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ACCURATE SOLUTION FOR SLIDING BURGER FLUID FLOW

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Abstract. This article addresses the influence of partial slip condition in the hydromagnetic flow of Burgers fluid in a rotating frame of reference.The flows are induced by oscillation of a boundary. Two problems for oscillatory flows are considered. Exact solutions to the resulting boundary value problems are constructed. Analysis has been carried out in the presence of magnetic field. Physical interpretation is made through the plots for various embedded parameters.

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1. Introduction

Over the past few decades, the rotating flows of electrically conducting fluids have received a lot of interest due to its applications in cosmical fluid dynamics, in gaseous and nuclear reactors, in meteorology and in geophysical fluid dynamics. The Coriolis force is found very significant in comparison to the inertial and viscous forces. Greenspan and Howard [1] presented the classical work related to the flows in a rotating system. Afterwards many studies have been given for analysis of rotating flows involving viscous fluids (see [2-5] and several refs. therein). Recent researchers also focused their attention on the investigation of rotating flows with non-Newtonian fluids (see [6-9]). In continuation, Hayat [10] analyzed the hydromagnetic flows of Burgers' fluid in a rotating frame. In view of this fact the aim of the current attempt is to securnitized the slip effects on the rotating flows of a Burgers' fluid (a subclass of rate type fluids). An incompressible, homogeneous and electrically conducting fluid occupied the porous half space. Two problems are studied here for the exact solutions. The first problem deals with the rotating flow by general periodic oscillations of a

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plate. Second problem corresponds to the flow analysis by elliptic harmonic oscillations. The graph for solutions are made and analyzed.

2. Problem development

We consider an incompressible Burgers' fluid in a half space $z > 0$. The fluid is bounded by a rigid plate. The whole system comprising plate and fluid are in a rotating frame of reference through uniform angular velocity. $\tilde{\mathbf{\Omega}} = \tilde{\mathbf{\Omega}}\hat{k}$ (where $\hat{k}$ denotes unit vector parallel to z-axis). A magnetic field (with strength B_0) acts in the $z-$ direction. The influence of MHD is taken into the account. The flow under consideration satisfies the following expressions [10]

$$\rho\left[\frac{\partial u}{\partial t} - 2\Omega v\right] = -\frac{\partial \hat{p}}{\partial x} + \frac{\partial}{\partial z}S_{xz} - \sigma B_0^2 u, \tag{1}$$

$$\rho[\frac{\partial v}{\partial t} - 2\Omega u] = -\frac{\partial \hat{p}}{\partial y} + \frac{\partial}{\partial z}S_{yz} - \sigma B_0^2 v, \tag{2}$$

with

$$\left(1 + \lambda_1\frac{\partial}{\partial t} + \lambda_2\frac{\partial^2}{\partial t^2}\right)S_{xx} + \lambda_1\left[\frac{\partial}{\partial t}S_{xx} - 2S_{xz}\frac{\partial u}{\partial z}\right] = -2\mu\lambda_3\left(\frac{\partial u}{\partial z}\right)^2, \tag{3}$$

$$\left(1 + \lambda_1\frac{\partial}{\partial t} + \lambda_2\frac{\partial^2}{\partial t^2}\right)S_{xy} + \lambda_1\left[\frac{\partial}{\partial t}S_{xy} - S_{yz}\frac{\partial u}{\partial z} - S_{xz}\frac{\partial v}{\partial z}\right] = -2\mu\lambda_3\left(\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial z}\right), \tag{4}$$

$$\left(1 + \lambda_1\frac{\partial}{\partial t} + \lambda_2\frac{\partial^2}{\partial t^2}\right)S_{xz} + \lambda_1\frac{\partial}{\partial t}S_{xz} = \mu\frac{\partial u}{\partial z} + \mu\lambda_3\left(\frac{\partial^2 u}{\partial z\partial t}\right), \tag{5}$$

$$\left(1 + \lambda_1\frac{\partial}{\partial t} + \lambda_2\frac{\partial^2}{\partial t^2}\right)S_{yy} + \lambda_1\left[\frac{\partial}{\partial t}S_{yy} - 2S_{yz}\frac{\partial v}{\partial z}\right] = -2\mu\lambda_3\left(\frac{\partial v}{\partial z}\right)^2, \tag{6}$$

$$\left(1 + \lambda_1\frac{\partial}{\partial t} + \lambda_2\frac{\partial^2}{\partial t^2}\right)S_{yz} + \lambda_1\frac{\partial}{\partial t}S_{yz} = \mu\frac{\partial v}{\partial z} + \mu\lambda_3\left(\frac{\partial^2 v}{\partial z\partial t}\right), \tag{7}$$

Here $\hat{p}$ is the modified pressure given by $\hat{p} = \hat{p} - \frac{1}{2}\rho\ \Omega^2(x^2 + y^2)$ and $\frac{\partial p}{\partial z} = 0$. From Eqs (1) and (2) we have

$$\begin{aligned}\rho\left(1 + \lambda_1\tfrac{\partial}{\partial t} + \lambda_2\tfrac{\partial^2}{\partial t^2}\right)\left(\tfrac{\partial^2 u}{\partial z\partial t} - 2\Omega\tfrac{\partial v}{\partial z}\right) + \sigma B_0^2\left(1 + \lambda_1\tfrac{\partial}{\partial t} + \lambda_2\tfrac{\partial^2}{\partial t^2}\right)\tfrac{\partial u}{\partial z} \\ = \left(1 + \lambda_3\tfrac{\partial}{\partial t}\right)\tfrac{\partial^3}{\partial z^3}S_{xz},\end{aligned} \tag{8}$$

$$\begin{aligned}\rho\left(1 + \lambda_1\tfrac{\partial}{\partial t} + \lambda_2\tfrac{\partial^2}{\partial t^2}\right)\left(\tfrac{\partial^2 v}{\partial z\partial t} - 2\Omega\tfrac{\partial u}{\partial z}\right) + \sigma B_0^2\left(1 + \lambda_1\tfrac{\partial}{\partial t} + \lambda_2\tfrac{\partial^2}{\partial t^2}\right)\tfrac{\partial v}{\partial z} \\ = \left(1 + \lambda_3\tfrac{\partial}{\partial t}\right)\tfrac{\partial^3}{\partial z^3}S_{yz}.\end{aligned}$$

Letting

$$F = u + iv \tag{9}$$

and using Eqs. (5) and (7),we can combine Eqs. (8) and (9) as follows:

$$\rho\left(1 + \lambda_1\tfrac{\partial}{\partial t} + \lambda_2\tfrac{\partial^2}{\partial t^2}\right)\left(\tfrac{\partial^2 F}{\partial z\partial t} + 2i\Omega\tfrac{\partial F}{\partial z}\right) + \sigma B_0^2\left(1 + \lambda_1\tfrac{\partial}{\partial t} + \lambda_2\tfrac{\partial^2}{\partial t^2}\right)\tfrac{\partial F}{\partial z}$$

$$= \mu \tfrac{\partial^3 F}{\partial z^3} + \mu\lambda_3 \tfrac{\partial^4 F}{\partial z^3 \partial t}. \tag{10}$$

In the forthcoming analysis we are interested to develope closed form solutions for two types of plate oscillations.

3. First problem

In this problem the flow is induced because of general periodic oscillation of a plate. Moreover, the plate exhibits the slip effect. The boundary conditions thus are:

$$u(0,t) - \frac{\gamma}{\mu} S_{xz} = \cup_0 f(t),\ v(0,t) - \frac{\gamma}{\mu} S_{yz} = 0, \tag{11}$$

$$u, v \longrightarrow 0 \text{ as } z \longrightarrow \infty, \tag{12}$$

where γ is the slip parameter and $f(t)$ is the general periodic function with non-zero frequency $n = (2\pi)/T_0$ (where T_0 is the time period). The function $f(t)$ has a Fourier series in the form

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{iknt}, \tag{13}$$

in which

$$a_k = \frac{1}{T_0} \int f(t) e^{-iknt} \mathbf{d}t \tag{14}$$

where $\{a_k\}$ denotes Fourier series coefficients of $f(t)$ and $\cup_0$ is the constant velocity. Through Eqs (5), (7), (9), (11), and (12) we can write,

$$\left(1 + \lambda_1 \tfrac{\partial}{\partial t} + \lambda_2 \tfrac{\partial^2}{\partial t^2}\right) F(z,t) - \gamma \left(1 + \lambda_3 \tfrac{\partial}{\partial t}\right) \tfrac{\partial F}{\partial z}$$
$$= \cup_0 \textstyle\sum_{k=-\infty}^{\infty} a_k \left(1 + \lambda_1 \tfrac{\partial}{\partial t} + \lambda_2 \tfrac{\partial^2}{\partial t^2}\right) e^{iknt}, \quad at\ z = 0, \tag{15}$$

$$F(\infty, t) = 0. \tag{16}$$

We introduce the following diemensionless quantities

$$\begin{aligned} z^* &= \frac{z\cup_0}{\nu},\ F^* = \frac{F}{\cup_0},\ t^* = \frac{t\cup_0^2}{\nu},\ \omega_0^* = \frac{\omega_0 \nu}{\cup_0^2}, \\ \lambda_1^* &= \frac{\lambda_1 \cup_0^2}{\nu},\ \lambda_2^* = \frac{\lambda_2 \cup_0^4}{\nu^2},\ \lambda_3^* = \frac{\lambda_3 \cup_0^2}{\nu},\ \Omega^* = \frac{\Omega \nu}{\cup_0^2}, \\ M^{*2} &= \frac{\sigma B_0^2}{\rho \cup_0},\ \gamma^* = \frac{\cup_0 \gamma}{\nu}, \end{aligned} \tag{17}$$

The resulting diemensionless problem becomes

$$\left(1 + \lambda_1 \tfrac{\partial}{\partial t} + \lambda_2 \tfrac{\partial^2}{\partial t^2}\right) \left(\tfrac{\partial^2 F}{\partial z \partial t} + 2i\Omega F\right)$$
$$= \left(1 + \lambda_3 \tfrac{\partial}{\partial t}\right) \tfrac{\partial^3 F}{\partial z^3} - M^2 \left(1 + \lambda_1 \tfrac{\partial}{\partial t} + \lambda_2 \tfrac{\partial^2}{\partial t^2}\right) \tfrac{\partial F}{\partial z}, \tag{18}$$

$$\left(1 + \lambda_1 \tfrac{\partial}{\partial t} + \lambda_2 \tfrac{\partial^2}{\partial t^2}\right) F(z,t) - \gamma \left(1 + \lambda_3 \tfrac{\partial}{\partial t}\right) \tfrac{\partial F}{\partial z}$$

$$= \sum_{k=-\infty}^{\infty} a_k \left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right) e^{iknt}, \quad at\ z = 0, \tag{19}$$

$$F(\infty, t) = 0. \tag{20}$$

The solution by temporal Fourier transform is

$$F(z,t) = \sum_{k=-\infty}^{\infty} \frac{a_k \left(1 + (ik\omega_0)\lambda_1 - (k\omega_0)^2 \lambda_2\right) e^{-m_k z + i(k\omega_0 t - n_k z)}}{\left(1 + (ik\omega_0)\lambda_1 - (k\omega_0)^2 \lambda_2\right) + \gamma (m_k + in_k)(1 + (ik\omega_0)\lambda_3)}, \tag{21}$$

$$m_k = \sqrt{\frac{a_{1k} \pm \sqrt{a_{1k}^2 + a_{2k}^2}}{2}}, \quad n_k = \sqrt{\frac{a_{2k}}{2\left[a_{1k} + \sqrt{a_{1k}^2 + a_{2k}^2}\right]}}, \tag{22}$$

$$a_{1k} = \left[\frac{M^2\left(1 - \lambda_2 (k\omega_0)^2 + \lambda_1\lambda_3 (k\omega_0)^2\right) - k\omega\,(k\omega + 2\Omega)\left(\lambda_1 - \lambda_3 + \lambda_2\lambda_3 (k\omega_0)^2\right)}{(1 + (k\omega)^2\lambda_3^2)}\right], \tag{23}$$

$$a_{2k} = \left[\frac{M^2 k\omega\left(\lambda_1 - \lambda_3 + \lambda_2\lambda_3 (k\omega_0)^2\right) + (k\omega + 2\Omega)\left(1 - \lambda_2 (k\omega_0)^2 + \lambda_1\lambda_3 (k\omega_0)^2\right)}{(1 + (k\omega)^2\lambda_3^2)}\right]. \tag{24}$$

Note that Eq. (21) provides the solution of the problem for general periodic oscillation of a plate. Flow fields in special case can be written through the appropriate Fourier coefficients (a_k) which give rise to different plate oscillations. For example, the flow fields F_j $(j = 1-5)$ due to five oscillations $\exp(i\omega_0 t)$, $\cos\omega_0 t$, $\sin\omega_0 t$, $\left\{\begin{array}{c} 1,\ |t| < T_1 \\ 0,\ T_1 < |t| < T_0/2 \end{array}\right\}$, $\sum_{k=-\infty}^{\infty} \delta(t - kT_0)$, are respectively given as

$$F_1(z,t) = \frac{a_k \left(1 + (ik\omega_0)\lambda_1 - (k\omega_0)^2 \lambda_2\right) e^{-m_k z + i(k\omega_0 t - n_k z)}}{\left(1 + (ik\omega_0)\lambda_1 - (k\omega_0)^2 \lambda_2\right) + \gamma (m_k + in_k)(1 + \lambda_3 (ik\omega_0))}, \tag{25}$$

$$F_2(z,t) = \frac{1}{2}\left[\begin{array}{c} \frac{\left(1+(ik\omega_0)\lambda_1-(k\omega_0)^2\lambda_2\right)e^{-m_1 z+i(\omega_0 t-n_1 z)}}{\left(1+(ik\omega_0)\lambda_1-(k\omega_0)^2\lambda_2\right)+\gamma(m_1+in_1)(1+\lambda_3(i\omega_0))} \\ +\frac{\left(1+(ik\omega_0)\lambda_1-(k\omega_0)^2\lambda_2\right)e^{-m_{-1} z-i(\omega_0 t+n_{-1} z)}}{\left(1+(ik\omega_0)\lambda_1-(k\omega_0)^2\lambda_2\right)+\gamma(m_{-1}+in_{-1})(1+\lambda_3(i\omega_0))} \end{array}\right], \tag{26}$$

$$F_3(z,t) = \frac{1}{2i}\left[\begin{array}{c} \frac{\left(1+(ik\omega_0)\lambda_1-(k\omega_0)^2\lambda_2\right)e^{-m_1 z+i(\omega_0 t-n_1 z)}}{\left(1+(ik\omega_0)\lambda_1-(k\omega_0)^2\lambda_2\right)+\gamma(m_1+in_1)(1+\lambda_3(i\omega_0))} \\ -\frac{\left(1+(ik\omega_0)\lambda_1-(k\omega_0)^2\lambda_2\right)e^{-m_{-1} z-i(\omega_0 t+n_{-1} z)}}{\left(1+(ik\omega_0)\lambda_1-(k\omega_0)^2\lambda_2\right)+\gamma(m_{-1}+in_{-1})(1+\lambda_3(i\omega_0))} \end{array}\right], \tag{27}$$

$$F_4(z,t) = \sum_{k=-\infty}^{\infty} \frac{\sin k\omega_0 T_1}{k\pi}\left[\frac{\left(1 + (ik\omega_0)\lambda_1 - (k\omega_0)^2\lambda_2\right) e^{-m_k z + i(k\omega_0 t - n_k z)}}{(1 + (ik\omega_0)\lambda_1 - (k\omega_0)^2\lambda_2) + \gamma (m_k + in_k)(1 + \lambda_3 (ik\omega_0))}\right]$$

$$,\ k \neq 0 \tag{28}$$

$$F_5(z,t) = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \frac{\left(1 + (ik\omega_0)\,\lambda_1 - (k\omega_0)^2\,\lambda_2\right) e^{-m_k z + i(k\omega_0 t - n_k z)}}{\left(1 + (ik\omega_0)\,\lambda_1 - (k\omega_0)^2\,\lambda_2\right) + \gamma\,(m_k + in_k)\,(1 + \lambda_3(ik\omega_0))}. \tag{29}$$

4. Second Problem

The flow in this subsection is due to elliptic harmonic oscillation of a plate with partial slip. The resulting boundary condition thus is representating in the form

$$\begin{aligned}&\left(1 + \lambda_1 \tfrac{\partial}{\partial t} + \lambda_2 \tfrac{\partial^2}{\partial t^2}\right) F(z,t) - \gamma\left(1 + \lambda_3 \tfrac{\partial}{\partial t}\right) \tfrac{\partial F}{\partial z} \\ &= \left(1 + \lambda_1 \tfrac{\partial}{\partial t} + \lambda_2 \tfrac{\partial^2}{\partial t^2}\right) \left[a e^{int} + b e^{-int}\right], \quad at\ z = 0\end{aligned} \tag{30}$$

We write the solution of the form

$$F = aF_1(z)e^{int} + bF_2(z)e^{-int}, n > 0 \tag{31}$$

Considering (31) and then solving ODEs problems from Eqs. (18) and (22) and (30) we have

$$F = \left(\frac{a\left(1 + in\lambda_1 - n^2\lambda_2\right) e^{-m_1 z} e^{int}}{(1 + in\lambda_1 - n^2\lambda_2) + \gamma m_1\,(1 + in\lambda_3)} + \frac{b\left(1 + in\lambda_1 - n^2\lambda_2\right) e^{-m_2 z} e^{-int}}{(1 + in\lambda_1 - n^2\lambda_2) + \gamma m_2\,(1 - in\lambda_3)} \right), \tag{32}$$

where

$$m_i z = \Psi_i\,(\alpha_i + i\beta_i)\,, \tag{33}$$

$$\alpha_i = \frac{1}{\sqrt{2}}\left[S_i + \sqrt{S_i^2 + 1}\right]^{1/2},\ \beta_i = \frac{1}{\sqrt{2}}\left[-S_i + \sqrt{S_i^2 + 1}\right]^{1/2},\ i = 1,2 \tag{34}$$

$$\Psi_1 = \left(\frac{B_1 - \lambda_3 n A_1}{(1 + \lambda_3^2 n^2)}\right)^{1/2} z,\ \Psi_2 = \left(\frac{B_2 + \lambda_3 n A_2}{(1 + \lambda_3^2 n^2)}\right)^{1/2} z$$

$$S_1 = \frac{A_1 + \lambda_3 n B_1}{B_1 - \lambda_3 n A_1},\ S_2 = \frac{A_2 - \lambda_3 n B_2}{B_2 + \lambda_3 n A_2},$$

$$\begin{aligned} A_1 &= M^2(1 - n^2\lambda_2) - n\lambda_1\,(n + 2\Omega)\,, \\ B_1 &= n\lambda_1 M^2 + (2\Omega + n)(1 - n^2\lambda_2), \\ A_2 &= M^2(1 - n^2\lambda_2) + n\lambda_1\,(2\Omega - n)\,, \\ B_2 &= -M^2\lambda_1 n + (2\Omega - n)(1 - n^2\lambda_2). \end{aligned}$$

If $a = a_1 + ia_2$, $b = b_1 + ib_2$ then we can write the real and imaginary parts from Eqs. (35) and (36) as

$$u = \begin{bmatrix} C_{1R}e^{-\Psi_1\alpha_1}\{a_1\cos(\Psi_1\beta_1 - nt) + a_2\sin(\Psi_1\beta_1 - nt)\} \\ +C_{2R}e^{-\Psi_2\alpha_2}\{b_1\cos(\Psi_2\beta_2 + nt) + b_2\sin(\Psi_2\beta_2 + nt)\} \end{bmatrix}, \tag{35}$$

$$v=\begin{bmatrix} C_{1I}e^{-\Psi_1\alpha_1}\{a_2\cos(\Psi_1\beta_1-nt)-a_1\sin(\Psi_1\beta_1-nt)\} \\ +C_{2I}e^{-\Psi_2\alpha_2}\{b_2\cos(\Psi_2\beta_2+nt)-b_1\sin(\Psi_2\beta_2+nt)\}\end{bmatrix}, \tag{36}$$

where

$$C_{1R}=\frac{\left(1-n^2\lambda_2\right)^2+(n\lambda_1)^2+\gamma\left[\alpha_1\left(1-n^2\lambda_2\right)+n\lambda_1\beta_1+n\lambda_3\left(\beta_1\left(1-n^2\lambda_2\right)-\alpha_1 n\lambda_1\right)\right]}{\left[1-n^2\lambda_2+\gamma\left(\alpha_1-n\lambda_3\beta_1\right)\right]^2+\left[n\lambda_1+\gamma\left(\alpha_1 n\lambda_3+\beta_1\right)\right]^2},$$

$$C_{1I}=\frac{\gamma\left[\left(1-n^2\lambda_2\right)\left(\alpha_1 n\lambda_3-\beta_1\right)+n\lambda_1\left(\alpha_1-\beta_1 n\lambda_3\right)\right]}{\left[1-n^2\lambda_2+\gamma\left(\alpha_1-n\lambda_3\beta_1\right)\right]^2+\left[n\lambda_1+\gamma\left(\alpha_1 n\lambda_3+\beta_1\right)\right]^2},$$

$$C_{2R}=\frac{\left(1-n^2\lambda_2\right)^2+(n\lambda_1)^2+\gamma\left[\left(1-n^2\lambda_2\right)\left(\alpha_2-\beta_2 n\lambda_3\right)+n\lambda_1\left(\beta_2-\alpha_2 n^2\lambda_3\right)\right]}{\left[1-n^2\lambda_2+\gamma\left(\alpha_2-n\lambda_3\beta_2\right)\right]^2+\left[n\lambda_1-\gamma\left(\beta_2-\alpha_2 n\lambda_3\right)\right]^2},$$

$$C_{2I}=\frac{\gamma\left[\left(1-n^2\lambda_2\right)\left(\alpha_2 n\lambda_3-\beta_2\right)-n\lambda_1\left(\alpha_2-\beta_2 n\lambda_3\right)\right]}{\left[1-n^2\lambda_2+\gamma\left(\alpha_2-n\lambda_3\beta_2\right)\right]^2+\left[n\lambda_1-\gamma\left(\beta_2-\alpha_2 n\lambda_3\right)\right]^2},$$

and the solutions for $n>2\Omega$ are

$$u=\begin{bmatrix} C_{1R}e^{-\Psi_1\alpha_1}\{a_1\cos(\Psi_1\beta_1-nt)+a_2\sin(\Psi_1\beta_1-nt)\} \\ +C_{2R}e^{-\Psi_3\alpha_3}\{b_1\cos(\Psi_3\beta_3+nt)+b_2\sin(\Psi_3\beta_3+nt)\}\end{bmatrix}, \tag{37}$$

$$v=\begin{bmatrix} C_{1I}e^{-\Psi_1\alpha_1}\{a_2\cos(\Psi_1\beta_1-nt)-a_1\sin(\Psi_1\beta_1-nt)\} \\ +C_{2I}e^{-\Psi_3\alpha_3}\{b_2\cos(\Psi_3\beta_3+nt)-b_1\sin(\Psi_3\beta_3+nt)\}\end{bmatrix}, \tag{38}$$

where

$$C_{1R}=\frac{\left(1-n^2\lambda_2\right)^2+(n\lambda_1)^2+\gamma\left[\alpha_1\left(1-n^2\lambda_2\right)+n\lambda_1\beta_1+n\lambda_3\left(\beta_1\left(1-n^2\lambda_2\right)-\alpha_1 n\lambda_1\right)\right]}{\left[1-n^2\lambda_2+\gamma\left(\alpha_1-n\lambda_3\beta_1\right)\right]^2+\left[n\lambda_1+\gamma\left(\alpha_1 n\lambda_3+\beta_1\right)\right]^2},$$

$$C_{1I}=\frac{\gamma\left[\left(1-n^2\lambda_2\right)\left(\alpha_1 n\lambda_3-\beta_1\right)+n\lambda_1\left(\alpha_1-\beta_1 n\lambda_3\right)\right]}{\left[1-n^2\lambda_2+\gamma\left(\alpha_1-n\lambda_3\beta_1\right)\right]^2+\left[n\lambda_1+\gamma\left(\alpha_1 n\lambda_3+\beta_1\right)\right]^2},$$

$$C_{2R}=\frac{\left(1-n^2\lambda_2\right)^2+(n\lambda_1)^2+\gamma\left[\left(1-n^2\lambda_2\right)\left(\alpha_3-\beta_3 n\lambda_3\right)+n\lambda_1\left(\beta_3-\alpha_3 n^2\lambda_3\right)\right]}{\left[1-n^2\lambda_2+\gamma\left(\alpha_3-n\lambda_3\beta_3\right)\right]^2+\left[n\lambda_1-\gamma\left(\beta_3-\alpha_3 n\lambda_3\right)\right]^2},$$

$$C_{2I}=\frac{\gamma\left[\left(1-n^2\lambda_2\right)\left(\alpha_3 n\lambda_3-\beta_3\right)-n\lambda_1\left(\alpha_3-\beta_3 n\lambda_3\right)\right]}{\left[1-n^2\lambda_2+\gamma\left(\alpha_3-n\lambda_3\beta_3\right)\right]^2+\left[n\lambda_1-\gamma\left(\beta_3-\alpha_3 n\lambda_3\right)\right]^2},$$

and

$$\alpha_3=\frac{1}{\sqrt{2}}\left[S_3+\sqrt{S_3^2+1}\right]^{1/2},\ \beta_3=\frac{1}{\sqrt{2}}\left[-S_3+\sqrt{S_3^2+1}\right]^{1/2},$$

$$\Psi_3=\left(\frac{A_3-\lambda_2 nB_3}{(1+\lambda_2^2n^2)}\right)^{1/2}z,$$

$$S_3=\frac{A_3+\lambda_2 nB_3}{\lambda_2 nA_1-B_3},$$

$$A_3=M^2-\lambda_1 n\,(n-2\Omega)\,,\ B_3=M^2\lambda_1 n+n-2\Omega,$$

The expressions of velocity components in the resonant case ($n = 2\Omega$) are

$$u = \begin{bmatrix} C_{1R}e^{-\Psi_1\alpha_1}\{a_1\cos(\Psi_1\beta_1 - nt) + a_2\sin(\Psi_1\beta_1 - nt)\} \\ +C_{2R}e^{-\Psi_0\alpha_0}\{b_1\cos(\Psi_0\beta_0 + nt) + b_2\sin(\Psi_0\beta_0 + nt)\} \end{bmatrix}, \tag{39}$$

$$v = \begin{bmatrix} C_{1I}e^{-\Psi_1\alpha_1}\{a_2\cos(\Psi_1\beta_1 - nt) - a_1\sin(\Psi_1\beta_1 - nt)\} \\ +C_{2I}e^{-\Psi_0\alpha_0}\{b_2\cos(\Psi_0\beta_0 + nt) - b_1\sin(\Psi_0\beta_0 + nt)\} \end{bmatrix}, \tag{40}$$

where

$$C_{1R} = \frac{\left(1-n^2\lambda_2\right)^2 + (n\lambda_1)^2 + \gamma\left[\alpha_1\left(1-n^2\lambda_2\right) + n\lambda_1\beta_1 + n\lambda_3\left(\beta_1\left(1-n^2\lambda_2\right) - \alpha_1 n\lambda_1\right)\right]}{\left[1-n^2\lambda_2 + \gamma\left(\alpha_1 - n\lambda_3\beta_1\right)\right]^2 + \left[n\lambda_1 + \gamma\left(\alpha_1 n\lambda_3 + \beta_1\right)\right]^2},$$

$$C_{1I} = \frac{\gamma\left[\left(1-n^2\lambda_2\right)\left(\alpha_1 n\lambda_3 - \beta_1\right) + n\lambda_1\left(\alpha_1 - \beta_1 n\lambda_3\right)\right]}{\left[1-n^2\lambda_2 + \gamma\left(\alpha_1 - n\lambda_3\beta_1\right)\right]^2 + \left[n\lambda_1 + \gamma\left(\alpha_1 n\lambda_3 + \beta_1\right)\right]^2},$$

$$C_{2R} = \frac{\left(1-n^2\lambda_2\right)^2 + (n\lambda_1)^2 + \gamma\left[\left(1-n^2\lambda_2\right)\left(\alpha_0 - \beta_0 n\lambda_3\right) + n\lambda_1\left(\beta_0 - \alpha_0 n^2\lambda_3\right)\right]}{\left[1-n^2\lambda_2 + \gamma\left(\alpha_0 - n\lambda_3\beta_0\right)\right]^2 + \left[n\lambda_1 - \gamma\left(\beta_0 - \alpha_0 n\lambda_3\right)\right]^2},$$

$$C_{2I} = \frac{\gamma\left[\left(1-n^2\lambda_2\right)\left(\alpha_0 n\lambda_3 - \beta_0\right) - n\lambda_1\left(\alpha_0 - \beta_0 n\lambda_3\right)\right]}{\left[1-n^2\lambda_2 + \gamma\left(\alpha_0 - n\lambda_3\beta_0\right)\right]^2 + \left[n\lambda_1 - \gamma\left(\beta_0 - \alpha_0 n\lambda_3\right)\right]^2},$$

and

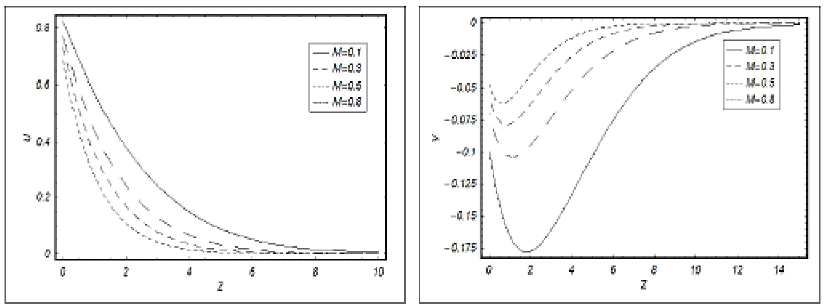
$$\alpha_0 = \frac{1}{\sqrt{2}}\left[S_0 + \sqrt{S_0^2 + 1}\right]^{1/2}, \ \beta_0 = \frac{1}{\sqrt{2}}\left[-S_0 + \sqrt{S_0^2 + 1}\right]^{1/2},$$

$$\Psi_0 = \left(\frac{nM^2\left[n\lambda_3\left(1-n^2\lambda_2\right) - n\lambda_1\right)\right]}{\left(1+n^2\lambda_3^2\right)}\right)^{1/2} z,$$

$$S_0 = \frac{\left(1-n^2\lambda_2\right) + \lambda_1\lambda_3 n^2}{n\lambda_3\left(1-n^2\lambda_2\right) - n\lambda_1\right)},$$

5. Graphical results and discussion

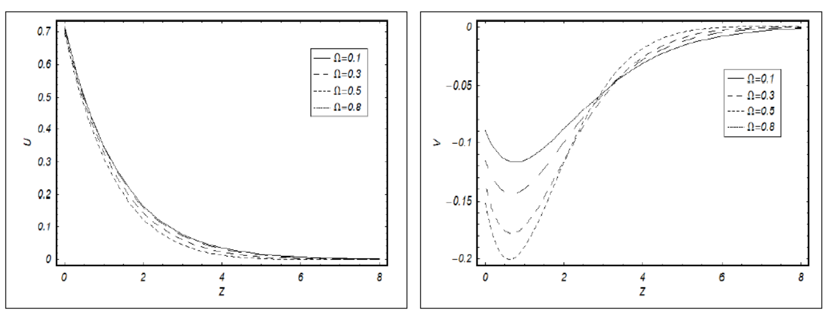
Here we are interseted to explain the influences of Hartmann number M, rotation Ω and slip γ parameters on both real and imagniary parts of the velocity profiles. Figs. 1 (a) and 3 (b) have been plotted. In order to get such purpose specially, Figs. 1 (a) and 1 (b) shows the effect of Hartmann number M on u and v. It is observed that the u and v components of velocity decreases by increasing the Hartmann parameter M. as excepted the magnetic force provides resistence to the flow that is why velocity profile decreases.



(a) (b)

FIGURE 1. The variation of M on u for general periodic oscillations when $\omega_0 = 0.2, \gamma = 0.5, t = 1, \Omega = 0.1, \lambda_1 = 2, \lambda_2 = 1$ and $\lambda_3 = 1$.

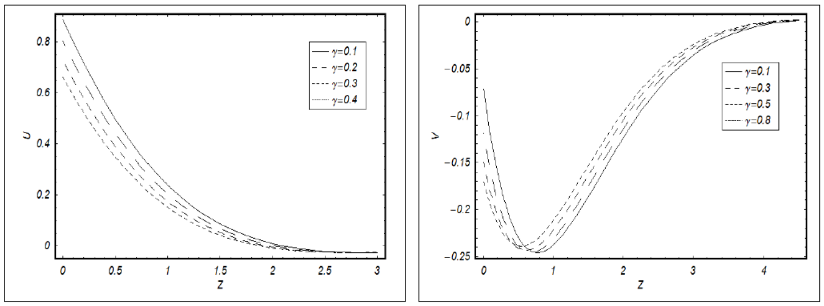
Figs. 2 (a) and 2 (b) has been sketched for the variation of rotation parameter for general periodic oscillations. Here we found that the u component of the velocity decreases near the plate when there is an increase in rotation parameter Ω. However v component of velocity increases with the increase in rotation Ω .



(a) (b)

FIGURE 2. The variation of Ω on u for general periodic oscillations when $\omega_0 = 0.2, \gamma = 0.5, t = 1, M = 0.1, \lambda_1 = 2, \lambda_2 = 1$ and $\lambda_3 = 1$.

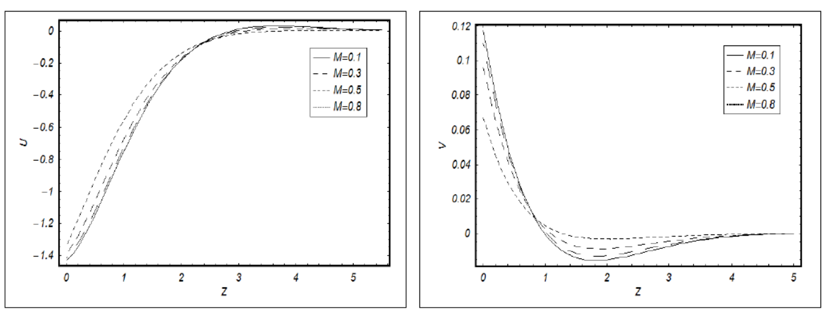
The variation of slip γ on u and v is depicted in the Figs. 3 (a) and 3 (b). These Figs. represent that u and associated boundary layer thickness decrease when γ increases. However v increases near the wall and decreases away from the wall.



(a) (b)

FIGURE 3. The variation of γ on u for general periodic oscillations when $\omega_0 = 0.2, \gamma = 0.5, t = 1, \Omega = 0.1, \lambda_1 = 2, \lambda_2 = 1$ and $\lambda_3 = 1$.

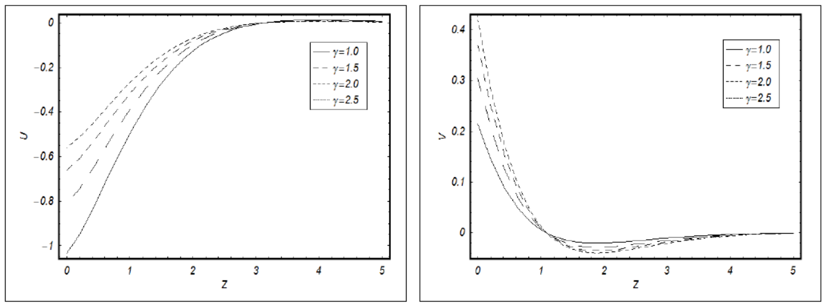
The effects of Hartmann number M, rotation Ω and slip γ on the velocity components u and v in case of elliptic harmonic oscillations are plotted in the Figs. $4\,(a) - 12\,(b)$. In these Figs. the flow has been discussed for non-resonant $(n \neq 2\Omega)$ and resonant $(n = 2\Omega)$ cases. For non-resonant case we have the situations for $n < 2\Omega$ and $n > 2\Omega$. Figs. $4\,(a) - 6\,(b)$ corresponds to the flow for $n < 2\Omega$. It is noticed from Fig. 4 u and v decreases near the plate while these increase away from the plate.



(a) (b)

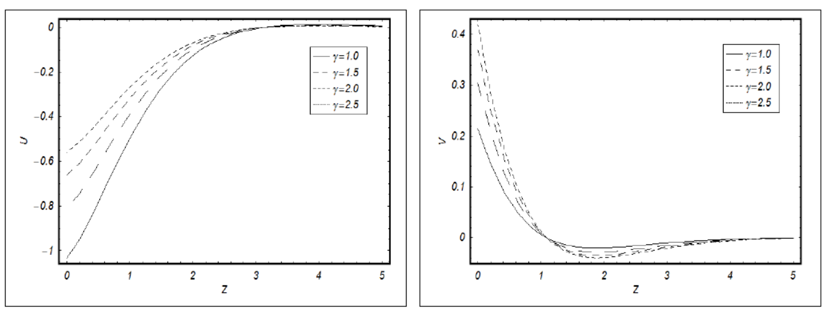
FIGURE 4. The variation of M on u for general periodic oscillations when $\omega_0 = 0.2, \gamma = 0.5, t = 1, \Omega = 0.7, \lambda_1 = 2, \lambda_2 = 1$, $\lambda_3 = 1.$ and $n < 2\Omega$.

When magnetic parameter increases Fig. 5 elucidates the effect of rotation Ω on u and v Both velocity components u and v and associated layer thickness are decreasing function of Ω. Fig. 6 characterizes the variation of slip parameter γ on the flow field.



(a) (b)

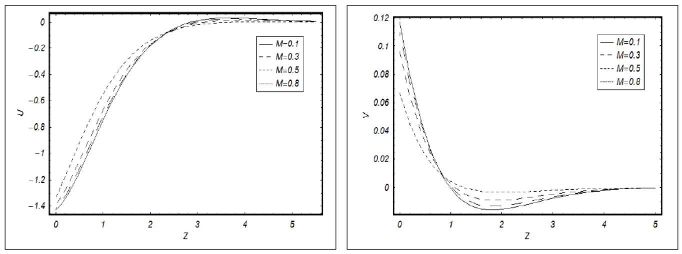
FIGURE 5. The variation of Ω on u for general periodic oscillations when $\omega_0 = 0.2, \gamma = 0.5, t = 1, M = 0.5, \lambda_1 = 2, \lambda_2 = 1,$ $\lambda_3 = 1.$ and $n < 2\Omega$.



(a) (b)

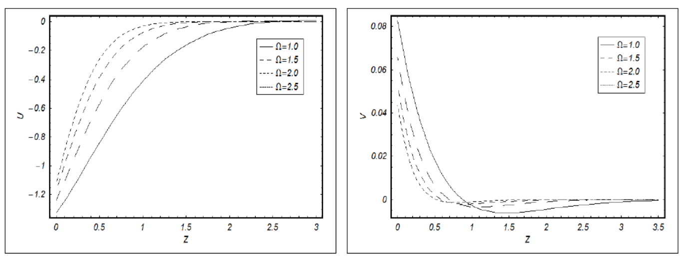
FIGURE 6. The variation of γ on u for general periodic oscillations when $\omega_0 = 0.2, \Omega = 0.5, t = 1, M = 0.5, \lambda_1 = 2, \lambda_2 = 1,$ $\lambda_3 = 1.$ and $n < 2\Omega$.

Here u decreases and v increases near the plate and reverse behavior is observed away from the wall when γ increases. Fig. $7\,(a) - 9\,(b)$ are sketched to see the influence of Hartmann number M, rotation parameter Ω and slip parameter γ on u and v when $n > 2\Omega$. Fig. 7 is plotted to see the influence of Hartmann number M on velocity profiles. It is observed that u and v components decrease near the wall while far away from the plate reverse behavior is noted within the increase of Hartmann number M. Fig. 8 is drawn to see the effect of rotation parameter on both u and v components of the velocity. Obviously the rotation Ω increases u component of velocity far away while v component decreases near the plate and it increases near the plate. Effects of slip parameter γ on velocity profile u and v are seen in Fig. 9.



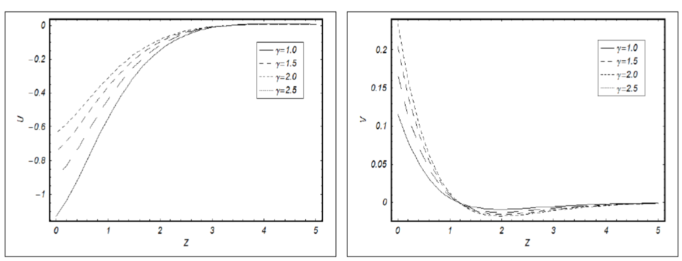
(a) (b)

FIGURE 7. The variation of M on u for general periodic oscillations when $\omega_0 = 0.1, \gamma = 0.7, t = 1, \Omega = 0.5, \lambda_1 = 2, \lambda_2 = 1,$ $\lambda_3 = 1.$ and $n > 2\Omega$.



(a) (b)

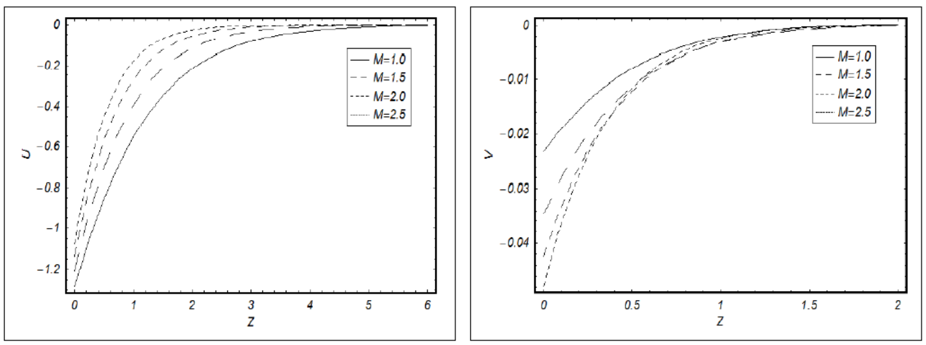
FIGURE 8. The variation of Ω on u for general periodic oscillations when $\omega_0 = 0.1, \gamma = 0.7, t = 1, M = 0.5, \lambda_1 = 2, \lambda_2 = 1,$ $\lambda_3 = 1.$ and $n > 2\Omega$.



(a) (b)

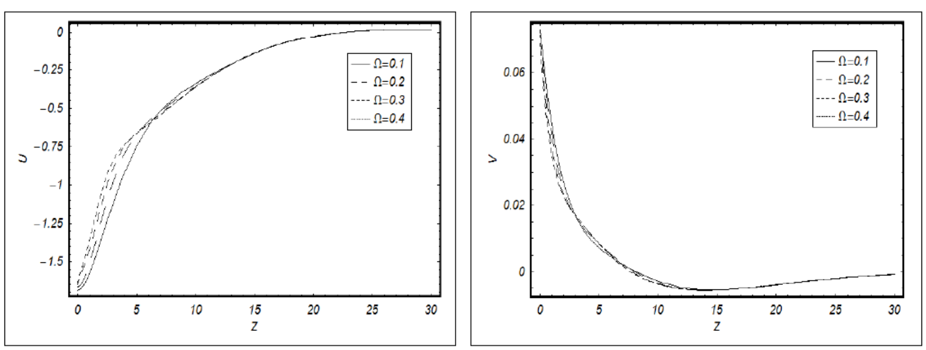
FIGURE 9. The variation of γ on u for general periodic oscillations when $\omega_0 = 0.1, \gamma = 0.7, t = 1, \Omega = 0.5, \lambda_1 = 2, \lambda_2 = 1,$ $\lambda_3 = 1.$ and $n > 2\Omega$.

It is noted that the reverse behavior of γ is observed on both u and v components of velocity profiles. Fig. $10\,(a) - 12\,(b)$ are sketched for velocities in resonant case ($n = 2\Omega$). From Fig. 10 it is noticed that u and v show reverse behaviour with an increase in Hartmann number M. However via Ω, u and v both decreases near the boundary (see Fig. 11). For an increase in γ there is decrease in u and an increase in v. (see Fig. 12 (a) and 12 (b)).



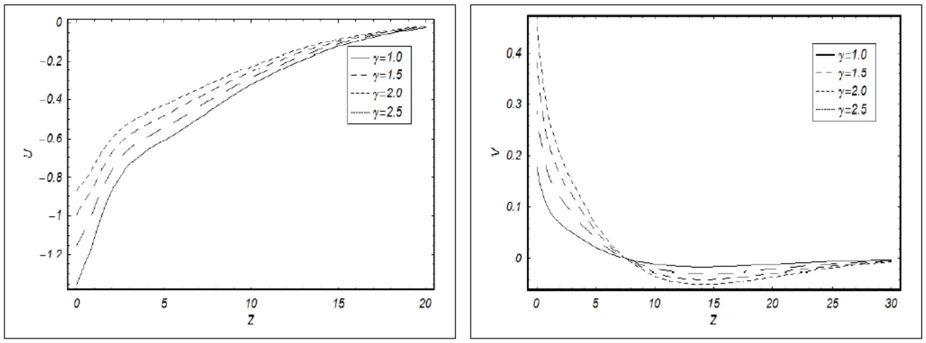
(a) (b)

FIGURE 10. The variation of M on u for general periodic oscillations when $\omega_0 = 0.2, \gamma = 0.7, t = 1, \Omega = 0.3, \lambda_1 = 2, \lambda_2 = 1$, $\lambda_3 = 1.$ and $n = 2\Omega$.



(a) (b)

FIGURE 11. The variation of Ω on u for general periodic oscillations when $\omega_0 = 0.2, \gamma = 0.7, t = 1, M = 0.3, \lambda_1 = 2, \lambda_2 = 1$, $\lambda_3 = 1.$ and $n = 2\Omega$.



(a) (b)

FIGURE 12. The variation of γ on u for general periodic oscillations when $\omega_0 = 0.2, M = 0.5, t = 1, \Omega = 0.3, \lambda_1 = 2, \lambda_2 = 1,$ $\lambda_3 = 1.$ and $n = 2\Omega$.

6. Conclusion

The exact solutions for rotating flows of an Burgers' fluid are constructed in the presence of slip condition. From the performed analysis, the following observations have been noted.

(1) It is noted that consideration of angular velocity generates oscillatory character in the flow.
(2) The layer thickness is decreasing function of applied magnetic field.
(3) Variation of slip parameter on the $x-$ component of velocity show decreasing behavior in general periodic oscillation case.
(4) Meaningful solutions exist in both resonant and non-resonant cases when fluid is magnetohydrodynamic.
(5) The results corresponding to no-slip condition are recovered when $\gamma = 0$.

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