. Then applying Theorem 1.7, X is isomorphic to $\ell_1(\Gamma)$, for some index set Γ. But X is a complemented subspace of ℓ_1, Γ should be countable set. This proves the theorem. $\square$

For one more application of theorem which was given by Lewis and Stegall in [10], we need to introduce a known theorem in [6, 7].

THEOREM 2.4. *([6, 7]). Let X be a Banach space with the Radon-Nikodým property. Then for $\epsilon > 0$, for any Banach space Y, and for a surjective linear map q from Y onto X, if $T : L_1(\mu) \to X$ is a bounded linear map, then T has a lifting $\tilde{T} : L_1(\mu) \to Y$ such that $\|\tilde{T}\| \leq (1+\epsilon)\|T\|$, and $q \circ \tilde{T} = T$.*

Now by using Theorem 1.7 and Theorem 2.4, we can give another proof of the following theorem originally due to Lewis and Stegall in [10] and in [2, pp.114].

THEOREM 2.5. *Let (Ω, Σ, μ) be a finite measure space and X be a complemented infinite dimensional subspace of $L_1(\mu)$. If X has the Radon-Nikodým property, then X is isomorphic to ℓ_1.*

Proof. Let $P : L_1(\mu) \to X$ be a bounded linear projection with norm $\|P\| = 1$. For Banach spaces Y and Z, let $T : X \to Y$ be any bounded linear operator and let $q : Z \to Y$ be a surjective linear map. Then consider the following diagram;

$$\begin{array}{ccccc} L_1(\mu) & \overset{P}{\rightleftarrows} & X & \xrightarrow{T} & Y \\ & \searrow & \downarrow & \nearrow q & \\ & & Z & & \end{array} \tag{2.15}$$

Since X has the RNP, we can apply Theorem 2.4 in [6, 7] for projection $T \circ P$ to get the lifting $\widehat{T \circ P} : L_1(\mu) \to Z$ such that $q \circ \widehat{T \circ P} = T \circ P$ and

$$\begin{aligned} \|\widehat{T \circ P}\| &\leq (1+\epsilon)\|T\|\|P\| \\ &= (1+\epsilon)\|T\|. \end{aligned} \tag{2.16}$$

Then define a lifting $\tilde{T}$ of T, by $\tilde{T} = \widehat{T \circ P} \circ J$ with $J : X \to L_1(\mu)$ inclusion operator. Then for all $x \in X$, we have

$$\begin{aligned} q \circ \tilde{T}(x) &= q \circ \widehat{T \circ P} \circ J(x) \\ &= T \circ P \circ J(x) \\ &= T \circ Id_X(x) = T(x) \end{aligned} \tag{2.17}$$

and

$$\begin{aligned}
\|\tilde{T}\| &= \|\widehat{T \circ P} \circ J\| \\
&\le \|\widehat{T \circ P}\| \|J\| \\
&\le \|\widehat{T \circ P}\| \qquad \text{by (2.16)} \\
&= (1+\epsilon)\|T\|. \qquad (2.18)
\end{aligned}$$

Hence X has the lifting property. This implies X is isomorphic to ℓ_1, by Theorem 1.7. This proves the theorem. □

References

[1] J. Diestel, H. Jarchow and A. Tonge, *Absolutely summing operators*, Vol. 43. Cambridge University Press, 1995

[2] J. Diestel and J. J. Uhl. Jr, *Vector Measures*, Math. Surveys Monographs 15, AMS, Providence RI.

[3] A. Grothendieck, *Une caraterisation vectorielle-métrique* L_1, Canad J. Math. **7**(1955), 552–562 MR17.

[4] W.B. Johnson and J. Lindenstrauss, *Handbook of the Geometry of Banach Spaces*, Vol. 2, Elsevier Science B.V. (2001).

[5] W.B. Johnson and M. Zippin, *Extension of operators from subspaces of* $c_0(\Gamma)$ *into* $C(\mathbf{K})$ *space*, Proc. Amer. Math. Soc. **107** (1989), 751–754.

[6] J.H. Kang, *Lifting properties on* $L^1(\mu)$, Comm. Korean Math. **16** (1) (2001), 119–124.

[7] J.H. Kang, *Lifting operators on some Banach spaces*, Korean, J. Math. **23** (3) (2015), 447–456.

[8] J.H. Kang, *Lifting on G.T. Banach Spaces with unconditional basis*, International J. Math. Analysis, **10** (14) (2016), 677–686.

[9] G. Köthe, *Hebbare lokakonvex Raume* , Math. Ann., 4651. **165** (1993), 188–195.

[10] D. R. Lewis and C. Stegall, *Banach spaces whose duals are isomorphic to* $\ell_1(\Gamma)$, J. Functional Analysis **12** (1973), 177–187.

[11] J. Lindenstrauss, *A remark on* ℓ_1 *spaces*, Israel J. Math. **8** (1970), 80–82.

[12] J. Lindenstrauss and A. Pelczyński, *Contributions to the theory of classical Banach spaces*, J. Funct. Anal. **8** (1971), 225–249.

[13] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I,*, Springer-Verlag, Berlin and New York 1977.

[14] B. Maurey, *Un theorem de prolongement*, C.R. Acad. Sci. Paris A **279** (1974), 329–332.

[15] E. Michael, *Continuous selections*, Ann. of Math. **63** (1956), 361–382.

[16] L. Nachbin, *A theorem of Hahn-Banach type for linear tranformation*, Trans. Amer. Math. Soc. **68** (1950), 28–46.

[17] A. Pelczynski, *Projections in certain Banach spaces*, Studia Math. Vol. 19 (1960), 209-228.

[18] G. Pisier, *Factorizations of linear operators and geometry of Banach spaces*, C.B.M.S. Amer. Math. Soc. **60** (1985).

[19] M. Zippin, *Applications of E. Michael's continuous selection theorem to operator extension problems*, Proc. Amer. Soc. **127** (1999).

Jeong Heung Kang
Department of Mathematics
Korea Military Academy
Seoul, Korea
E-mail: `jkang@kma.ac.kr`