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# EXACT SOLUTIONS OF GENERALIZED STOKES' PROBLEMS FOR AN INCOMPRESSIBLE COUPLE STRESS FLUID FLOWS

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Abstract. The ground for this paper is to examine the generalized Stokes' first and second issues for an incompressible couple pressure liquid under isothermal conditions. Exact solutions for each problem are acquired by using the Laplace transform (LT) with respect to the time variable $t$ and the sine Fourier transform (FT) with respect to the y-variable. Further, a comparison is given of the obtained results and the results of Devakar and Lyengar [1] and by using the four inverse Laplace transform algorithms (Stehfest's, Tzou's, Talbot, Fourier series) in the space time domain utilizing a numerical methodology. Moreover, velocity profiles are plotted and considered for various occasions and distinctive estimations of couple stress parameters. At the end, the outcomes are exhibited by graphs and in tabular forms.

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## 1. Introduction

The essential governing equations for the flow of a liquid of Newtonian sort are the Navier-Stokes equations. This arrangement of non-linear partial differential conditions has no broad solution, what's more, just a set number of correct solutions can be found in writing. Then again, exact solutions are vital they address to the essential flow marvels as well as likewise solutions gotten by different procedures can be verified by these fundamental solutions. For circumstances where flow of liquid is non-Newtonian in nature, it turns out to be fairly hard to acquire correct arrangements. This trouble happens in view of the highly non-linear terms in the viscous portion of the flow equations.

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The majority of the liquids in nature don't comply with the straight connection between the stress and the rate of strain. Such liquids speak to the non-Newtonian class of liquids. A few models have been exhibited to clarify the conduct of these liquids. From most recent couple of decades, in the hypothesis of non-Newtonian liquids, the couple pressure liquid presented by Stokes [2] has picked up impressive consideration and has been generally contemplated by scientists [3-14]. Couple stress liquid has been the subject of enthusiasm, because of its various mechanical and logical applications for example, expulsion of polymer liquids, cementing of fluid gems and creature bloods.

The couple stress hypothesis considers the impact of the polar effects such as couple stresses and body couples on the motion of the fluid. These couple stresses result due to the mechanical association in the fluid medium, where the stress tensor is not symmetric [15]. The hypothesis fully depicts the possible effects of such stresses assuming that the fluid has no microstructure at the kinematical level and the velocity field determines the kinematics of motion. The resultant equations from this hypothesis are like the Navier-stokes equations of motion but with the order of differential equations expanded by two. An excellent preface of this hypothesis is accessible in the monograph "Theories of Fluids bwith Microstructure" [15] composed by Stokes, which contains an extensive study about these fluids. The works of Naduvinamani et al. [16–19] and Jaw-Ren Lin et al. [20] expose that, the couple stress fluid can be taken as a oil as it has bigger load conveying limit and bring down coefficient of friction in correlation with the Newtonian fluid.

The present paper is given to the investigation of generalized Stoke's first and second problems concerning an incompressible couple stress fluid. A few analysts took a shot at generalized Stokes problems, however not with this name, for Newtonian and differing non-Newtonian fluids. The works of Gupta and Arora [21], Hassanien and Mansour [22], Hayat et al. [23], Erdogan and Imrak [24] and Fetecau and Fetecau [25] show the ongoing interest on the generalized Stokes problems.

In this work, we investigate the exact solutions of generalized Stokes' first and second problems for an incompressible couple stress fluid by Laplace and Fourier transoms. To the best knowledge of authors, the exact solutions of generalized Stokes' problems have not been solved in literature. The basic governing equations for couple stress fluids are given in Section 2, and the formulation of the problem is given in section 3. Exact solutions of Stokes' first and second problems are obtained in section 4. In section 5, the conclusion and discussion is given also the comparison of our results is shown with the results of Devakar and Lyengar [1] and with the semi analytical solutions of the problems which, are obtained by employing the different numerical inversion techniques (Stehfest's, Tzou's, Talbot, Fourier series) [26] for the inverse Laplace transform by graphically and tabulated form.

## 2. Basic Equations

The simple equations overseeing the flow of an incompressible couple stress fluid are [7, 13, 27-30]

$$div(\mathbf{V}), \tag{1}$$

$$\rho\frac{\delta \mathbf{V}}{\delta t} = div(\mathbf{T}) - \eta\nabla^4\mathbf{V} + \rho\mathbf{f}, \tag{2}$$

where $\mathbf{V}$ is the velocity vector, $\rho$ is the constant density, $\mathbf{f}$ is the body force per unit mass, $\mathbf{T}$ is the Cauchy stress tensor, $\eta$ is the couple stress parameter and the operator $\delta/\delta t$ denotes the material derivative which is defined as:

$$\frac{\delta}{\delta t}(*) = \left(\frac{\partial}{\partial t} + V\cdot\nabla\right)(*).$$

The Cauchy stress tensor $\mathbf{T}$ can be defined as:

$$\mathbf{T} = -p\boldsymbol{I} + \tau, \quad \tau = \mu\mathbf{A}_1,$$

where $p$ is the dynamic pressure, $\boldsymbol{I}$ is the unit tensor, $\mu$ is the coefficient of viscosity and $\mathbf{A}_1$ is the first Rivlin-Ericksen tensor defined as: $\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T$, $\mathbf{L}$ is the gradient of $\mathbf{V}$ and $\mathbf{L}^T$ is the transpose of $\mathbf{L}$.
To solve the couple stress fluid flow problems with regular suspicion of no-slip condition, it is assumed that the couple stresses vanish at the boundary.

## 3. Formulation of the problem

Consider the unsteady flow of an incompressible, couple stress fluid among two infinite horizontal parallel plates at a distance $h$ apart. At the first, we except that both plates and fluid are at rest. Elect a Cartesian coordinates system of with origin on the lower plate and y-axis perpendicular to the plates. Both plates are denoted by $y = 0$ and $y = h$. At time $t = 0^+$, whether we enable the lower plate to start with a constant velocity $U(t)$ along x-axis or oscillate with velocity $U\cos(\omega t)$ or $U\sin(\omega t)$ the flow occurs only in x-direction and keep the plate $y = h$ fixed. Along these lines, the velocity is expected to be in the form $\mathbf{V} = (u(y,t), 0, 0)$ and it consequently satisfies the continuity equation (1). The governing equation, is currently observed to be

$$\rho\frac{\partial u(y,t)}{\partial t} = \mu\frac{\partial^2 u(y,t)}{\partial y^2} - \eta\frac{\partial^4 u(y,t)}{\partial y^4}. \tag{3}$$

Introducing the following non-dimensional variables

$$\mathrm{u} = \frac{u}{U},\ \xi = \frac{y}{l},\ \tau = \frac{U}{l}t,$$

into Eq. (3), we get:

$$Re\frac{\partial \mathrm{u}(\xi,\tau)}{\partial \tau} = \frac{\partial^2 \mathrm{u}(\xi,\tau)}{\partial \xi^2} - a^2\frac{\partial^4 \mathrm{u}(\xi,\tau)}{\partial \xi^4}, \tag{4}$$

where, $Re = \frac{\rho U l}{\mu}$ is the Reynold number and $a^2 = \frac{l^2}{h^2}$ is the couple stress parameter, where $l^2 = \frac{\eta}{\mu}$.
It is simple that we need to solve the above equation utilizing the suitable boundary conditions relying upon whether we are managing generalization Stokes' first or second problem.

## 4. Solution of the problem

### 4.1. Generalized Stokes' first problem.

Initially, both fluid and plates are at rest. At time $\tau = 0^+$, the lower plate $\xi = 0$ start to moving with constant velocity $U(\tau) = U$. The non-dimensional initial and boundary conditions to be satisfied for this problem are

$$\begin{aligned} &\mathrm{u}(\xi,0) = 0, && \text{for all } \xi, \text{ (initial condition)} \\ &\mathrm{u}(0,\tau) = 1,\ \mathrm{u}(1,\tau) = 0 && \text{for all } \ \tau > 0, \\ & && \text{(no-slip condition on the boundary)} \\ &\tfrac{\partial^2(\xi,\tau)}{\partial \xi^2} = 0, \ \text{ at } \ \xi = 0 \text{ and } \xi = 1 && \text{for any } \ \tau > 0. \\ & && \text{(vanishing of couple stress on the boundary)} \end{aligned} \tag{5}$$

Taking LT to Eqs. (4), (5) and using the initial condition $(5)_1$, we get

$$\frac{\partial^4 \overline{\mathrm{u}}(\xi,q)}{\partial \xi^4} - \frac{1}{a^2}\frac{\partial^2 \overline{\mathrm{u}}(\xi,q)}{\partial \xi^2} + \frac{Re}{a^2} q \overline{\mathrm{u}}(\xi,q) = 0, \tag{6}$$

with conditions

$$\begin{aligned} \overline{\mathrm{u}}(0,q) = \frac{1}{q},\ \overline{\mathrm{u}}(1,q) = 0, \\ \frac{\partial^2 \overline{\mathrm{u}}(\xi,q)}{\partial \xi^2} = 0, \ \text{ at } \ \xi = 0 \text{ and } \xi = 1. \end{aligned} \tag{7}$$

Applying the sine FT to Eq. (6) and considering the conditions (7), we get

$$\overline{\mathrm{u}}_{sn}(n,q) = \frac{(n\pi)(1 + a^2(n\pi)^2)}{q[a^2(n\pi)^4 + (n\pi)^2 + Req]}.$$

Taking inverse LT to the last relation, we obtain

$$\mathrm{u}_{sn}(n,t) = \frac{1}{(n\pi)} - \frac{1}{(n\pi)} \exp\left(-\frac{(n\pi)(1 + a^2(n\pi)^3)}{Re} t\right). \tag{8}$$

Now, employing the inverse sine FT, we find

$$\mathrm{u}(\xi,\tau) = (1-\xi) - 2\sum_{0}^{\infty} \frac{\sin((n\pi)\xi)}{(n\pi)} \exp\left(-\frac{(n\pi)(1 + a^2(n\pi)^3)}{Re}\tau\right). \tag{9}$$

Our final solution is the sum of post-transient (steady) solution and transient solution, where the transient solution is

$$\mathrm{u}_\tau(\xi,\tau) = -2\sum_0^\infty \frac{\sin((n\pi)\xi)}{(n\pi)} \exp\left(-\frac{(n\pi)(1+a^2(n\pi)^3)}{Re}\tau\right).$$

We hightailed the property $\lim_{\tau\longrightarrow\infty.} \mathrm{u}_\tau(\xi,\tau) = 0$.

**4.2. Generalized Stokes' second problem.** Initially, both fluid and plate are rest. At time $\tau = 0^+$, it is assumed that the lower plate $\xi = 0$ start oscillating with velocity $U(\tau) = U\cos(\omega\tau)$ or $U(t) = U\sin(\omega\tau)$, (where $U$ is the amplitude of the motion and $\omega$ is the frequency of the vibration). Therefore, the non-dimensional conditions to be satisfied are

$$\begin{aligned} &\mathrm{u}(\xi,0) = 0, \qquad \text{for all } \ \xi,\\ \mathrm{u}(0,\tau) = \cos(\omega\tau) \text{ or } \mathrm{u}(0,\tau) = \sin(\omega\tau);\ &\mathrm{u}(1,\tau) = 0 \quad \text{ for all } \ \tau > 0,\\ \frac{\partial^2 \mathrm{u}(\xi,\tau)}{\partial\xi^2} &= 0, \ \text{ at } \ \xi = 0 \text{ and } \xi = 1, \ \text{ for any } \ \tau > 0. \end{aligned} \tag{10}$$

(vanishing of couple stress on the boundary)

As in the case of generalized Stokes first problem, taking LT of Eqs. (4), (10) and using initial condition $(10)_1$, we obtain Eq. (6) with boundary conditions

$$\begin{aligned} \overline{\mathrm{u}}(0,q) = \frac{q}{q^2+\omega^2} \ \text{ or } \ \overline{\mathrm{u}}(0,q) = \frac{\omega}{q^2+\omega^2};\ \overline{\mathrm{u}}(1,q) = 0,\\ \frac{\partial^2 \overline{\mathrm{u}}(\xi,q)}{\partial\xi^2} = 0, \ \text{ at } \ \xi = 0 \text{ and } \xi = 1. \end{aligned} \tag{11}$$

Employing the sine FT to Eq. (6) and taking into account the conditions (11), we get

$$\overline{\mathrm{u}}_{sn}(n,q) = \frac{(n\pi)(1+a^2(n\pi)^2)}{Req + (n\pi)(1+a^2(n\pi)^2} \frac{q}{q^2+\omega^2}.$$

An equivalent form is

$$\overline{\mathrm{u}}_{sn}(n,q) = \frac{1}{(n\pi)}\left\{\frac{q}{q^2+\omega^2} - \frac{1}{q+\frac{(n\pi)^2(1+a^2(n\pi)^2)}{Re}} + \frac{\omega}{q^2+\omega^2}\frac{\omega}{q+\frac{(n\pi)^2(1+a^2(n\pi)^2)}{Re}}\right\}.$$

Taking inverse LT to the above relation, we have

$$\begin{aligned} \mathrm{u}_{sn}(n,t) &= \frac{1}{(n\pi)}\left\{\cos(\omega\tau) - \exp\left(-\frac{(n\pi)^2(1+a^2(n\pi)^2)}{Re}\tau\right)\right.\\ &+ \left.\omega\int_0^\tau \sin(\omega(\tau-t))\exp\left(-\frac{(n\pi)^2(1+a^2(n\pi)^2)}{Re}t\right)dt\right\}. \end{aligned} \tag{12}$$

Now, employing the inverse sine FT, we obtain the solution corresponding to the cosine oscillation of the boundary

$$\begin{aligned} & \mathrm{u}_c(\xi,\tau) \\ = \quad & (1-\xi)\cos(\omega\tau) - 2\sum_{n=1}^{\infty}\frac{\sin((n\pi)\xi)}{(n\pi)}\exp\left(-\frac{(n\pi)^2(1+a^2(n\pi)^2)}{Re}\tau\right) \\ + \quad & 2\omega\sum_{n=1}^{\infty}\frac{\sin((n\pi)\xi)}{(n\pi)}\int_0^{\tau}\sin(\omega(\tau-t))\exp\left(-\frac{(n\pi)^2(1+a^2(n\pi)^2)}{Re}t\right)dt. \end{aligned} \tag{13}$$

With the similar procedure as adopt in this section, we obtain the following solution corresponding to the sine oscillation of the boundary

$$\begin{aligned} & \mathrm{u}_s(\xi,\tau) \\ = \quad & (1-\xi)\sin(\omega\tau) \\ & -2\omega\sum_{n=1}^{\infty}\frac{\sin((n\pi)\xi)}{(n\pi)}\int_0^{\tau}\cos(\omega(\tau-t))\exp\left(-\frac{(n\pi)^2(1+a^2(n\pi)^2)}{Re}t\right)dt. \end{aligned} \tag{14}$$

## 5. Conclusions and Numerical results

In the present study, the velocity field corresponding to the generalized Stokes' first and second problems for an incompressible couple stress fluid under isothermal conditions are determines by using the Laplace and Fourier sine transforms. Straightforward computations show that $\mathrm{u}(\xi,\tau)$ given by Eq. (9), as well as $\mathrm{u}_c(\xi,\tau)$ and $\mathrm{u}_s(\xi,\tau)$ give by Eqs. (13) and (14), satisfy both the governing equation and all imposed initial and boundary conditions. Devakar and Lyengar in [1] show these results graphically by employing the numerically inverse technique of inverse laplace transform. But here we find the exact solutions of the aforementioned problems and compared the obtained results with the numerically evaluated results of [1] and four other inverse Laplace transform algorithms (Stehfest's, Tzou's, Fourier series, Talbot) in the space time domain using a numerical approach [26]. In each case, physical aspects of the flow parameters on velocity field are shown by graphically and in tabular form. All the graphs and tables are presented in dimensional velocity profile and the special variable $\xi$.

**Generalized Stokes' first problem.** The variation of velocity u is presented through Fig.1 to Fig. 3. Fig. 1 is plotted against the velocity field and spatial variable $\xi$ to see the effect of time $\tau$ for fixed parameter $Re$ and $a$. It is observed that, at a fixed distance $\xi$, as time increases, the fluid velocity increases. it means the velocity profile is directly proportional to the time $\tau$. As $\xi$ stars from 0 to 1, as expected the velocity starts from 1 and decreases to 0. The same trend is seen in Fig (2) at a fixed time $\tau$ for a fixed $Re$ for any value of the couple stress parameter $a$. As the couple stress parameter $a$ increases for a fixed time $t$ and $Re$, the velocity shows an increasing trend initially near the moving plate and a decreasing trend subsequently near the stationary plate. Fig. 3, shows that for any fixed time $\tau$ and couple stress parameter $a$, as $Re$ is increasing, the velocity profile is seen to be decreasing for a fixed distance $\xi$. Fig. 4, shows the comparison of our obtained result (9) and the numerically evaluated results of [1] and four other Inverse Laplace transform algorithms in the space time domain using a numerical approach [26]. From Table 1, it can be observed that the Honig-Hirdes method [1], Stehfest's, Tzou's, Talbot and Fourier Series methods have good agreement with our obtained analytical results.

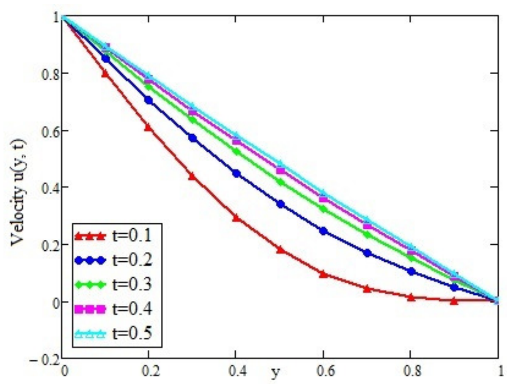


FIGURE 1. Variation of the velocity field $u(y,t)$ with distance at different times for $Re = 2.0,\ a = 0.2$.

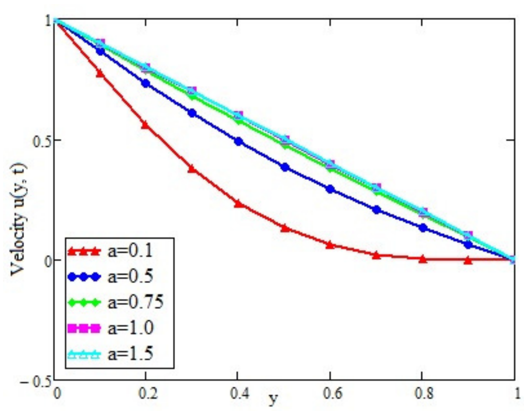


FIGURE 2. Variation of the velocity field $u(y,t)$ for different values of a at $Re = 2.0,\ t = 0.1$.

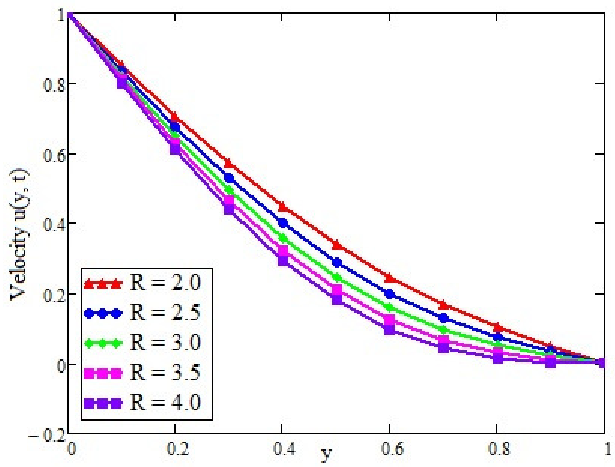


FIGURE 3. Variation of the velocity field $u(y,t)$ for different values of $Re$ at $a = 0.2,\ t = 0.2$.

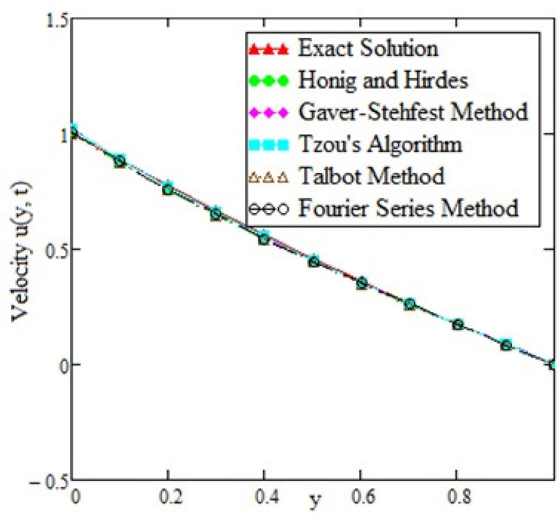


FIGURE 4. Comparison of our result (10) with the results of [1, 26] for $Re = 2.0,\ a = 0.1,\ t = 0.5$

TABLE 1. Validation of obtained numerical results with exact solution (10).

| y | R=0.5, t=1, ω=π/7 | | | | | |
|---|---|---|---|---|---|---|
| | Exact Solution $u_c(y, t)$ | Honig and Hirdes | Stehfest's | Tzaou's | Fouries Series | Talbot |
| 0.0 | 0.901 | 0.9 | 0.901 | 0.902 | 0.901 | 0.901 |
| 0.3 | 0.874 | 0.858 | 0.874 | 0.865 | 0.874 | 0.864 |
| 0.6 | 0.846 | 0.816 | 0.846 | 0.83 | 0.846 | 0.828 |
| 0.9 | 0.819 | 0.776 | 0.819 | 0.8 | 0.819 | 0.792 |
| 1.2 | 0.791 | 0.739 | 0.791 | 0.773 | 0.791 | 0.757 |
| 1.5 | 0.764 | 0.703 | 0.764 | 0.75 | 0.764 | 0.723 |
| 1.8 | 0.737 | 0.67 | 0.737 | 0.727 | 0.737 | 0.69 |
| 2.1 | 0.709 | 0.639 | 0.709 | 0.705 | 0.709 | 0.659 |
| 2.4 | 0.682 | 0.611 | 0.682 | 0.683 | 0.682 | 0.628 |
| 2.7 | 0.655 | 0.585 | 0.655 | 0.659 | 0.655 | 0.599 |
| 3.0 | 0.628 | 0.561 | 0.628 | 0.634 | 0.628 | 0.571 |

**Stokes' second problem.** The oscillating behavior of the velocity is seen in the Figs. 5, for fixed values of $Re$ and $a$, as can be expected. In Figs. 6, the variation of fluid velocity is plotted for different values of couple stress parameter $a$ at a fixed time $\tau$ and $Re$. As $a$ increasing, it can be seen that the velocity increasing for both cosine and sine oscillations. Fig. 7, shows that for any fixed time $\tau$ and couple stress parameter $a$, as $Re$ is increasing, the velocity profile is seen to be decreasing for a fixed distance $\xi$. Fig. 8 gives the comparison of our obtained analytical results (13) and (14), with the numerically evaluated results in [1, 26]. From table 2 and 3, it can seen that the all numerically evaluated results have good agreement with our obtained obtained analytical cosine (13) and sine (14) results.

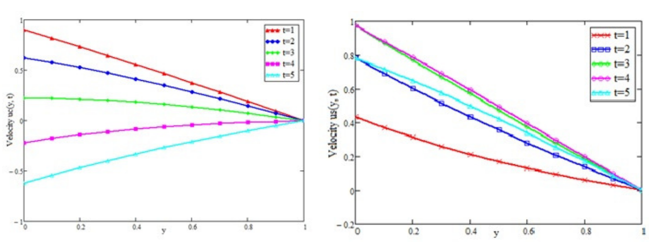


FIGURE 5. Variation of the velocity fields $u_c(y, t)$ and $u_s(y, t)$ for different values of $t$ at $Re = 2.0, a = 0.1$ and $\omega = \pi/7$.

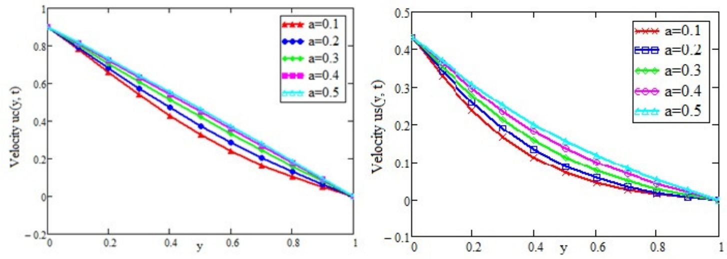


FIGURE 6. Variation of the velocity fields $u_c(y,t)$ and $u_s(y,t)$ for different values of a at $Re = 2.0, t = 1$ and $\omega = \pi/7$.

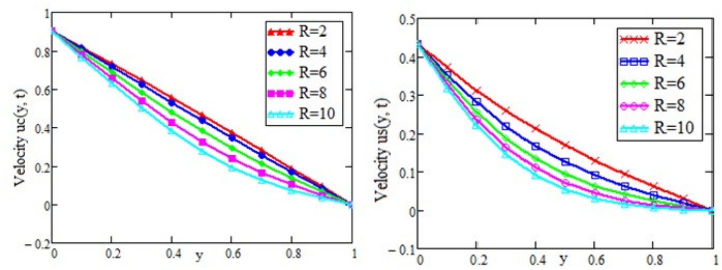


FIGURE 7. Variation of the velocity fields $u_c(y,t)$ and $u_s(y,t)$ for different values of $Re$ at $t = 1, a = 0.1$ and $\omega = \pi/7$.

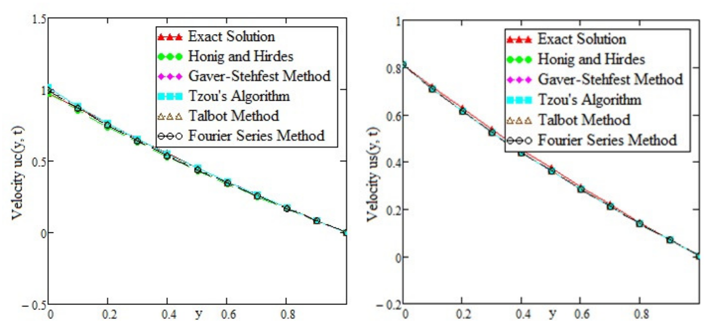


FIGURE 8. Comparison of our results (14) and (15) with the results of [1, 26] for $Re = 2.0, t = 0.5, a = 0.1$ and $\omega = \pi/7$.

TABLE 2. Validation of obtained numerical results with exact solution (14).

| y | R=2, t=0.5, $\omega=\pi/7$, a=0.1 | | | | | |
|---|---|---|---|---|---|---|
| | Exact Solution $u_c(y, t)$ | Honig and Hirdes | Stehfest's | Tzaou's | Talbot | Fourier Series |
| 0.0 | 0.975 | 0.975 | 1.012 | 1.012 | 1 | 0.992 |
| 0.1 | 0.868 | 0.851 | 0.884 | 0.884 | 0.873 | 0.867 |
| 0.2 | 0.761 | 0.737 | 0.765 | 0.765 | 0.755 | 0.75 |
| 0.3 | 0.655 | 0.629 | 0.653 | 0.653 | 0.645 | 0.64 |
| 0.4 | 0.552 | 0.528 | 0.548 | 0.548 | 0.541 | 0.538 |
| 0.5 | 0.453 | 0.432 | 0.449 | 0.449 | 0.443 | 0.44 |
| 0.6 | 0.356 | 0.341 | 0.354 | 0.354 | 0.349 | 0.347 |
| 0.7 | 0.264 | 0.253 | 0.262 | 0.262 | 0.259 | 0.257 |
| 0.7 | 0.174 | 0.167 | 0.173 | 0.173 | 0.171 | 0.17 |
| 0.9 | 0.086 | 0.083 | 0.086 | 0.086 | 0.085 | 0.085 |
| 1.0 | 0 | 0 | 0 | 0 | 0 | 0 |

TABLE 3. Validation of obtained numerical results with exact solution (15).

| y | R=2, t=0.5, $\omega=\pi/7$, a=0.1 | | | | | |
|---|---|---|---|---|---|---|
| | Exact Solution $u_s(y, t)$ | Honig and Hirdes | Stehfest's | Tzaou's | Talbot | Fourier Series |
| 0.0 | 0.812 | 0.811 | 0.811 | 0.812 | 0.811 | 0.812 |
| 0.1 | 0.717 | 0.709 | 0.708 | 0.709 | 0.708 | 0.709 |
| 0.2 | 0.626 | 0.613 | 0.613 | 0.614 | 0.613 | 0.613 |
| 0.3 | 0.538 | 0.524 | 0.523 | 0.524 | 0.523 | 0.524 |
| 0.4 | 0.453 | 0.44 | 0.439 | 0.44 | 0.439 | 0.44 |
| 0.5 | 0.372 | 0.36 | 0.359 | 0.36 | 0.359 | 0.36 |
| 0.6 | 0.294 | 0.284 | 0.283 | 0.284 | 0.283 | 0.284 |
| 0.7 | 0.219 | 0.21 | 0.21 | 0.21 | 0.21 | 0.21 |
| 0.7 | 0.145 | 0.139 | 0.139 | 0.139 | 0.139 | 0.139 |
| 0.9 | 0.072 | 0.069 | 0.069 | 0.069 | 0.069 | 0.069 |
| 1.0 | 0 | 0 | 0 | 0 | 0 | 0 |

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## References


1. M. Devakar and T.K.V. Iyengar, *Generalized Stokes Problems for an Incompressible Couple Stress Fluid*, Int. J of Math. Comp. Phy. Elec. and Comp. Eng. **8** (2014), 113-116.
2. V.K. Stokes, *Couple stresses in fluids, Phys. Fluids*, **9** (1966), 1709-1715.
3. G. Ramanaiah, *Squeeze films between finite plates lubricated by fluids with couple-stresses*, Wear. **54** (1979), 315–320.
4. S. Islam and C.Y. Zhou, *Exact solutions for two dimensional flows of couple stress fluids*, ZAMP **58** (2007), 1035–1048.
5. S. Islam, C.Y. Zhou and X.J. Ran, *Exact solutions for different vorticity functions of couple stress fluids*, J Zhejiang Univ Sci A **9** (2008), 672–680.
6. W. Khan and F. Yousafzai, *On the Exact Solutions of Couple Stress Fluids*, Advanced Trends in Mathematics **1** (2014), 27–32.
7. M. Farooq, M.T. Rahim, S. Islam and A.M. Siddiqui, *Steady Poiseuille flow and heat transfer of couple stress fluids between two parallel inclined plates with variable viscosity*, Journal of the Association of Arab Universities for Basic and Applied Sciences **14** (2013), 9-18.
8. A.K. Aggarwal and A. Verma, *Effect of hall currents on thermal instability of dusty couple stress fluid*, archives of thermodynamics **37** (2016), 3-18.
9. S. Islam, I. Ali, X.J. Ran, A. Shah and A.M. Siddiqui, *Effects of couple stresses on Couette and Poiseuille flow*, Int. J. of Non-linear Sciences and Numerical Simmulations **10** (2009), 99-112.
10. A.K. Aggarwal and A. Verma, *Effect of hall currents on thermal instability of dusty couple stress fluid*, archives of thermodynamics **37** (2016), 3-18.
11. K.N. Memon, A.M. Siddiqui and S.F. Shah, *Exact Solution of Tank Drainage through the Circular Pipe for Couple Stress Fluid*, J. Appl. Environ. Biol. Sci. **7** (2017), 27–34.
12. A.K. Aggarwal and A. Verma, *Effect of hall currents on thermal instability of dusty couple stress fluid*, archives of thermodynamics **37** (2016), 3-18.
13. A.R. Hadjesfandiari, A. Hajesfandiari and G.F. Dargush, *Size-dependent couple stress Timoshenko beam theory*, Acta Mechanica **226** (2015), 871–895.
14. S. Akhtar, *Flows between two parallel plates of couple stress fluids with time-fractional Caputo and Caputo-Fabrizio derivatives*, Eur. Phys. J. Plus. **131** (2016), 401–413.
15. V.K. Stokes, *Theories of Fluids with Microstructure*, Springer, New York, 1984.
16. N.B. Naduvinamani, P.S. Hiremath and G. Gurubasavaraj, *Squeeze film lubrication of a short porous journal bearing with Couple stress fluids*, Tribology International **34** (2001), 739–747.
17. N.B. Naduvinamani, P.S. Hiremath and G. Gurubasavaraj, *Surface roughness effects in a short porous journal bearing with a Couple stress fluid*, Fluid Dynamics Research **31** (2002), 333–354.
18. N.B. Naduvinamani, P.S. Hiremath and G. Gurubasavaraj, *Effects of surface roughness on the Couple stress squeeze film between a sphere and a flat Plate*, Tribology International **38** (2005), 451–458.
19. N.B. Naduvinamani, Syeda Tasneem Fathima and P.S. Hiremath, *Hydrodynamic lubrication of rough slider bearings with Couple stress fluids*, Tribology International **36** (2003), 949–959.
20. Jaw-Ren Lin and Chi-Ren Hung, *Combined effects of non-Newtonian couple stresses and fluid inertia on the squeeze film characteristics between a long cylinder and an infinite plate*, Fluid Dynamics Research **39** (2007), 616–639.
21. P.R. Gupta and K.L. Arora, *Hydromagnetic flow between two parallel planes, one is oscillating and the other fixed*, Pure. Appl. Geophy. **112** (1974), 498–505.
22. I.A. Hassanien and M.A. Mansour, *Unsteady magnetohydrodynamic flow through a porous medium between two infinite parallel plates*, Astrophy. Spac. Scie. **163** (1990), 241–246.

23. T. Hayat, S. Asghar and A.M. Siddiqui, *Some unsteady unidirectional flows of a non-Newtonian fluid*, Int. J. Engg. Sci. **38** (2000), 337-346.
24. M.E. Erdogan and C.E. Imrak, *On some unsteady flows of a non-Newtonian fluid*, Appl. Math. Model.**31** (2007), 170–180.
25. C. Fetecau and C. Fetecau, *Starting solutions for some unsteady unidirectional flows of second grade fluid*, Int. J. Engg. Sci. **43** (2005), 781–789.
26. K.L. Kuhlman, *Review of inverse Laplace transform algorithms for Laplace-space numerical approaches*, Numerical Algorithms **63** (2013), 339–355.
27. M. Devakar and T.K.V. Iyengar, *Run up flow of a Couple Stress Fluid between parallel plates*, Nonlinear Analysis: Modelling and Control **15** (2010), 29-37.
28. S. Islam, I. Ali, X.J. Ran, A. Shah and A.M. Siddiqui, *Effects of couple stresses on Couette and Poiseuille flow*, Int. J. of Non-linear Sciences and Numerical Simmulations **10** (2009), 99-112.
29. M. Devakar, D. Sreenivasu and B. Shankar, *Analytical solutions of couple stress fluid flows with slip boundary conditions*, Alexanderia Engineering Journal **1** (2008), 181-190.
30. Sahin Ahmed, O.A. Beg and S.K. Ghosh, *A couple stress fluid modeling on free convection oscillatory hydromagnetic flow in an inclined rotating channel*, Ain Shams Engineering Journal **5** (2014). 1249-1265.

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