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AN EXISTENCE AND UNIQUENESS THEOREM OF STOCHASTIC DIFFERENTIAL EQUATIONS AND THE PROPERTIES OF THEIR SOLUTION†

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Abstract. In this paper, we show the existence and uniqueness of solution to stochastic differential equations under weakened Hölder condition and a weakened linear growth condition. Furthermore, the properties of their solutions investigated and estimate for the error between Picard iterations $x_n(t)$ and the unique solution $x(t)$ of SDEs.

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1. Introduction

The inclusion of random effects in differential equation leads to two distinct classes of equations, for which the solution processes have differentiable and non-differentiable sample paths, respectively. They require fundamentally different methods of analysis. The first, and simpler, class arises when an ordinary differential equation has random coefficients, a random initial value or is forced by a fairly regular stochastic process, or when any combination of these holds. The equations are called *random differential equations* and are solved sample path by sample path as ordinary differential equations.

The second class occurs when the forcing is an irregular stochastic process such as Gaussian white noise. The equations are written symbolically as stochastic differentials, but are interpreted as integral equations with Itô or Stratonovich stochastic integrals. They are called *stochastic differential equations,* which

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we shall abbreviate by SDEs, and in general their solutions inherit the non-differentiability of sample paths from the Wiener processes in the stochastic integrals.

Mao Xuerong had investigated the following SDEs;

$$dx(t) = f\left(x(t), t\right) dt + g\left(x(t), t\right) dB(t), \tag{1}$$

on the closed interval $[t_0, T]$, $t_0 \leq T$ in his book [1], and be obtained that if Lipschitz condition (2) and linear growth condition (3) hold, namely, for any $x, y \in R^d$ and $t \in [t_0, T]$, it follows that

$$|f(x,t) - f(y,t)|^2 \vee |g(x,t) - g(y,t)|^2 \leq \overline{K}|x-y|^2, \quad \overline{K} > 0. \tag{2}$$

For any $(x,t) \in R^d \times [t_0, T]$, it follows that

$$|f(x,t)|^2 \vee |g(x,t)|^2 \leq K\left(1 + |x|^2\right), \quad K > 0 \tag{3}$$

then (1) had a unique solution $x(t)$, moreover $x(t) \in \mathcal{M}^2\left([t_0, T]; R^d\right)$.

After that the studies of the existence and uniqueness theorem of the SDEs has been conducted in [1], [3]-[9]. Motivated by [7], we will investigate the existence and uniqueness theorem of the solution for SDEs at a phase space $\mathcal{M}^2\left([t_0, T]; R^d\right)$ in this paper. We still take $t_0 \in R$ as our initial time throughout this paper. And we want to prove our main results as follows; first, under the weakened Hölder condition and the weakened linear growth condition, we estimate bounded of the solution for SDEs. Next, we prove the existence and uniqueness theorem of the solution for SDEs. Finally, we derived the estimate for the error between Picard iterations $x_n(t)$ and the unique solution $x(t)$ of SDEs.

2. Preliminary

Let one norm $|\cdot|$ denote Euclidean in R^n. If A is a matrix or vector its transpose is denoted by A^T; If A is a matrix, its trace norm is represented by $|A| = \sqrt{trace\,(A^TA)}$. Throughout this paper unless otherwise specified, let t_0 be a positive constant and $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq t_0}$ satisfying the usual conditions.(i.e it is right continuous and $\mathcal{F}_{t_0}$ contains all P-null sets.) Assume that $B(t) = (B_1(t), B_2(t), \cdots, B_m(t))^T$ be consider the d-dimensional stochastic differential equation of Itô type

$$dx(t) = f\left(x(t), t\right) dt + g\left(x(t), t\right) dB(t), \quad on \;\; t_0 \leq t \leq T \tag{4}$$

with initial value $x(t_0) = x_0$, where $f : R^d \times [t_0, T] \to R^d$ and $g : R^d \times [t_0, T] \to R^{d \times m}$ be both Borel measurable. By the definition of stochastic differential, this equation is equivalent to the following stochastic equation:

$$x(t) = x_0 + \int_{t_0}^{t} f(x(x), s) ds + \int_{t_0}^{t} g(x(s), s) dB(s) \quad on \;\; t_0 \leq t \leq T. \tag{5}$$

To be more precise, we give the definition of the solution of the equation (4) with initial data.

Definition 2.1. R^d-valued stochastic process $x(t)$ defined on $t_0 \leq t \leq T$ is called the solution (4), if $x(t)$ has the following properties;
(i) $x(t)$ is continuous and $\{x(t)\}_{t_0 \leq t \leq T}$ is $\mathcal{F}_t$-adapted;
(ii) $\{f(x(t),t)\} \in \mathcal{L}^1([t_0,T];R^d)$ and $\{g(x(t),t)\} \in \mathcal{L}^2([t_0,T];R^{d\times m})$;
(iii) equation (5) holds for every $t \in [t_0,T]$ with probability 1.

A solution $\{x(t)\}$ is said to be unique if any other solution $\{\overline{x}(t)\}$ is indistinguishable from $\{x(t)\}$, that is

$$P\{x(t) = \overline{x}(t) \;\; for\ all\ t_0 \leq t \leq T\} = 1.$$

The following lemmas are known as special name for stochastic integrals which was appear in [1] or [2].

Lemma 2.2. *(Stachurska's inequality)* [2] *Let $u(t)$ and $k(t)$ be nonnegative continuous functions for $t \geq \alpha$, and let $u(t) \leq a(t) + b(t)\int_\alpha^t k(s)u^p(s)ds$, $t \in J = [\alpha,\beta)$, where $\frac{a}{b}$ is nondecreasing function and $0 < p < 1$. Then*

$$u(t) \leq a(t)\left(1-(p-1)\left[\frac{a(t)}{b(t)}\right]^{p-1}\int_\alpha^t k(s)b^p(s)ds\right)^{\frac{-1}{p-1}}.$$

Lemma 2.3. *(Hölder's inequality)* [1] *If $\frac{1}{p}+\frac{1}{p}=1$ for any $p,q>1$, $f \in \mathcal{L}^p$, and $g \in \mathcal{L}^q$, then $fg \in \mathcal{L}^1$ and $\int_a^b fgdx \leq \left(\int_a^b |f|^p dx\right)^{\frac{1}{p}}\left(\int_a^b |g|^q dx\right)^{\frac{1}{q}}$.*

Lemma 2.4. *(Moment inequality)* [1] *If $p \geq 2$, $g \in \mathcal{M}^2([t_0,T]:R^{d\times m})$ such that $E\int_{t_0}^T |g(s)|^p ds < \infty$, then*

$$E\left(\sup_{0\leq t\leq T}\left|\int_{t_0}^T g(s)dB(s)\right|^p\right) \leq \left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}} T^{\frac{p-2}{2}} E\int_{t_0}^T |g(s)|^p ds.$$

In order to attain the solution of (4) we impose following assumptions.

(H1) (Weakened Hölder condition) For any $x,y \in R^d$ and $t \in [t_0,T]$, we assume that

$$|f(x,t)-f(y,t)|^2 \vee |g(x,t)-g(y,t)|^2 \leq \overline{K}|x-y|^{2\alpha},$$

where $\overline{K}$ is a positive constant and $0 < \alpha \leq 1$ is a constant.

(H2) (Weakened linear growth condtion) For any $t \in [t_0,T]$ it follows that $f(0,t), g(0,t) \in \mathcal{L}^2([t_0,T])$ it follows that

$$|f(0,t)|^2 \vee |g(0,t)|^2 \leq K,$$

where K is a positive constant.

3. Main Results

In order to obtain the existence of solutions to SDEs, let $x_0(t) = x_0$ for $t_0 \le t \le T$. For each $n = 1, 2, \cdots$, and define Picard sequence

$$x_n(t) = x_0 + \int_{t_0}^{t} f\left(x_{n-1}(s), s\right) ds + \int_{t_0}^{t} g\left(x_{n-1}(s), s\right) dB(s). \tag{6}$$

Now we give the existence and uniqueness theorem to the solution of equation (4) by approximate solutions by means of Picard sequence.

Theorem 3.1. *Assume that (H1), (H2) hold. Then there exists a unique solution to the SDEs (4). Moreover, the solution belongs to* $\mathcal{M}^2\left([t_0, T]; R^d\right)$.

We prepare two lemmas in order to prove this theorem.

Lemma 3.2. *Let* $u(t)$ *and* $a(t)$ *be continuous functions on* $[0, T]$. *Let* $k \ge 1$ *and* $0 < p \le 1$ *be constants. If* $u(t) \le k + \int_{t_0}^{t} a(s) u^p(s) ds$ *for* $t \in [t_0, T]$ *then*

$$u(t) \le k \exp\left(\int_{t_0}^{t} a(s) ds\right)$$

for $t \in [t_0, T]$

Lemma 3.3. *Let the assumption (H1) and (H2) hold. If* $x(t)$ *is a solution of (4), then*

$$E\left(\sup_{t_0 \le t \le T} |x(t)|^2\right) \le C \exp\left(6(T - t_0 + 4)\overline{K}(T - t_0)\right),$$

where $C = 3E|x_0|^2 + 6(T - t_0 + 4)K(T - t_0)$ *with* $C \ge 1$.

Proof. For each number $n \ge 1$, define the stopping time

$$\tau_n = T \wedge \inf\{t \in [t_0, T] : |x(t)| \ge n\}.$$

Clearly, as $n \to \infty$, $\tau_n \uparrow T$ a.s. Let $x_n(t) = x(t \wedge \tau_n)$, $t \in [t_0, T]$.Then $x_n(t)$ satisfies the following equation

$$x_n(t) = x_0 + \int_{t_0}^{t} f\left(x_n(s), s\right) I_{[t_0, \tau_n]}(s) ds + \int_{t_0}^{t} g\left(x_n(s), s\right) I_{[t_0, \tau_n]}(s) dB(s).$$

Using the elementary $(a + b + c)^2 \le 3(a^2 + b^2 + c^2)$, one gets

$$\begin{aligned} &|x_n(t)|^2 \\ &\le 3\left[|x_0|^2 + \left|\int_{t_0}^{t} f(x_n(s), s) I_{[t_0, \tau_n]}(s) ds\right|^2 + \left|\int_{t_0}^{t} g(x_n(s), s) I_{[t_0, \tau_n]}(s) dB(s)\right|^2\right]. \end{aligned}$$

Taking the expectation on both sides, one sees that

$$E\left(\sup_{t_0 \le s \le t} |x_n(s)|^2\right)$$

$$\leq 3\left[E|x_0|^2 + E\left(\sup_{t_0\leq s\leq t}\left|\int_{t_0}^{s} f(x_n(r),r)I_{[t_0,\tau_n]}(r)dr\right|^2\right)\right.$$
$$\left. + E\left(\sup_{t_0\leq s\leq t}\left|\int_{t_0}^{s} g(x_n(r),r)I_{[t_0,\tau_n]}(r)dr\right|^2\right)\right].$$

By Hölder inequality and lemma 2.3, one can show that

$$E\left(\sup_{t_0\leq s\leq t}|x_n(s)|^2\right)$$
$$\leq 3\left[E|x_0|^2 + (T-t_0)E\int_{t_0}^{t}|f(x_n(s),s)|^2\,ds + 4E\int_{t_0}^{t}|g(x_n(s),s)|^2\,ds\right].$$

By the condition (H1) and (H2), one can show that

$$E\left(\sup_{t_0\leq s\leq t}|x_n(s)|^2\right)$$
$$\leq 3\left[E|x_0|^2 + (T-t_0)E\int_{t_0}^{t}|f(x_n(s),s)-f(0,s)+f(0,s)|^2\,ds\right.$$
$$\left. +4E\int_{t_0}^{t}|g(x_n(s),s)-g(0,s)+g(0,s)|^2\,ds\right]$$
$$\leq 3\left[E|x_0|^2 + (T-t_0)E\int_{t_0}^{t}2\left(|f(x_n(s),s)-f(0,s)|^2+|f(0,s)|^2\right)ds\right.$$
$$\left. +4E\int_{t_0}^{t}\left(2|g(x_n(s),s)-g(0,s)|^2+2|g(0,s)|^2\right)ds\right]$$
$$\leq 3\left[E|x_0|^2 + 2(T-t_0+4)\overline{K}E\int_{t_0}^{t}|x_n(s)|^{2\alpha}ds + 2(T-t_0+4)E\int_{t_0}^{t}Kds\right]$$
$$\leq 3E|x_0|^2 + 6(T-t_0+4)K(T-t_0) + 6(T-t_0+4)\overline{K}\int_{t_0}^{t}E|x_n(s)|^{2\alpha}ds$$
$$\leq C + 6(T-t_0+4)\overline{K}\int_{t_0}^{t}E\left(\sup_{t_0\leq r\leq s}|x_n(r)|^{2\alpha}\right)ds,$$

where $C = 3E|x_0|^2 + 6(T-t_0+4)K(T-t_0)$. By the lemma 3.2

$$E\left(\sup_{t_0\leq s\leq t}|x_n(s)|^2\right) \leq C\exp\left(6(T-t_0+4)\overline{K}(T-t_0)\right)$$

with $C \geq 1$. We deduce that

$$E\left(\sup_{t_0\leq s\leq t}|x_n(s\wedge\tau_n)|^2\right) \leq C\exp\left(6(T-t_0+4)\overline{K}(T-t_0)\right).$$

Consequently the required inequality follows by letting $n\to\infty$. □

Proof of Theorem 3.1. To check the uniqueness, let $x(t)$ and $\overline{x}(t)$ be any two solutions of (4). By Lemma 3.3, $x(t), \overline{x}(t) \in \mathcal{M}^2\left([t_0, T]; R^d\right)$. Note that

$$\begin{aligned} & x(t) - \overline{x}(t) \\ & = \int_{t_0}^{t} [f(x(s),s) - f(\overline{x}(s),s)]ds + \int_{t_0}^{t} [g(x(s),s) - g(\overline{x}(s),s)]dB(s). \end{aligned}$$

By the elementary inequality $(a+b)^2 = 2(a^2+b^2)$, one then gets

$$\begin{aligned} & |x(t) - \overline{x}(t)|^2 \\ & \leq 2\left|\int_{t_0}^{t} [f(x(s),s) - f(\overline{x}(s),s)]ds\right|^2 + 2\left|\int_{t_0}^{t} [g(x(s),s) - g(\overline{x}(s),s)]dB(s)\right|^2. \end{aligned}$$

Taking the expectation on both sides, one sees that

$$\begin{aligned} & E\left(\sup_{t_0 \leq s \leq t} |x(t) - \overline{x}(t)|^2\right) \\ & \leq 2E\left|\int_{t_0}^{t} [f(x(s),s) - f(\overline{x}(s),s)]ds\right|^2 \\ & +2E\left(\sup_{t_0 \leq s \leq t}\left|\int_{t_0}^{t} [g(x(s),s) - g(\overline{x}(s),s)]dB(s)\right|^2\right). \end{aligned}$$

By the Hölder inequality and Lemma 2.3 one can show that

$$\begin{aligned} & E\left(\sup_{t_0 \leq s \leq t} |x(t) - \overline{x}(t)|^2\right) \\ & \leq 2(T - t_0)E\int_{t_0}^{t} |f(x(s),s) - f(\overline{x}(s),s)|^2 ds \\ & \qquad +4E\int_{t_0}^{t} |g(x(s),s) - g(\overline{x}(s),s)|^2 ds. \end{aligned} \tag{7}$$

By the condition (H1), one can show that

$$E\left(\sup_{t_0 \leq s \leq t} |x(s) - \overline{x}(s)|^2\right) \leq 2(T - t_0 + 4)E\int_{t_0}^{t} \overline{K}|x(s) - \overline{x}(s)|^{2\alpha} ds.$$

This yields that

$$\begin{aligned} & E\left(\sup_{t_0 \leq s \leq t} |x(s) - \overline{x}(s)|^2\right) \\ & \leq 2\overline{K}(T - t_0 + 4)\int_{t_0}^{t} E\sup_{t_0 \leq r \leq s} |x(r) - \overline{x}(r)|^{2\alpha} ds. \end{aligned}$$

Therefore, by the Stachurska's inequality, we have

$$E\left(\sup_{t_0 \leq s \leq t} |x(s) - \overline{x}(s)|^2\right) = 0.$$

Hence, we get $x(t) = \overline{x}(t)$ for $t_0 \leq t \leq T$ a.s. The uniqueness has been proved.

Now we check the existence of the solution using Picard sequence (6). Obviously, from the Picard iterations, we have $x_0 \in \mathcal{M}^2\left([t_0, T]; R^d\right)$. By induction $x_n(t) \in \mathcal{M}^2\left([t_0, T]; R^d\right)$, in fact

$$|x_n(t)|^2 \leq 3\left[|x_0|^2 + \left|\int_{t_0}^{t} f\left(x_{n-1}(s), s\right) ds\right|^2 + \left|\int_{t_0}^{t} g\left(x_{n-1}(s), s\right) dB(s)\right|^2\right].$$

Taking the expectation on both sides, one see that

$$\begin{aligned} E\left(\sup_{t_0 \leq s \leq t} |x_n(s)|^2\right) \leq \;& 3\left[E|x_0|^2 + E\left(\sup_{t_0 \leq s \leq t}\left|\int_{t_0}^{s} f\left(x_{n-1}(r), r\right) dr\right|^2\right)\right. \\ & \left. + E\left(\sup_{t_0 \leq s \leq t}\left|\int_{t_0}^{s} g\left(x_{n-1}(r), r\right) dB(r)\right|^2\right)\right]. \end{aligned}$$

By Hölder inequality and lemma 2.3, one can show that

$$\begin{aligned} & E\left(\sup_{t_0 \leq s \leq t} |x_n(s)|^2\right) \\ & \leq 3\left[E|x_0|^2 + (T - t_0)E\int_{t_0}^{t} \left|f\left(x_{n-1}(s), s\right)\right|^2 ds + 4E\int_{t_0}^{t} \left|g\left(x_{n-1}(s), s\right)\right|^2 ds\right]. \end{aligned}$$

By the condition (H1) and (H2), one can show that

$$\begin{aligned} & E\left(\sup_{t_0 \leq s \leq t} |x_n(s)|^2\right) \\ & \leq 3\left[E|x_0|^2 + (T - t_0)E\int_{t_0}^{t} \left|f\left(x_{n-1}(s), s\right) - f(0, s) + f(0, s)\right|^2 ds\right. \\ & \quad \left. + 4E\int_{t_0}^{t} \left|g\left(x_{n-1}(s), s\right) - g(0, s) + g(0, s)\right|^2 ds\right] \\ & \leq 3\left[E|x_0|^2 + (T - t_0)E\int_{t_0}^{t} \left(2\left|f\left(x_{n-1}(s), s\right) - f(0, s)\right|^2 + 2|f(0, s)|^2\right) ds\right. \\ & \quad \left. + 4E\int_{t_0}^{t} \left(2\left|g\left(x_{n-1}(s), s\right) - g(0, s)\right|^2 + 2|g(0, s)|^2\right) ds\right] \\ & \leq 3\left[E|x_0|^2 + 2(T - t_0 + 4)\left(\overline{K}E\int_{t_0}^{t} |x_{n-1}(s)|^{2\alpha} ds + E\int_{t_0}^{t} K ds\right)\right] \\ & \leq 3E|x_0|^2 + 6(T - t_0 + 4)K(T - t_0) + 6(T - t_0 + 4)\overline{K}\int_{t_0}^{t} E|x_{n-1}(s)|^{2\alpha} ds \\ & \leq C + 6(T - t_0 + 4)\overline{K}\int_{t_0}^{t} E\left(\sup_{t_0 \leq r \leq s} |x_{n-1}(r)|^{2\alpha}\right) ds, \end{aligned}$$

where $C = 3E|x_0|^2 + 6(T - t_0 + 4)K(T - t_0)$. It also follows note that for any $k \geq 1$,

$$\begin{aligned}
&\max_{1\leq n\leq k} E\left(\sup_{t_0\leq s\leq t} |x_{n-1}(s)|^{2\alpha}\right)\\
&= \max\left\{E|x_0|^{2\alpha}, E\left(\sup_{t_0\leq s\leq t}|x_1(s)|^{2\alpha}\right), \cdots, E\left(\sup_{t_0\leq s\leq t}|x_{k-1}(s)|^{2\alpha}\right)\right\}\\
&\leq \max\left\{E|x_0|^{2\alpha}, E\left(\sup_{t_0\leq s\leq t}|x_1(s)|^{2\alpha}\right), \cdots, E\left(\sup_{t_0\leq s\leq t}|x_k(s)|^{2\alpha}\right)\right\}\\
&\leq E|x_0|^{2\alpha} + \max_{1\leq n\leq k} E\left(\sup_{t_0\leq s\leq t}|x_n(s)|^{2\alpha}\right).
\end{aligned}$$

Therefore, one can derive that

$$\begin{aligned}
&\max_{1\leq n\leq k} E\left(\sup_{t_0\leq s\leq t}|x_n(s)|^2\right)\\
&\leq C + 6(T - t_0 + 4)E|x_0|^{2\alpha}\\
&\quad +6(T - t_0 + k)\overline{K}\int_{t_0}^{t}\max_{1\leq n\leq k} E\left(\sup_{t_0\leq r\leq s}|x_n(s)|^{2\alpha}\right)ds\\
&:= \gamma + 6(T - t_0 + k)\overline{K}\int_{t_0}^{t}\max_{1\leq n\leq k} E\left(\sup_{t_0\leq r\leq s}|x_n(s)|^{2\alpha}\right)ds,
\end{aligned}$$

where $\gamma = C + 6(T - t_0 + 4)E|x_0|^{2\alpha}$. By lemma 3.2, we have

$$\max_{1\leq n\leq k} E\left(\sup_{t_0\leq s\leq t}|x_n(s)|^2\right) \leq \gamma\exp\left(6\overline{K}(T - t_0 + 4)(T - t_0)\right)$$

with $\gamma \geq 1$. Since k is arbitrary, for all $n = 0, 1, 2, \cdots$, we deduce that

$$E\left(\sup_{t_0\leq s\leq t}|x_n(s)|^2\right) \leq \gamma\exp\left(6\overline{K}(T - t_0 + 4)(T - t_0)\right),$$

which shows the boundedness of the sequence $\{x_n(t), n \geq 0\}$.

Next, we check that the sequence $\{x_n(t)\}$ is Cauchy sequence. For all $n \geq 0$ and $t_0 \leq t \leq T$, we have

$$\begin{aligned}
&x_{n+1}(t) - x_n(t)\\
&= \int_{t_0}^{t}[f(x_n(s), s) - f(x_{n-1}(s), s)]ds + \int_{t_0}^{t}[g(x_n(s), s) - g(x_{n-1}(s), s)]dB(s).
\end{aligned}$$

Using an elementary inequality $(a + b)^2 \leq 2a^2 + 2b^2$ and taking the expectation on both sides, we derive that

$$\begin{aligned}
&E\left(\sup_{t_0\leq s\leq t}|x_{n+1}(s) - x_n(s)|^2\right)\\
&\leq 2E\left(\sup_{t_0\leq s\leq t}\left|\int_{t_0}^{s}[f(x_n(r), r) - f(x_{n-1}(r), r)]dr\right|^2\right)
\end{aligned}$$

$$+2E\left(\sup_{t_0\le s\le t}\left|\int_{t_0}^{s}[g(x_n(r),r)-g(x_{n-1}(r),r)]dB(r)\right|^2\right).$$

By Hölder inequality and lemma 2.3 and condition (H1), one can show that

$$\begin{aligned}
&E\left(\sup_{t_0\le s\le t}|x_{n+1}(s)-x_n(s)|^2\right)\\
&\le 2(T-t_0)\overline{K}E\int_{t_0}^{t}\sup_{t_0\le r\le s}|x_n(r)-x_{n-1}(r)|^{2\alpha}ds\\
&\quad+8\overline{K}E\int_{t_0}^{t}\sup_{t_0\le r\le s}|x_n(r)-x_{n-1}(r)|^{2\alpha}ds.
\end{aligned}$$

This yields that

$$\begin{aligned}
&E\left(\sup_{t_0\le s\le t}|x_{n+1}(s)-x_n(s)|^2\right)\\
&\le 2(T-t_0+4)\overline{K}\int_{t_0}^{t}E\left(\sup_{t_0\le r\le s}|x_{n+1}(r)-x_n(r)|^{2\alpha}\right)ds.
\end{aligned}$$

Let $z(t)=\limsup_{n\to\infty}E\left(\sup_{t_0\le s\le t}|x_{n+1}(s)-x_n(s)|^2\right)$, we get

$$z(t)\le 2(T-t_0+4)\overline{K}\int_{t_0}^{t}z^{\alpha}(s)ds.$$

By stachurska's inequality, we get $z(t)=0$. This shows the sequence $\{x_n(t),n\ge 0\}$ is Cauchy sequence in $\mathcal{L}^2$. Hence, as $n\to\infty$, $x_n(t)\to x(t)$, that is $E|x_n(t)-x(t)|^2\to 0$. Therefore, we obtain that $x(t)\in\mathcal{M}^2([t_0,t];R^d)$. Now to show that $x(t)$ satisfy (5)

$$\begin{aligned}
&E\left|\int_{t_0}^{t}[f(x_n(s),s)-f(x(s),s)]ds+\int_{t_0}^{t}[g(x_n(s),s)-g(x(s),s)]dB(s)\right|^2\\
&\le 2\left[(T-t_0)E\int_{t_0}^{t}|f(x_n(s),s)-f(x(s),s)|^2ds\right.\\
&\quad\left.+4E\int_{t_0}^{t}|g(x_n(s),s)-g(x(s),s)|^2ds\right]\\
&\le 2(T-t_0+4)\int_{t_0}^{t}E\left(\sup_{t_0\le r\le s}|x_n(r)-x(r)|^{2\alpha}\right)ds.
\end{aligned}$$

Noting that sequence $\{x_n(t)\}$ is uniformly converge on $[t_0,T]$, it means that

$$E\left(\sup_{t_0\le s\le t}|x_n(s)-x(s)|^2\right)\to 0$$

as $n \to \infty$. Hence, taking limits on both sides in the Picard sequence, we obtain that

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)ds + \int_{t_0}^{t} g(x(s), s)dB(s). \quad on \quad t_0 \le t \le T.$$

The above expression demonstrates that $x(t)$ is a solution of equation (4). So far, the existence of theorem in complete. □

Lemma 3.4. *Assume that (H1) and (H2) hold. Let $x_n(t)$ be the Picard iterations defined by (6). Then*

$$E\left(\sup_{t_0\le t\le T}|x_{n+1}(t) - x_n(t)|^2\right) \le (M(T-t_0))^{\frac{1-\alpha^n}{1-\alpha}} C^{\alpha^n} \prod_{i=1}^{n} \frac{(1-\alpha)^{\alpha^{n-i}}}{(1-\alpha^i)^{\alpha^{n-i}}} \tag{8}$$

where $M = 2\overline{K}(T - t_0 + 1)$.

Proof. We note that

$$\begin{aligned} |x_1(t) - x_0(t)|^2 &= |x_1(t) - x_0|^2 \\ &\le 2\left|\int_{t_0}^{t} f(x_0, s)ds\right|^2 + 2\left|\int_{t_0}^{t} g(x_0, s)dB(s)\right|^2. \end{aligned}$$

Taking the expectation on both sides, we derive that

$$E|x_1(t) - x_0|^2 \le 2E\left|\int_{t_0}^{t} f(x_0, s)ds\right|^2 + 2E\left|\int_{t_0}^{t} g(x_0, s)dB(s)\right|^2.$$

From Hölder inequality and condition (H1) one can show that

$$\begin{aligned} E|x_1(t) - x_0|^2 &\le 2(T - t_0)E\int_{t_0}^{t} |f(x_0, s)|^2 ds + 2E\int_{t_0}^{t} |g(x_0, s)|^2 dB(s) \\ &\le 2(T - t_0)E\int_{t_0}^{t} |f(x_0, s) - f(0, s) + f(0, s)|^2 ds \\ &+2E\int_{t_0}^{t} |g(x_0, s) - g(0, s) + g(0, s)|^2 ds \\ &\le 2(T - t_0)E\int_{t_0}^{t} 2\left(|f(x_0, s) - f(0, s)|^2 + |f(0, s)|^2\right) ds \\ &\quad +2E\int_{t_0}^{t} 2((|g(x_0, s) - g(0, s)|^2 + |g(0, s)|^2) ds \\ &\le 4(T - t_0 + 1)E\int_{t_0}^{t} \left(\overline{K}|x_0|^{2\alpha} + K\right) ds \le C, \end{aligned}$$

where $C = 4(T - t_0 + 1)(T - t_0)\left(K + \overline{K}E(|x_0|^2)\right)$. That is

$$E|x_1(t) - x_0|^{2\alpha} \le (E|x_1(t) - x_0|)^{2\alpha} \le C^{\alpha},$$

where $0 < \alpha \leq 1$. By the same ways as above, we compute

$$\begin{aligned}
&E|x_2(t) - x_1(t)|^2 \\
&\leq 2E\left|\int_{t_0}^{t}[f(x_1(s),s) - f(x_0,s)]ds\right|^2 + 2E\left|\int_{t_0}^{t}[g(x_1(s),s) - g(x_0,s)]dB(s)\right|^2 \\
&\leq 2(T-t_0)E\int_{t_0}^{t}|f(x_1(s),s) - f(x_0,s)|^2 ds + 2E\int_{t_0}^{t}|g(x_1(s),s) - g(x_0,s)|^2 ds \\
&\leq 2(T-t_0+1)E\int_{t_0}^{t}\overline{K}|x_1(s) - x_0|^{2\alpha}ds \\
&\leq 2(T-t_0+1)\overline{K}(t-t_0)C^{\alpha} \\
&\leq M(t-t_0)C^{\alpha},
\end{aligned}$$

where $M = 2\overline{K}(T - t_0 + 1)$. That is $E|x_2(t) - x_1(t)|^{2\alpha} \leq M^{\alpha}(t-t_0)^{\alpha}C^{\alpha^2}$. Now we claim that for all $n \geq 1$,

$$E|x_{n+1}(t) - x_n(t)|^2 \leq (M(T-t_0))^{\frac{1-\alpha^n}{1-\alpha}}\, C^{\alpha^n}\prod_{i=1}^{n}\frac{(1-\alpha)^{\alpha^{n-i}}}{(1-\alpha^i)^{\alpha^{n-i}}}. \tag{9}$$

When $n = 0, 1$ inequality (9) holds. We suppose that (9) holds for some $n \geq 1$, then, we derive that for $n + 1$,

$$\begin{aligned}
&E|x_{n+2}(t) - x_{n+1}(t)|^2 \\
&\leq 2(T-t_0)E\int_{t_0}^{t}\overline{K}|x_{n+1}(s) - x_n(s)|^{2\alpha}ds \\
&\leq ME\int_{t_0}^{t}M^{\frac{\alpha(1-\alpha^n)}{1-\alpha}}(s-t_0)^{\frac{\alpha(1-\alpha^n)}{1-\alpha}}C^{\alpha^{n+1}}\prod_{i=1}^{n}\frac{(1-\alpha)^{\alpha^{n-i+1}}}{(1-\alpha^i)^{\alpha^{n-i+1}}}ds \\
&= (M(t-t_0))^{\frac{(1-\alpha^{n+1})}{1-\alpha}}\, C^{\alpha^{n+1}}\prod_{i=1}^{n+1}\frac{(1-\alpha)^{\alpha^{n+1-i}}}{(1-\alpha^i)^{\alpha^{n+1-i}}}
\end{aligned}$$

That is (9) holds for $n + 1$. Hence, by induction, (8) hold for all $n \geq 1$.

Theorem 3.5. *Assume that (H1) and (H2) hold. Let $x(t)$ be the unique solution $x(t)$ of equation (4) and $x_n(t)$ be the Picard iteration defined by (6). Then*

$$E\left(\sup_{t_0\leq t\leq T}|x_n(t) - x(t)|^2\right) \leq \gamma_1 \exp\left(2M(T-t_0)\right)$$

for all $n \geq 1$, where $C = 4(T - t_0 + 1)(T - t_0)(K + \overline{K}E|x_0|^2)$ and $M = 2\overline{K}(T - t_0 + 1)$ and $\gamma_1 = 2\left(M(t-t_0)\right)^{\frac{1-\alpha^n}{1-\alpha}} C^{\alpha^n}\prod_{i=1}^{n}\frac{(1-\alpha)^{\alpha^{n-i}}}{(1-\alpha^i)^{\alpha^{n-i}}}$.

Proof. From the Picard iteration and unique solution, we have

$$\begin{aligned}
&x_n(t) - x(t) \\
&= \int_{t_0}^{t}[f(x_{n-1}(s),s) - f(x(s),s)]ds + \int_{t_0}^{t}[g(x_{n-1}(s),s) - g(x(s),s)]dB(s).
\end{aligned}$$

Taking the expectation and by Hölder inequality and (H2) thus we have

$$\begin{aligned}
& E|x_n(t)-x(t)|^2 \\
& \leq 2(T-t_0)E\int_{t_0}^{t}|f(x_{n-1}(s),s)-f(x(s),s)|^2 ds \\
& +2E\int_{t_0}^{t}|g(x_{n-1}(s),s)-g(x(s),s)|^2 ds \\
& \leq 2(T-t_0)E\int_{t_0}^{t}2\left(f(x_{n-1}(s),s)-f(x_n(s),s)|^2\right. \\
& \left.+|f(x_n(s),s)-f(x(s),s)|^2\right)ds \\
& +2E\int_{t_0}^{t}2\left(|g(x_{n-1}(s),s)-g(x_n(s),s)|^2\right. \\
& \left.+|g(x_n(s),s)-g(x(s),s)|^2\right)ds \\
& \leq 4(T-t_0+1)\overline{K}\int_{t_0}^{t}\left(E|x_n(s)-x_{n-1}(s)|^{2\alpha}+E|x_n(s)-x(s)|^{2\alpha}\right)ds.
\end{aligned}$$

Substituting (9) into this yields that

$$\begin{aligned}
& E\left(\sup_{t_0\leq s\leq t}|x_n(s)-x(s)|^2\right) \\
& \leq 2M\int_{t_0}^{t}M^{\frac{\alpha(1-\alpha^{n-1})}{1-\alpha}}(s-t_0)^{\frac{\alpha(1-\alpha^{n-1})}{1-\alpha}}C^{\alpha^n}\prod_{i=1}^{n-1}\frac{(1-\alpha)^{\alpha^{n-i}}}{(1-\alpha^i)^{\alpha^{n-i}}}ds \\
& +2M\int_{t_0}^{t}E|x_n(s)-x(s)|^{2\alpha}ds \\
& \leq 2\left(M(T-t_0)\right)^{\frac{1-\alpha^n}{1-\alpha}}C^{\alpha^n}\prod_{i=1}^{n}\frac{(1-\alpha)^{\alpha^{n-i}}}{(1-\alpha^i)^{\alpha^{n-i}}} \\
& +2M\int_{t_0}^{t}E\left(\sup_{t_0\leq r\leq s}|x_n(s)-x(s)|^{2\alpha}\right)ds
\end{aligned}$$

where $\gamma_1 = 2\left(M(T-t_0)\right)^{\frac{1-\alpha^n}{1-\alpha}}C^{\alpha^n}\prod_{i=1}^{n}\frac{(1-\alpha)^{\alpha^{n-i}}}{(1-\alpha^i)^{\alpha^{n-i}}}$. By lemma 3.4, we have

$$E\left(\sup_{t_0\leq s\leq t}|x_n(s)-x(s)|^2\right)\leq \gamma_1\exp(2M(T-t_0))$$

with $\gamma_1\geq 1$. The proof is complete.

Theorem 3.6. *Assume that*
(i) *(Linear growth condition) For all $t\in[t_0,T]$ and $x\in R^d$, there exists a positive number K such that*

$$|f(x,t)|^2\vee|g(x,t)|^2\leq K\left(1+|x|^2\right). \tag{10}$$

(ii) *(Local Hölder condition) For each integer $n \geq 1$, there exists a positive constant number K_n such that for all $t \in [t_0, T]$ and all $x, y \in R^d$, with $|x| \vee |y| \leq n$, it follows that*

$$|f(x,t) - f(y,t)|^2 \vee |g(x,t) - g(y,t)|^2 \leq K_n |x-y|^{2\alpha}. \tag{11}$$

Then there exists a unique solution $x(t)$, moreover $x(t) \in M^2([t_0, T]; R^d)$.

Proof. For each $n \geq 1$, define truncation functions f_n and g_n as follows,

$$f_n(x,t) = \begin{cases} f(x,t), & |x| \leq n, \\ f\left(\frac{nx}{|x|}, t\right), & |x| > n, \end{cases}$$

$$g_n(x,t) = \begin{cases} g(x,t), & |x| \leq n, \\ g\left(\frac{nx}{|x|}, t\right), & |x| > n, \end{cases}$$

then f_n and g_n satisfy condition (H1) and (H2). By Theorem 3.1, equation

$$x_n(t) = x_0 + \int_{t_0}^t f_n(x_n(s), s)ds + \int_{t_0}^t g_n(x_n(s), s)dB(s), \quad t \in [t_0, T] \tag{12}$$

has a unique solution $x_n(t)$, moreover $x_n(t) \in \mathcal{M}^2([t_0, T]; R^d)$. Of course, $x_{n+1}(t)$ is the unique solution of equation

$$x_{n+1}(t) = x_0 + \int_{t_0}^t f_{n+1}(x_{n+1}(s), s)ds + \int_{t_0}^t g_{n+1}(x_{n+1}(s), s)dB(s)$$

on $t_0 \leq t \leq T$ and $x_{n+1}(t) \in \mathcal{M}^2([t_0, T]; R^d)$. Define the stopping time

$$\tau_n = T \wedge \inf\{t \in [t_0, T] : |x_n(t)| \geq n\}.$$

Taking the expectation, and by the Hölder inequality, it deduces that

$$\begin{aligned}
&E|x_{n+1}(t) - x_n(t)|^2 \\
&\leq 2E\left|\int_{t_0}^t [f_{n+1}(x_{n+1}(s), s) - f_n(x_n(s), s)]ds\right|^2 \\
&+2E\left|\int_{t_0}^t [g_{n+1}(x_{n+1}(s), s) - g_n(x_n(s), s)]dB(s)\right|^2 \\
&\leq 2(T-t_0)E\int_{t_0}^t |f_{n+1}(x_{n+1}(s), s) - f_n(x_n(s), s)|^2 ds \\
&\qquad +2E\int_{t_0}^t |g_{n+1}(x_{n+1}(s), s) - g_n(x_n(s), s)|^2 ds \\
&\leq 4(T-t_0)E\int_{t_0}^t \Big[|f_{n+1}(x_{n+1}(s), s) - f_{n+1}(x_n(s), s)|^2 \\
&\quad + |f_{n+1}(x_n(s), s) - f_n(x_n(s), s)|^2\Big] ds
\end{aligned}$$

$$+4E\int_{t_0}^{t}\Big[|g_{n+1}(x_{n+1}(s),s)-g_{n+1}(x_n(s),s)|^2$$
$$+|g_{n+1}(x_n(s),s)-g_n(x_n(s),s)|^2\Big]\,ds.$$

For $t_0\le t\le\tau_n$, we have known that

$$f_{n+1}(x_n(s),s)=f_n(x_n(s),s)=f(x_n(s),s),$$
$$g_{n+1}(x_n(s),s)=g_n(x_n(s),s)=g(x_n(s),s).$$

Moreover one then gets that

$$E\left(\sup_{t_0\le s\le t}|x_{n+1}(s)-x_n(s)|^2\right)$$
$$\le 4(T-t_0)E\int_{t_0}^{t}|f_{n+1}(x_{n+1}(s),s)-f_{n+1}(x_n(s),s)|^2\,ds$$
$$+4E\int_{t_0}^{t}|g_{n+1}(x_{n+1}(s),s)-g_{n+1}(x_n(s),s)|^2\,ds$$
$$\le 4(T-t_0+1)E\int_{t_0}^{t}K_n|x_{n+1}(s)-x_n(s)|^{2\alpha}ds.$$

This yields that

$$E\left(\sup_{t_0\le s\le t}|x_{n+1}(s)-x_n(s)|^2\right)$$
$$\le 4(T-t_0+1)K_n\int_{t_0}^{t}E\left(\sup_{t_0\le r\le s}|x_{n+1}(r)-x_n(r)|^{2\alpha}\right)ds.$$

Therefore, by the Stachurska's inequality, one see that

$$E\left(\sup_{t_0\le s\le t}|x_{n+1}(s)-x_n(s)|^2\right)=0,\qquad t_0\le t\le\tau_n$$

this means that for $t_0\le t\le\tau_n$, we always have

$$x_n(t)=x_{n+1}(t). \tag{13}$$

It then deduces that τ_n is increasing, that is as $n\to\infty$, $\tau_n\to T$ a.s. By linear growth condition, for almost all $\omega\in\Omega$, there exists an integer $n_0=n_0(\omega)$ such that $\tau_n=T$ as $n\ge n_0$. Now define $x(t)$ by $x(t)=x_{n_0}(t)$, $t\in[t_0,T]$.

Next to verify that $x(t)$ is the solution of (4). By (12), $x(t\wedge\tau_n)=x_n(t\wedge\tau_n)$, and by (11), it follows that

$$\begin{aligned}x(t\wedge\tau_n)&=x_0+\int_{t_0}^{t\wedge\tau_n}f_n(x(s),s)ds+\int_{t_0}^{t\wedge\tau_n}g_n(x(s),s)dB(s)\\&=x_0+\int_{t_0}^{t\wedge\tau_n}f(x(s),s)ds+\int_{t_0}^{t\wedge\tau_n}g(x(s),s)dB(s).\end{aligned}$$

Letting $n \to \infty$ then yields

$$x(t \wedge T) = x_0 + \int_{t_0}^{t \wedge T} f(x(s), s)ds + \int_{t_0}^{t \wedge T} g(x(s), s)dB(s)$$

that is

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)ds + \int_{t_0}^{t} g(x(s), s)dB(s).$$

Letting $n \to \infty$, we see that $x(t)$ is the solution of (4), and $x(t) \in \mathcal{M}^2([t_0, T]; R^d)$. So far, the existence is complete. The uniqueness is obtained by stopping our process. The proof is complete. □

Remark 3.1. Theorem 3.1 and Theorem 3.6 shown that the Picard iteration sequence $x^n(t)$ converge to the unique solution $x(t)$ of the SDEs (4). In Theorem 3.5 we gives an estimate on how fast converges is. This theorem shows that one can use the Picard iteration procedure to obtain the approximate solution of the systems give the estimate for the error of the approximation. Also it clearly shall show that one can use the iteration sequence procedure to obtain the approximate solutions to the SDEs.

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