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FIXED POINT THEOREMS FOR CERTAIN CONTRACTIVE MAPPINGS OF INTEGRAL TYPE

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Abstract. Some fixed point theorems and properties of diminishing orbital diameters for a few contractive mappings of integral type in complete metric spaces are proved. Four nontrivial examples are included.

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1. Introduction

In 1969, Belluce and Kirk [3] introduced the concept of diminishing orbital diameters and obtained some fixed point theorems for nonexpansive mappings, which deal with diminishing orbital diameters. Since then, an increasing number of researchers have generalized Belluce and Kirk's results, for instance, see [10, 12–16, 21]. In 1996, Liu [12] presented the following fixed point theorem for contractive mappings with diminishing orbital diameters.

Theorem 1.1. [12] *Let f be a self mapping of a complete bounded metric space (X, d) and $b \in (0, 1)$ satisfying*

$$d(fx, fy) \leq b\delta(O_f(x, y)), \quad \forall x, y \in X. \tag{1.1}$$

Then

(a1) *f has diminishing orbital diameters;*
(a2) *f has a unique fixed point.*

In 1999, Liu [13] proved the following fixed point theorems for general contractive type mappings.

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Theorem 1.2. [13] *Let f be a self mapping of a metric space (X,d) and $p,q \in \mathbb{N}$. Suppose that there exist $u \in X$ and $\varphi \in \Phi_6$ such that*

$$O_f(u) \text{ has a cluster point } v \in X \text{ and } \delta(O_f(u,v)) < +\infty; \tag{1.2}$$

$$f \text{ is closed at } v; \tag{1.3}$$

$$d(f^p x, f^q y) \le \varphi(\delta(O_f(x,y))), \quad \forall x,y \in O_f(u,v). \tag{1.4}$$

Then f has a unique fixed point $v \in \overline{O_f}(u)$ and $f^n u \to v$ as $n \to \infty$.

Theorem 1.3. [13] *Let f be a self mapping of a metric space (X,d) and $q \in \{1,2\}$. Suppose that there exist $\varphi \in \Phi_6$ and $u,v \in X$ such that*

$$O_f(u) \text{ has a cluster point } v \in X \text{ and } \delta(O_f(u,v)) < +\infty; \tag{1.5}$$

$$d(fx, f^q y) \le \varphi(\delta(O_f(x,y))), \quad \forall x,y \in O_f(u,v). \tag{1.6}$$

Then f has a unique fixed point $v \in \overline{O_f}(u)$ and $f^n u \to v$ as $n \to \infty$.

On the other hand, in 2002, Branciari [4] introduced the notion of contractive mappings of integral type in metric spaces and proved the following fixed point theorem, which generalizes the Banach fixed point theorem.

Theorem 1.4. [4] *Let f be a mapping from a complete metric space (X,d) into itself satisfying*

$$\int_0^{d(fx,fy)} \varphi(t)dt \le c \int_0^{d(x,y)} \varphi(t)dt, \quad \forall x,y \in X, \tag{1.7}$$

where $\varphi \in \Phi_1$ and $c \in (0,1)$ is a constant. Then f has a unique fixed point $u \in X$ such that $\lim_{n\to\infty} f^n x = u$ for each $x \in X$.

Afterwards many researchers continued the study of Branciari and obtained a lot of fixed point theorems for various contractive mappings of integral type, see, for example, [1,5,9,15,17,19] and the references cited therein.

Motivated by the results in [1–21], in this paper we introduce a few contractive mappings of integral type and show the existence and uniqueness of fixed point and properties of diminishing orbital diameters for these mappings in complete metric spaces. The results obtained in this paper extend Theorem 5 in [12]. Four examples are given.

2. Preliminaries

Throughout this paper, we assume that $\mathbb{R}^+ = [0,+\infty)$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers and

$\Phi_1 = \big\{\phi \mid \phi : \mathbb{R}^+ \to \mathbb{R}^+$ is Lebesgue integrable and summable in each compact subset of $\mathbb{R}^+$ and $\int_0^\varepsilon \phi(t)dt > 0, \forall \varepsilon > 0\big\}$;

$\Phi_2 = \big\{\phi \mid \phi : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, upper semicontinuous in $\mathbb{R}^+ \setminus \{0\}$ and $\phi(t) < t, \forall t > 0\big\}$;

$\Phi_3 = \big\{\phi \mid \phi : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, continuous in $\mathbb{R}^+$ and $\phi(t) > 0, \forall t > 0\big\}$;

$\Phi_4 = \{\phi \mid \phi$ belongs to Φ_1 and $\int_0^{a+b} \phi(t)dt \le \int_0^a \phi(t)dt + \int_0^b \phi(t)dt, \forall a, b > 0\}$;

$\Phi_5 = \{\phi \mid \phi : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, continuous in $\mathbb{R}^+, \phi^{-1}(0) = \{0\}$ and $\phi(a+b) \le \phi(a) + \phi(b), \forall a, b > 0\}$;

$\Phi_6 = \{\phi \mid \phi : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing and $\lim_{n\to\infty} \phi^n(t) = 0, \forall t > 0\}$.

Let (X, d) be a metric space. For $f : X \to X$, $x, y \in X$ and $A \subseteq X$, put

$$O_f(x) = \{f^n x : n \in \mathbb{N}_0\}, \quad O_f(x,y) = O_f(x) \cup O_f(y),$$
$$\delta(A) = \sup\{d(a,b) : \forall a \in A, b \in A\}$$

and $\overline{O_f}(x)$ denotes the closure of $O_f(x)$.

Definition 2.1. Let (X, d) be a metric space and $A \subseteq X$. A mapping $f : A \to A$ is said to have *diminishing orbital diameters* in A if $\lim_{n\to\infty} \delta(O_f(f^n x)) < \delta(O_f(x))$ for all $x \in A$ with $\delta(O_f(x)) > 0$.

Definition 2.2. [5] Let (X, d) be a metric space, $A \subseteq X$ and $A_n \subseteq X$ for $n \in \mathbb{N}$. The sequence $\{A_n\}_{n\in\mathbb{N}}$ is said to *converge* to the set A if

(b1) each point $a \in A$ is the limit of some convergent sequence $\{a_n : a_n \in A_n$ for each $n \in \mathbb{N}\}$;

(b2) for arbitrary $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $A_n \subseteq A_\varepsilon$ for $n > k$, where A_ε is the union of all open spheres with centers in A and radius ε.

Lemma 2.3. [16] *Let $\varphi \in \Phi_1$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence with $\lim_{n\to\infty} r_n = a$. Then*

$$\lim_{n\to\infty} \int_0^{r_n} \varphi(t)dt = \int_0^a \varphi(t)dt.$$

Lemma 2.4. [16] *Let $\varphi \in \Phi_1$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence. Then*

$$\lim_{n\to\infty} \int_0^{r_n} \varphi(t)dt = 0$$

if and only if

$$\lim_{n\to\infty} r_n = 0.$$

3. Fixed point theorems

Our main results are as follows.

Theorem 3.1. *Let f be a self mapping of a metric space (X, d). Assume that there exist $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$, $u \in X$ and $p, q \in \mathbb{N}$ such that*

$$O_f(u) \text{ has a cluster point } v \in X \text{ and } \delta(O_f(u,v)) < +\infty; \tag{3.1}$$

$$f \text{ is closed at } v; \tag{3.2}$$

$$\int_0^{\psi(d(f^p x, f^q y))} \varphi(t)dt \le \phi\left(\int_0^{\psi(\delta(O_f(x,y)))} \varphi(t)dt \right), \quad \forall x, y \in O_f(u,v). \tag{3.3}$$

Then

(c1) *f has diminishing orbital diameters in $O_f(u,v)$;*

(c2) *f has a unique fixed point v in $O_f(u,v)$;*
(c3) $\lim_{n\to\infty} f^n x = v$, $\forall x \in O_f(u,v)$;
(c4) $\{\overline{O_f}(f^n u)\}_{n\in\mathbb{N}}$ *converges to* $\{v\}$.

Proof. Let $r = \max\{p,q\}$ and x be in $O_f(u,v)$. Since $\delta(O_f(f^{n+1}x)) \le \delta(O_f(f^n x))$ for each $n \in \mathbb{N}_0$, it follows that $\{\delta(O_f(f^n x))\}_{n\in\mathbb{N}_0}$ converges to some $a_0 \ge 0$. Now we claim that $a_0 = 0$. If not, then $a_0 > 0$ and $\lim_{n\to\infty} \delta(O_f(f^n x)) = a_0$. For all $m,n \in \mathbb{N}$ and $i,j \in \mathbb{N}_0$, by (3.3) and $(\varphi,\phi,\psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$, we arrive at

$$\begin{aligned}\int_0^{\psi(d(f^{n+r+i}x,f^{m+r+j}x))} \varphi(t)dt &\le \phi\bigg(\int_0^{\psi(\delta(O_f(f^{n+r-p+i}x,f^{m+r-q+j}x)))} \varphi(t)dt\bigg)\\ &\le \phi\bigg(\int_0^{\psi(\delta(O_f(f^h x)))} \varphi(t)dt\bigg),\end{aligned}$$

where $h = \min\{m,n\}$. It follows that

$$\int_0^{\psi(\delta(O_f(f^{n+r}x)))} \varphi(t)dt \le \phi\bigg(\int_0^{\psi(\delta(O_f(f^n x)))} \varphi(t)dt\bigg), \quad \forall n \in \mathbb{N}. \tag{3.4}$$

Letting n tend to infinity in (3.4) and using $(\varphi,\phi,\psi) \in \Phi_1\times\Phi_2\times\Phi_3$ and Lemma 2.3, we have

$$\begin{aligned}0 &< \int_0^{\psi(a_0)} \varphi(t)dt = \limsup_{n\to\infty} \int_0^{\psi(\delta(O_f(f^{n+r}x)))} \varphi(t)dt\\ &\le \limsup_{n\to\infty} \phi\bigg(\int_0^{\psi(\delta(O_f(f^n x)))} \varphi(t)dt\bigg) \le \phi\bigg(\int_0^{\psi(a_0)} \varphi(t)dt\bigg)\\ &< \int_0^{\psi(a_0)} \varphi(t)dt,\end{aligned}$$

which is impossible. That is, $a_0 = 0$. Therefore f has diminishing orbital diameters in $O_f(u,v)$ and $\{f^n u\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. (3.1) ensures that $f^n u \to v$ as $n \to \infty$. By means of (3.2), we deduce that $v = fv$. That is, v is a fixed point of the mapping f in $O_f(u,v)$. Suppose that f has a fixed point w in $O_f(u,v)$ with $w \ne v$. In view of (3.3) and $(\varphi,\phi,\psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$, we find that

$$\begin{aligned}0 &< \int_0^{\psi(d(w,v))} \varphi(t)dt = \int_0^{\psi(d(f^p w,f^q v))} \varphi(t)dt\\ &\le \phi\bigg(\int_0^{\psi(\delta(O_f(w,v)))} \varphi(t)dt\bigg) = \phi\bigg(\int_0^{\psi(d(w,v))} \varphi(t)dt\bigg)\\ &< \int_0^{\psi(d(w,v))} \varphi(t)dt,\end{aligned}$$

which is a contradiction.

Notice that $\overline{O_f}(f^n u) = \{v\} \cup O_f(f^n u)$ for each $n \in \mathbb{N}$, $\lim_{n\to\infty} \delta(O_f(f^n u)) = 0$ and $\lim_{n\to\infty} f^n u = v$. It follows that for each $\varepsilon > 0$, there exists $P \in \mathbb{N}$

satisfying

$$d(f^n u, v) < \varepsilon, \quad \forall n > P$$

and

$$\overline{O_f}(f^n u) \subseteq B(v, \varepsilon) = \{x \in X : d(x, v) < \varepsilon\}, \quad \forall n > P.$$

That is, $\{\overline{O_f}(f^n u)\}_{n \in \mathbb{N}}$ converges to $\{v\}$. This completes the proof. □

Remark 3.1. The following example testifies that Theorem 3.1 is different from Theorem 1.4.

Example 3.2. Let $X = \{0, 1\} \cup \{\frac{1}{n} : n \geq 3\} \cup \{\frac{n+1}{n} : n \in \mathbb{N}\} \cup [3, +\infty)$ with the usual metric $d = |\cdot|$, $p = q = 2$, $f : X \to X$ and $\varphi, \phi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$fx = \begin{cases} x, & \forall x \in \{0, 1\} \cup [3, +\infty), \\ n, & x = \frac{1}{n}, \quad \forall n \geq 3, \\ \frac{n+2}{n+1}, & x = \frac{n+1}{n}, \quad \forall n \in \mathbb{N}, \end{cases}$$

$$\varphi(t) = \frac{1}{1+t}, \quad \phi(t) = t, \quad \psi(t) = 2t, \quad \forall t \in \mathbb{R}^+.$$

Clearly, (X, d) is a metric space, f is closed at 1 and $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$.

Put $u = 2$ and $v = 1$. Note that $O_f(2, 1) = \{1\} \cup \{\frac{n+1}{n} : n \in \mathbb{N}\}$. It is easy to verify that (3.1) holds. Let $x, y \in O_f(u, v)$ with $x < y$. In order to prove (3.3), we have to consider the following possible cases:

Case 1. $x = 1$ and $y = f^n 2$ for some $n \in \mathbb{N}_0$. It follows that

$$\begin{aligned} \int_0^{\psi\left(d\left(f^2 1, f^{2+n} 2\right)\right)} \varphi(t)dt &= \int_0^{\psi\left(d\left(1, \frac{n+4}{n+3}\right)\right)} \varphi(t)dt = \int_0^{\frac{2}{n+3}} \frac{1}{1+t} dt \\ &= \ln\left(1 + \frac{2}{n+3}\right) < \ln\left(1 + \frac{2}{n+1}\right) \\ &= \phi\left(\int_0^{\psi\left(d\left(1, \frac{n+2}{n+1}\right)\right)} \varphi(t)dt\right) \\ &= \phi\left(\int_0^{\psi(\delta(O_f(1, f^n 2)))} \varphi(t)dt\right); \end{aligned}$$

Case 2. $x = f^m 2$ with $y = f^n 2$ for some $m, n \in \mathbb{N}_0$ with $m > n$. Note that

$$\begin{aligned} \int_0^{\psi\left(d\left(f^{2+m} 2, f^{2+n} 2\right)\right)} \varphi(t)dt &= \int_0^{\psi\left(d\left(\frac{m+4}{m+3}, \frac{n+4}{n+3}\right)\right)} \varphi(t)dt \\ &= \ln\left(1 + \frac{2(m-n)}{(m+3)(n+3)}\right) < \ln\left(1 + \frac{2}{n+1}\right) \\ &= \phi\left(\int_0^{\psi\left(\frac{1}{n+1}\right)} \varphi(t)dt\right) \\ &= \phi\left(\int_0^{\psi(\delta(O_f(f^m 2, f^n 2)))} \varphi(t)dt\right). \end{aligned}$$

That is, (3.3) holds. Consequently, all the conditions of Theorem 3.1 are fulfilled. It follows from Theorem 3.1 that f has a fixed point in $O_f(2,1)$.

Unfortunately, Theorem 1.4 is useless in presenting the existence of fixed points of the mapping f in X. Suppose that the conditions of Theorem 1.4 are satisfied. That is, there exist $c \in (0,1)$ and $\varphi \in \Phi_1$ satisfying (1.7). Set $(x_0, y_0) = (0,1)$. By means of (1.7), $c \in (0,1)$ and $\varphi \in \Phi_1$, we conclude that

$$\begin{aligned} 0 < \int_0^1 \varphi(t)dt = \int_0^{d(fx_0,fy_0)} \varphi(t)dt \le c\int_0^{d(x_0,y_0)} \varphi(t)dt \\ = c\int_0^1 \varphi(t)dt < \int_0^1 \varphi(t)dt, \end{aligned}$$

which is absurd.

Following Theorem 3.1, we gain immediately that

Theorem 3.3. *Let f be a self mapping of a complete bounded metric space (X,d). Assume that there exist $p,q \in \mathbb{N}$ and $(\varphi,\phi,\psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ such that*

$$f \text{ is closed in } X; \tag{3.5}$$

$$\int_0^{\psi(d(f^p x, f^q y))} \varphi(t)dt \le \phi\Big(\int_0^{\psi(\delta(O_f(x,y)))} \varphi(t)dt\Big), \quad \forall x,y \in X. \tag{3.6}$$

Then

(d1) *f has diminishing orbital diameters in X;*
(d2) *f has a unique fixed point $v \in X$ and $\lim_{n\to\infty} f^n x = v$, $\forall x \in X$;*
(d3) *$\{\overline{O_f}(f^n x)\}_{n\in\mathbb{N}}$ converges to $\{v\}$, $\forall x \in X$.*

Remark 3.2. The following example reveals that the condition that f be closed when $p,q \ge 2$ is necessary in Theorems 3.1 and 3.3.

Example 3.4. Let p,q be in $\mathbb{N}$ and $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ with the usual metric $d = |\cdot|$. Define $f : X \to X$ and $\varphi,\phi,\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$fx = \begin{cases} 1, & x = 0, \\ \frac{1}{4n}, & x = \frac{1}{n}, \quad \forall n \in \mathbb{N}, \end{cases}$$

$$\varphi(t) = \frac{1}{1+t}, \quad \phi(t) = t, \quad \psi(t) = 2t, \quad \forall t \in \mathbb{R}^+.$$

Clearly, (X,d) is a complete bounded metric space and $(\varphi,\phi,\psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$. For any $m,n \in \mathbb{N}$, we get that $f^m 0 = \frac{1}{4^{m-1}}$ and $f^m \frac{1}{n} = \frac{1}{4^m n}$. It follows that $\delta(O_f(x,y)) = 1$ for at least one of $x,y \in \{0,1\}$ and $\delta(O_f(x,y)) = \max\{x,y\}$ if $x,y \notin \{0,1\}$. Let $x,y \in X$. In order to verify (3.6), we consider the following cases:

Case 1. $x = 0$ and $y = \frac{1}{n}$ for some $n \in \mathbb{N}$. It is easy to see that

$$\begin{aligned}
&\int_0^{\psi(d(f^p 0, f^q \frac{1}{n}))} \varphi(t)dt \\
&= \int_0^{\psi\left(d\left(\frac{1}{4^{p-1}}, \frac{1}{4^q n}\right)\right)} \varphi(t)dt = \ln\left(1 + 2\left|\frac{1}{4^{p-1}} - \frac{1}{4^q n}\right|\right) \\
&\le \ln\left(1 + 2\max\left\{\frac{1}{4^{p-1}}, \frac{1}{4^{q-1}}\right\}\right) = \ln\left(1 + \frac{2}{4^{\min\{p-1,q-1\}}}\right) \\
&\le \ln(1+2) = \phi\left(\int_0^{\psi(1)} \varphi(t)dt\right) \\
&= \phi\left(\int_0^{\psi(\delta(O_f(0,\frac{1}{n})))} \varphi(t)dt\right);
\end{aligned}$$

Case 2. $x = \frac{1}{n}$ for some $n \in \mathbb{N}$ and $y = 0$. It follows that

$$\begin{aligned}
&\int_0^{\psi(d(f^p \frac{1}{n}, f^q 0))} \varphi(t)dt \\
&= \int_0^{\psi\left(d\left(\frac{1}{4^p n}, \frac{1}{4^{q-1}}\right)\right)} \varphi(t)dt = \ln\left(1 + 2\left|\frac{1}{4^p n} - \frac{1}{4^{q-1}}\right|\right) \\
&\le \ln\left(1 + 2\max\left\{\frac{1}{4^{p-1}}, \frac{1}{4^{q-1}}\right\}\right) = \ln\left(1 + \frac{2}{4^{\min\{p-1,q-1\}}}\right) \\
&\le \ln(1+2) = \phi\left(\int_0^{\psi(1)} \varphi(t)dt\right) \\
&= \phi\left(\int_0^{\psi(\delta(O_f(\frac{1}{n},0)))} \varphi(t)dt\right);
\end{aligned}$$

Case 3. $x = y = 0$. Note that

$$\begin{aligned}
&\int_0^{\psi(d(f^p 0, f^q 0))} \varphi(t)dt \\
&= \int_0^{\psi\left(d\left(\frac{1}{4^{p-1}}, \frac{1}{4^{q-1}}\right)\right)} \varphi(t)dt = \ln\left(1 + 2\left|\frac{1}{4^{p-1}} - \frac{1}{4^{q-1}}\right|\right) \\
&\le \ln\left(1 + 2\max\left\{\frac{1}{4^{p-1}}, \frac{1}{4^{q-1}}\right\}\right) = \ln\left(1 + \frac{2}{4^{\min\{p-1,q-1\}}}\right) \\
&\le \ln(1+2) = \phi\left(\int_0^{\psi(1)} \varphi(t)dt\right) \\
&= \phi\left(\int_0^{\psi(\delta(O_f(0,0)))} \varphi(t)dt\right);
\end{aligned}$$

Case 4. $x = \frac{1}{m}$ with $y = \frac{1}{n}$ for some $m, n \in \mathbb{N}$. Observe that

$$\begin{aligned}
&\int_0^{\psi(d(f^p\frac{1}{m}, f^q\frac{1}{n}))} \varphi(t)dt \\
&= \int_0^{\psi\left(d\left(\frac{1}{4^p m}, \frac{1}{4^q n}\right)\right)} \varphi(t)dt = \ln\left(1 + 2\left|\frac{1}{4^p m} - \frac{1}{4^q n}\right|\right) \\
&\le \ln\left(1 + \frac{2}{4^{\min\{p,q\}}} \cdot \max\left\{\frac{1}{m}, \frac{1}{n}\right\}\right) \\
&\le \ln\left(1 + 2\max\left\{\frac{1}{m}, \frac{1}{n}\right\}\right) = \phi\left(\int_0^{\psi(\max\{\frac{1}{m}, \frac{1}{n}\})} \varphi(t)dt\right) \\
&= \phi\left(\int_0^{\psi(\delta(O_f(\frac{1}{m}, \frac{1}{n})))} \varphi(t)dt\right).
\end{aligned}$$

Therefore, all conditions of Theorems 3.1 and 3.3 are satisfied except the closedness assumption. However, f has no fixed points.

Remark 3.3. Example 3.6 proves that completeness of X is necessary in Theorem 3.3.

Example 3.5. Let $X = (0, 1]$ with the usual metric $d = |\cdot|$, $p = q = 2$, $\varphi, \phi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ and $f : X \to X$ be defined by

$$fx = \frac{x}{\sqrt{3}}, \quad \forall x \in (0, 1],$$

$$\varphi(t) = 2t, \quad \phi(t) = \frac{t}{9}, \quad \psi(t) = 3t, \quad \forall t \in \mathbb{R}^+.$$

Apparently, (X, d) is a bounded metric space, f is closed in X and $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$, but X is not complete. Note that $p = q$, which yields that (3.6) is symmetric in x and y, and (3.6) holds for all $x = y \in X$. Let $x, y \in X$ with $x < y$. Notice that

$$\begin{aligned}
\int_0^{\psi(d(f^p x, f^q y))} \varphi(t)dt &= \left(3\left|\frac{x}{3} - \frac{y}{3}\right|\right)^2 < y^2 \\
&= \phi\left(\int_0^{\psi(y)} \varphi(t)dt\right) = \phi\left(\int_0^{\psi(\delta(O_f(x,y)))} \varphi(t)dt\right).
\end{aligned}$$

That is, (3.6) holds. Consequently, the conditions of Theorem 3.3 are satisfied except for completeness assumption. However, f has no fixed points.

The following result presents that the closedness of f in Theorems 3.1 and 3.3 is unnecessary if $p = 1$ and $q \in \{1, 2\}$.

Theorem 3.6. *Let f be a self mapping of a metric space (X, d), $q \in \{1, 2\}$ and $(\phi, \varphi, \psi) \in \Phi_2 \times \Phi_4 \times \Phi_5$. Assume that there exist $u, v \in X$ satisfying* (3.1) *and*

$$\int_0^{\psi(d(fx, f^q y))} \varphi(t)dt \le \phi\left(\int_0^{\psi(\delta(O_f(x,y)))} \varphi(t)dt\right), \quad \forall x, y \in O_f(u, v), \tag{3.7}$$

then the conclusions of Theorem 3.1 *hold.*

Proof. Let $s_n = \delta(O_f(f^n u, f^n v))$ and $d_n(x) = \delta(O_f(f^n x))$ for all $(n, x) \in \mathbb{N}_0 \times O_f(u, v)$. Similar to the proof of Theorem 3.1, we obtain that

$$\lim_{n\to\infty} d_n(x) = 0, \quad \forall x \in O_f(u, v) \tag{3.8}$$

and

$$\lim_{n\to\infty} f^n u = v. \tag{3.9}$$

Since $\{s_n\}_{n\in\mathbb{N}_0}$ is decreasing and bounded below by 0, it follows that it converges to some number $\varepsilon_0 \geq 0$. Suppose that $\varepsilon_0 > 0$. By virtue of (3.7) and $(\phi, \varphi, \psi) \in \Phi_2 \times \Phi_4 \times \Phi_5$, we gain that, $\forall i, j, k, n \in \mathbb{N}$ with $i, j, k \geq n$

$$\begin{aligned}
&\int_0^{\psi(d(f^i u, f^j v))} \varphi(t)dt \\
&\leq \int_0^{\psi(d(f^i u, f^{k+q} v) + d(f^j v, f^{k+q} v))} \varphi(t)dt \\
&\leq \int_0^{\psi(d(f^i u, f^{k+q} v)) + \psi(d(f^j v, f^{k+q} v))} \varphi(t)dt \\
&\leq \int_0^{\psi(d(f^i u, f^{k+q} v))} \varphi(t)dt + \int_0^{\psi(d(f^j v, f^{k+q} v))} \varphi(t)dt \\
&\leq \phi\Big(\int_0^{\psi(\delta(O_f(f^{i-1} u, f^k v)))} \varphi(t)dt\Big) + \phi\Big(\int_0^{\psi(\delta(O_f(f^{j-1} v, f^k v)))} \varphi(t)dt\Big) \\
&\leq \phi\Big(\int_0^{\psi(\delta(O_f(f^{n-1} u, f^{n-1} v)))} \varphi(t)dt\Big) + \phi\Big(\int_0^{\psi(\delta(O_f(f^{n-1} v)))} \varphi(t)dt\Big) \\
&= \phi\Big(\int_0^{\psi(s_{n-1})} \varphi(t)dt\Big) + \phi\Big(\int_0^{\psi(d_{n-1}(v))} \varphi(t)dt\Big)
\end{aligned} \tag{3.10}$$

and, $\forall y \in \{u, v\}, i, j, n \in \mathbb{N}$ with $j, i \geq n \geq 2$

$$\begin{aligned}
\int_0^{\psi(d(f^i y, f^j y))} \varphi(t)dt &\leq \phi\Big(\int_0^{\psi(\delta(O_f(f^{i-1} y, f^{j-q} y)))} \varphi(t)dt\Big) \\
&\leq \phi\Big(\int_0^{\psi(\delta(O_f(f^{n-1} y, f^{n-2} y)))} \varphi(t)dt\Big) \\
&= \phi\Big(\int_0^{\psi(\delta(O_f(f^{n-2} y)))} \varphi(t)dt\Big) = \phi\Big(\int_0^{\psi(d_{n-2}(y))} \varphi(t)dt\Big),
\end{aligned}$$

which give that

$$\begin{aligned}\int_0^{\psi(s_n)}\varphi(t)dt &= \int_0^{\psi(\max\{d_n(u),d_n(v),\sup\{d(f^iu,f^jv):\forall i,j\in\mathbb{N}\text{ with } i,j\geq n\}\})}\varphi(t)dt\\
&= \max\Big\{\int_0^{\psi(d_n(u))}\varphi(t)dt, \int_0^{\psi(d_n(v))}\varphi(t)dt,\\
&\qquad \sup\Big\{\int_0^{\psi(d(f^iu,f^jv))}\varphi(t)dt : \forall i,j\in\mathbb{N}\text{ with } i,j\geq n\Big\}\Big\}\\
&\leq \max\Big\{\phi\Big(\int_0^{\psi(d_{n-2}(u))}\varphi(t)dt\Big), \phi\Big(\int_0^{\psi(d_{n-2}(v))}\varphi(t)dt\Big),\\
&\qquad \phi\Big(\int_0^{\psi(s_{n-1})}\varphi(t)dt\Big)+\phi\Big(\int_0^{\psi(d_{n-1}(v))}\varphi(t)dt\Big)\Big\}, \quad \forall n\geq 2,\end{aligned}$$

which together with (3.8), $\lim_{n\to\infty} s_n = \varepsilon_0$, Lemmas 2.3 and 2.4 and $(\phi,\varphi,\psi)\in\Phi_2\times\Phi_4\times\Phi_5$ shows that

$$\begin{aligned}0 &< \int_0^{\psi(\varepsilon_0)}\varphi(t)dt = \limsup_{n\to\infty}\int_0^{\psi(s_n)}\varphi(t)dt\\
&\leq \limsup_{n\to\infty}\max\Big\{\phi\Big(\int_0^{\psi(d_{n-2}(u))}\varphi(t)dt\Big), \phi\Big(\int_0^{\psi(d_{n-2}(v))}\varphi(t)dt\Big),\\
&\qquad \phi\Big(\int_0^{\psi(s_{n-1})}\varphi(t)dt\Big)+\phi\Big(\int_0^{\psi(d_{n-1}(v))}\varphi(t)dt\Big)\Big\}\\
&\leq \max\Big\{\limsup_{n\to\infty}\phi\Big(\int_0^{\psi(d_{n-2}(u))}\varphi(t)dt)\Big), \limsup_{n\to\infty}\phi\Big(\int_0^{\psi(d_{n-2}(v))}\varphi(t)dt\Big),\\
&\qquad \limsup_{n\to\infty}\phi\Big(\int_0^{\psi(s_{n-1})}\varphi(t)dt\Big)+\limsup_{n\to\infty}\phi\Big(\int_0^{\psi(d_{n-1}(v))}\varphi(t)dt\Big)\Big\}\\
&\leq \max\Big\{0,0,\phi\Big(\int_0^{\psi(\varepsilon_0)}\varphi(t)dt\Big)\Big\} = \phi\Big(\int_0^{\psi(\varepsilon_0)}\varphi(t)dt\Big)\\
&< \int_0^{\psi(\varepsilon_0)}\varphi(t)dt,\end{aligned}$$

which is impossible. That is, $\lim_{n\to\infty} s_n = 0$.

It follows from (3.9) that

$$0\leq d(v,f^nv)\leq \delta(\overline{O_f}(f^nu,f^nv)) = \delta(O_f(f^nu,f^nv)) = s_n\to 0 \quad \text{as } n\to\infty,$$

which implies that

$$\lim_{n\to\infty} f^nv = v. \tag{3.11}$$

Suppose that $v\neq fv$. It follows that $d_0(v)=\delta(O_f(v))>0$. Thus (3.11) means that there exists $n_0\in\mathbb{N}$ satisfying

$$d(f^nv,v)\leq \frac{1}{2}\delta(O_f(v)) = \frac{1}{2}d_0(v), \quad \forall n\geq n_0. \tag{3.12}$$

By virtue of (3.7) and $(\phi,\varphi,\psi)\in\Phi_2\times\Phi_4\times\Phi_5$, we have

$$\begin{aligned}\int_0^{\psi(d(f^nv,f^jv))}\varphi(t)dt &\le \phi\Big(\int_0^{\psi(\delta(O_f(f^{n-1}v,f^{j-q}v)))}\varphi(t)dt\Big)\\ &\le \phi\Big(\int_0^{\psi(\delta(O_f(v)))}\varphi(t)dt\Big)\\ &= \phi\Big(\int_0^{\psi(d_0(v))}\varphi(t)dt\Big),\quad \forall n,j\in\mathbb{N}\text{ with } j>n,\end{aligned}$$

which implies that

$$\begin{aligned}\int_0^{\psi(d_1(v))}\varphi(t)dt = \int_0^{\psi(O_f(fv))}\varphi(t)dt &\le \phi\Big(\int_0^{\psi(d_0(v))}\varphi(t)dt\Big)\\ &< \int_0^{\psi(d_0(v))}\varphi(t)dt,\end{aligned}\tag{3.13}$$

which together with $(\varphi,\psi)\in\Phi_4\times\Phi_5$ yields that

$$d_1(v) < d_0(v).\tag{3.14}$$

Using (3.12) and (3.14), we get that

$$\begin{aligned}d_0(v) &= \max\{\sup\{d(f^nv,v): n\in\mathbb{N}\}, d_1(v)\}\\ &= \sup\{d(f^nv,v): n\in\mathbb{N}\}\\ &= \max\{d(f^nv,v): 1\le n<n_0\},\end{aligned}$$

which means that there exists $r\in\mathbb{N}$ with $1\le r<n_0$ satisfying

$$d_0(v) = \max\{d(f^nv,v): n<n_0\} = d(f^rv,v).\tag{3.15}$$

On account of (3.7), (3.10), (3.14), $(\phi,\varphi,\psi)\in\Phi_2\times\Phi_4\times\Phi_5$ and Lemma 2.3, we find that

$$\begin{aligned}0 &< \int_0^{\psi(d_0(v))}\varphi(t)dt = \int_0^{\psi(d(f^rv,v))}\varphi(t)dt\\ &= \limsup_{n\to\infty}\int_0^{\psi(d(f^rv,f^nv))}\varphi(t)dt\\ &\le \limsup_{n\to\infty}\phi\Big(\int_0^{\psi(\delta(O_f(f^{r-1}v,f^{n-q}v)))}\varphi(t)dt\Big)\\ &\le \phi\Big(\int_0^{\psi(d_0(v))}\varphi(t)dt\Big) < \int_0^{\psi(d_0(v))}\varphi(t)dt,\end{aligned}$$

which is absurd. That is, $v=fv$.

The rest of the proof is identical with the proof of Theorem 3.1. This completes the proof. □

Following Theorem 3.6, we have

Theorem 3.7. *Let f be a self mapping of a complete bounded metric space (X,d). Assume that there exist $q \in \{1,2\}$ and $(\phi,\varphi,\psi) \in \Phi_2 \times \Phi_4 \times \Phi_5$ satisfying*

$$\int_0^{\psi(d(fx,f^qy))} \varphi(t)dt \le \phi\bigg(\int_0^{\psi(\delta(O_f(x,y)))} \varphi(t)dt\bigg), \quad \forall x,y \in X, \tag{3.16}$$

then the conclusions of Theorem 3.3 hold.

Remark 3.4. Theorem 3.6 extends Theorem 5 in [12]. The following example shows that Theorem 3.7 differs from Theorem 1.4.

Example 3.8. Let $X = [0,8]$ with the usual metric $d = |\cdot|$, $\phi,\varphi,\psi : \mathbb{R}^+ \to \mathbb{R}^+$ and $f : X \to X$ satisfy that

$$fx = \begin{cases} 1, & x = 4, \\ 2, & \forall x \in X \backslash \{4\}, \end{cases}$$

$$\varphi(t) = \frac{1}{1+t}, \quad \phi(t) = \frac{t}{2}, \quad \psi(t) = \frac{t}{3}, \quad \forall t \in \mathbb{R}^+.$$

Evidently, (X,d) is a complete bounded metric space and $(\phi,\varphi,\psi) \in \Phi_2 \times \Phi_4 \times \Phi_5$. Let $q = 1$ and $x,y \in X$. In other to prove (3.16), we have to consider the following possible cases:

Case 1. $x = y \in X\backslash\{4\}$. It follows that

$$\int_0^{\psi(d(fx,fy))} \varphi(t)dt = 0 \le \phi\bigg(\int_0^{\psi(\delta(O_f(x,y)))} \varphi(t)dt\bigg);$$

Case 2. $x = y = 4$. Notice that

$$\int_0^{\psi(d(fx,fy))} \varphi(t)dt = 0 \le \phi\bigg(\int_0^{\psi(\delta(O_f(x,y)))} \varphi(t)dt\bigg);$$

Case 3. $x,y \in X$ with $x \ne y$. Now we consider the following subcases:

Subcase 1. $x \in [0,1]$ and $y = 4$. Note that

$$\begin{aligned} \int_0^{\psi(d(fx,fy))} \varphi(t)dt &= \ln\left(1+\frac{1}{3}\right) < \frac{1}{2}\ln 2 \\ &\le \frac{1}{2}\ln\left(1+\frac{4-x}{3}\right) = \phi\bigg(\int_0^{\psi(4-x)} \varphi(t)dt\bigg) \\ &= \phi\bigg(\int_0^{\psi(\delta(O_f(x,y)))} \varphi(t)dt\bigg); \end{aligned}$$

Subcase 2. $x \in (1,4)$ and $y = 4$. It is easy to conclude that

$$\begin{aligned}\int_0^{\psi(d(fx,fy))} \varphi(t)dt &= \ln\left(1+\frac{1}{3}\right) < \frac{1}{2}\ln\left(1+\frac{3}{3}\right)\\ &= \phi\bigg(\int_0^{\psi(3)} \varphi(t)dt\bigg)\\ &= \phi\bigg(\int_0^{\psi(\delta(O_f(x,y)))} \varphi(t)dt\bigg);\end{aligned}$$

Subcase 3. $x \in (4,8]$ and $y = 4$. It is evident to know that

$$\begin{aligned}\int_0^{\psi(d(fx,fy))} \varphi(t)dt &= \ln\left(1+\frac{1}{3}\right) < \frac{1}{2}\ln\left(1+\frac{x-1}{3}\right)\\ &= \phi\bigg(\int_0^{\psi(x-1)} \varphi(t)dt\bigg)\\ &= \phi\bigg(\int_0^{\psi(\delta(O_f(x,y)))} \varphi(t)dt\bigg);\end{aligned}$$

Subcase 4. $x = 4$ and $y \in [0,1]$. It follows that

$$\begin{aligned}\int_0^{\psi(d(fx,fy))} \varphi(t)dt &= \ln\left(1+\frac{1}{3}\right) < \frac{1}{2}\ln\left(1+\frac{4-y}{3}\right)\\ &= \phi\bigg(\int_0^{\psi(4-y)} \varphi(t)dt\bigg)\\ &= \phi\bigg(\int_0^{\psi(\delta(O_f(x,y)))} \varphi(t)dt\bigg);\end{aligned}$$

Subcase 5. $x = 4$ and $y \in (1,4)$. Notice that

$$\begin{aligned}\int_0^{\psi(d(fx,fy))} \varphi(t)dt &= \ln\left(1+\frac{1}{3}\right) < \frac{1}{2}\ln\left(1+\frac{3}{3}\right)\\ &= \phi\bigg(\int_0^{\psi(3)} \varphi(t)dt\bigg)\\ &= \phi\bigg(\int_0^{\psi(\delta(O_f(x,y)))} \varphi(t)dt\bigg);\end{aligned}$$

Subcase 6. $x = 4$ and $y \in (4,8]$. It is clear that

$$\begin{aligned}\int_0^{\psi(d(fx,fy))} \varphi(t)dt &= \ln\left(1+\frac{1}{3}\right) < \frac{1}{2}\ln\left(1+\frac{y-1}{3}\right)\\ &= \phi\bigg(\int_0^{\psi(y-1)} \varphi(t)dt\bigg)\\ &= \phi\bigg(\int_0^{\psi(\delta(O_f(x,y)))} \varphi(t)dt\bigg);\end{aligned}$$

Subcase 7. $x, y \in X\backslash\{4\}$ with $x \neq y$. It follows that

$$\int_0^{\psi(d(fx,fy))} \varphi(t)dt = 0 \leq \phi\bigg(\int_0^{\psi(\delta(O_f(x,y)))} \varphi(t)dt\bigg).$$

That is, (3.16) holds. Therefore, the conditions of Theorem 3.7 are satisfied. It follows from Theorem 3.7 that f has a unique fixed point $2 \in X$.

However, Theorem 1.4 is useless in proposing the existence of fixed points of the mapping f in X. Suppose that the conditions of Theorem 1.4 are satisfied. That is, there exist $c \in (0,1)$ and $\varphi \in \Phi_1$ satisfying (1.7). Take $(x_1, y_1) = (4, 5)$. In view of (1.7), $c \in (0,1)$ and $\varphi \in \Phi_1$, we infer that

$$\begin{aligned} 0 < \int_0^1 \varphi(t)dt &= \int_0^{d(fx_1,fy_1)} \varphi(t)dt \\ &\leq c\int_0^{d(x_1,y_1)} \varphi(t)dt = c\int_0^1 \varphi(t)dt < \int_0^1 \varphi(t)dt, \end{aligned}$$

which is impossible.

4. Conclusions

Combining the ideas of Belluce and Kirk [3] and Branciari [4], we suggest a few contractive mappings of integral type and prove, under certain conditions, the existence and uniqueness of fixed point and properties of diminishing orbital diameters for the contractive mappings. Our results have potential applications in nonlinear integral and differential equations and functional equations.

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