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FIXED POINT THEOREMS FOR SOME CONTRACTIVE MAPPINGS OF INTEGRAL TYPE WITH w-DISTANCE[†]

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Abstract. The existence, uniqueness and iterative approximations of fixed points for some contractive mappings of integral type defined in complete metric spaces with w-distance are proved. Four examples are given to demonstrate that the results in this paper extend and improve some well-known results in the literature.

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1. Introduction

Throughout this paper, we assume that $\mathbb{R} = (-\infty, +\infty), \mathbb{R}^+ = [0, +\infty)$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers and

$\Phi_1 = \{\varphi \mid \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies that φ is Lebesgue integrable, summable on each compact subset of $\mathbb{R}^+$ and $\int_0^\varepsilon \varphi(t)dt > 0$, $\forall \varepsilon > 0\}$;

$\Phi_2 = \{\varphi \mid \varphi$ belongs to Φ_1 and satisfies that $\int_0^u \varphi(t)dt < \int_0^v \varphi(t)dt$, $\forall u, v \in \mathbb{R}^+$ with $u < v\}$;

$\Phi_3 = \{\varphi \mid \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is lower semicontinuous and $\varphi^{-1}(0) = \{0\}\}$;

$\Phi_4 = \{\varphi \mid \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0\}$.

In 2001, Rhoades [17] proved the following fixed point theorem, which is a generalization of the Banach fixed point theorem.

Theorem 1.1. [17] *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) \le d(x, y) - \psi(d(x, y)), \quad \forall x, y \in X, \tag{1.1}$$

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where $\psi \in \Phi_4$. Then T has a unique fixed point.

In 2002, Branciari [1] gave an interesting integral version of the Banach fixed point theorem by introducing the concept of contractive mapping of integral type and discussed the existence of fixed points for the following contractive mapping of integral type in complete metric spaces.

Theorem 1.2. [1] *Let T be a mapping from a complete metric space (X, d) into itself satisfying*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \le c \int_0^{d(x,y)} \varphi(t)dt, \quad \forall x, y \in X, \tag{1.2}$$

where $c \in (0, 1)$ is a constant and $\varphi \in \Phi_1$. Then T has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} T^n x = a$ for each $x \in X$.

The existence of fixed points for various contractive mappings of integral type has been researched by many authors under different conditions, see, for example, [1, 7, 12–16] and the references cited therein. In 2015, Liu et al. [15] established two fixed point theorems for the contractive mappings of integral type.

Theorem 1.3. [15] *Let f be a mapping from a complete metric space (X, d) into itself satisfying*

$$\int_0^{d(fx,fy)} \varphi(t)dt \le \int_0^{d(x,y)} \varphi(t)dt - \psi(d(x, y)), \quad \forall x, y \in X, \tag{1.3}$$

where $(\varphi, \psi) \in \Phi_1 \times \Phi_3$. Then f has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} f^n x = a$ for each $x \in X$.

Theorem 1.4. [15] *Let f be a mapping from a complete metric space (X, d) into itself satisfying*

$$\int_0^{d(fx,fy)} \varphi(t)dt \le \int_0^{d(x,y)} \varphi(t)dt - \psi(d(fx, fy)), \quad \forall x, y \in X, \tag{1.4}$$

where $(\varphi, \psi) \in \Phi_1 \times \Phi_3$. Then f has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} f^n x = a$ for each $x \in X$.

In 1996, Kada et al. [9] introduced the concept of w-distance in metric spaces and proved some fixed point theorems for some contractive mappings under w-distance. The results in [9] are generalizations the results of Caristi [2], Ekeland [4] and Takahashi [18]. Afterwards, the researchers in [3, 5, 6, 8, 10, 11] obtained the existence of fixed points for some contractive mappings with w-distance.

The aim of this paper is to prove the existence, uniqueness and iterative approximations of fixed points for some contractive conditions of integral type with w-distance in complete metric spaces. The results presented in this paper generalize Theorems 1.1-1.4. Four illustrative examples are constructed.

2. Preliminaries

Recall that a self mapping f in a metric space X is called orbitally continuous if $\lim_{n\to\infty} f^n x = u$ implies $\lim_{n\to\infty} f^{n+1} x = fu$ for each $\{f^n x\}_{n\in\mathbb{N}_0} \subseteq X$ and $u \in X$.

Definition 2.1. [9] Let (X, d) be a metric space. A function $p : X \times X \to \mathbb{R}^+$ is called a *w-distance* in X if it satisfies the following

(p_1) $p(x, z) \le p(x, y) + p(y, z), \forall x, y, z \in X$;

(p_2) for each $x \in X$, a mapping $p(x, \cdot) : X \to \mathbb{R}^+$ is lower semi-continuous, that is, if $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in X with $\lim_{n\to\infty} y_n = y \in X$, then $p(x, y) \le \liminf_{n\to\infty} p(x, y_n)$;

(p_3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \le \delta$ and $p(z, y) \le \delta$ imply $d(x, y) \le \varepsilon$.

Lemma 2.2. [9] *Let X be a metric space with metric d and let p be a w-distance in X. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be sequences in X, let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}^+$ converging to 0, and let $x, y, z \in X$. Then the following hold:*

(a) If $p(x_n, y) \le \alpha_n$ and $p(x_n, z) \le \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$;

(b) If $p(x_n, y_n) \le \alpha_n$ and $p(x_n, z) \le \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}_{n\in\mathbb{N}}$ converges to z;

(c) If $p(x_n, x_m) \le \alpha_n$ for any $n, m \in \mathbb{N}$ with $n > m$, then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence;

(d) If $p(x, x_n) \le \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Lemma 2.3. [14] *Let $\varphi \in \Phi_1$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence with $\lim_{n\to\infty} r_n = a$. Then*

$$\lim_{n\to\infty} \int_0^{r_n} \varphi(t)dt = \int_0^a \varphi(t)dt.$$

Lemma 2.4. [14] *Let $\varphi \in \Phi_1$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence. Then $\lim_{n\to\infty} \int_0^{r_n} \varphi(t)dt = 0$ if and only if $\lim_{n\to\infty} r_n = 0$.*

Remark 2.1. (1.1) is a special case of (1.3).

In fact, put $\varphi(t) = 1$ for all $t \in \mathbb{R}^+$. It follows from $\Phi_4 \subseteq \Phi_3$ that (1.3) yields (1.1).

Remark 2.2. (1.2) is a special case of (1.3).

In fact, put $\psi(s) = (1 - c)\int_0^s \varphi(t)dt$ for all $s \in \mathbb{R}^+$, where $c \in (0, 1)$ is a constant and $\varphi \in \Phi_1$. It follows from Lemma 2.3 and $\varphi \in \Phi_1$ that $\psi \in \Phi_3$ and (1.3) reduces to (1.2).

3. Fixed point theorems for contractive mappings with w-distance

In this section we establish four fixed point theorems for contractive mappings (3.1), (3.19), (3.32) and (3.33) below.

Theorem 3.1. *Let (X,d) be a complete metric space and let p be a w-distance in X. Assume that $f : X \to X$ satisfies that*

$$\int_0^{p(fx,fy)} \varphi(t)dt \le \int_0^{p(x,y)} \varphi(t)dt - \psi(p(x,y)), \quad \forall x, y \in X, \tag{3.1}$$

where $(\varphi, \psi) \in \Phi_1 \times \Phi_3$. Then f has a unique fixed point $u \in X$ such that $p(u,u) = 0$, $\lim_{n\to\infty} p(f^n x_0, u) = 0$ and $\lim_{n\to\infty} f^n x_0 = u$ for each $x_0 \in X$.

Proof. Firstly we show that f has a fixed point $u \in X$. Pick an arbitrary point x_0 in X and put $x_n = f^n x_0$ for each $n \in \mathbb{N}_0$. Now we consider two cases as follows:

Case 1. $x_{n_0} = x_{n_0-1}$ for some $n_0 \in \mathbb{N}$. It follows that x_{n_0-1} is a fixed point of f and $\lim_{n\to\infty} f^n x_0 = x_{n_0-1}$. Suppose that $p(x_{n_0-1}, x_{n_0-1}) > 0$. Making use of (3.1) and $(\varphi, \psi) \in \Phi_1 \times \Phi_3$, we conclude immediately that

$$\begin{aligned}\int_0^{p(x_{n_0-1},x_{n_0-1})} \varphi(t)dt &= \int_0^{p(fx_{n_0-1},fx_{n_0-1})} \varphi(t)dt \\ &\le \int_0^{p(x_{n_0-1},x_{n_0-1})} \varphi(t)dt - \psi(p(x_{n_0-1}, x_{n_0-1})) \\ &< \int_0^{p(x_{n_0-1},x_{n_0-1})} \varphi(t)dt,\end{aligned}$$

which is a contradiction. Hence $p(x_{n_0-1}, x_{n_0-1}) = 0$, which means that

$$\lim_{n\to\infty} p(f^n x_0, x_{n_0-1}) = p(x_{n_0-1}, x_{n_0-1}) = 0;$$

Case 2. $x_n \ne x_{n-1}$ for all $n \in \mathbb{N}$. Suppose that

$$p(x_{n_0-1}, x_{n_0}) = 0 \quad \text{for some} \quad n_0 \in \mathbb{N}. \tag{3.2}$$

It follows from (3.1), (3.2) and $(\varphi, \psi) \in \Phi_1 \times \Phi_3$ that

$$\begin{aligned} 0 &\le \int_0^{p(x_{n_0},x_{n_0+1})} \varphi(t)dt = \int_0^{p(fx_{n_0-1},fx_{n_0})} \varphi(t)dt \\ &\le \int_0^{p(x_{n_0-1},x_{n_0})} \varphi(t)dt - \psi(p(x_{n_0-1}, x_{n_0})) = 0,\end{aligned}$$

which yields that

$$\int_0^{p(x_{n_0},x_{n_0+1})} \varphi(t)dt = 0,$$

which together with $\varphi \in \Phi_1$ gives that

$$p(x_{n_0}, x_{n_0+1}) = 0. \tag{3.3}$$

Combining (3.2), (3.3) and (p_1), we know that

$$0 \le p(x_{n_0-1}, x_{n_0+1}) \le p(x_{n_0-1}, x_{n_0}) + p(x_{n_0}, x_{n_0+1}) = 0,$$

that is,

$$p(x_{n_0-1}, x_{n_0+1}) = 0. \tag{3.4}$$

By virtue of (3.2), (3.4) and Lemma 2.2, we deduce that $x_{n_0} = x_{n_0+1}$, which is a contradiction and hence

$$p(x_{n-1}, x_n) > 0, \quad \forall n \in \mathbb{N}. \tag{3.5}$$

In light of (3.1), (3.5) and $(\varphi, \psi) \in \Phi_1 \times \Phi_3$, we conclude that

$$\begin{aligned}
&\int_0^{p(x_n,x_{n+1})} \varphi(t)dt = \int_0^{p(fx_{n-1},fx_n)} \varphi(t)dt \\
&\le \int_0^{p(x_{n-1},x_n)} \varphi(t)dt - \psi(p(x_{n-1}, x_n)) < \int_0^{p(x_{n-1},x_n)} \varphi(t)dt, \quad \forall n \in \mathbb{N}.
\end{aligned}$$

By means of (3.5) and $\varphi \in \Phi_1$, we get that

$$0 < p(x_n, x_{n+1}) < p(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}. \tag{3.6}$$

It follows from (3.6) that $\{p(x_n, x_{n+1})\}_{n\in\mathbb{N}_0}$ is a positive and strictly decreasing sequence. Hence there exists a constant $v \ge 0$ with

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = v. \tag{3.7}$$

Now we claim $v = 0$. Suppose that $v > 0$. In view of (3.1), (3.7), $(\varphi, \psi) \in \Phi_1 \times \Phi_3$ and Lemma 2.3, we infer that

$$\begin{aligned}
\int_0^{v} \varphi(t)dt &= \limsup_{n\to\infty} \int_0^{p(x_n,x_{n+1})} \varphi(t)dt = \limsup_{n\to\infty} \int_0^{p(fx_{n-1},fx_n)} \varphi(t)dt \\
&\le \limsup_{n\to\infty} \left(\int_0^{p(x_{n-1},x_n)} \varphi(t)dt - \psi(p(x_{n-1}, x_n)) \right) \\
&\le \limsup_{n\to\infty} \int_0^{p(x_{n-1},x_n)} \varphi(t)dt - \liminf_{n\to\infty} \psi(p(x_{n-1}, x_n)) \\
&\le \int_0^{v} \varphi(t)dt - \psi(v) < \int_0^{v} \varphi(t)dt,
\end{aligned}$$

which is ridiculous. Therefore, $v = 0$ and hence

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{3.8}$$

In a similar manner, we find that

$$\lim_{n\to\infty} p(x_{n+1}, x_n) = 0. \tag{3.9}$$

Now we claim that

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0. \tag{3.10}$$

Otherwise there exists a constant $\varepsilon > 0$ such that for each positive integer k, there are subsequences $\{m(k)\}_{k\in\mathbb{N}}$, $\{n(k)\}_{k\in\mathbb{N}}$ such that

$$m(k) > n(k) > k, \quad p(x_{n(k)}, x_{m(k)}) > \varepsilon, \quad \forall k \in \mathbb{N}. \tag{3.11}$$

For each positive integer k, let $m(k)$ denote the least integer exceeding $n(k)$ and satisfying (3.11). It is clear that

$$p(x_{n(k)}, x_{m(k)}) > \varepsilon \quad \text{and} \quad p(x_{n(k)}, x_{m(k)-1}) \le \varepsilon, \quad \forall k \in \mathbb{N}. \tag{3.12}$$

On account of (p_1) and (3.12), we get that, $\forall k \in \mathbb{N}$

$$\begin{aligned} \varepsilon &< p(x_{n(k)}, x_{m(k)}) \\ &\le p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{m(k)}) \\ &\le p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)-1}) \\ &\quad + p(x_{m(k)-1}, x_{m(k)}) \\ &\le p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{n(k)}) + \varepsilon + p(x_{m(k)-1}, x_{m(k)}). \end{aligned} \tag{3.13}$$

Letting $k \to \infty$ in (3.13) and using (3.8), (3.9) and (3.12), we know that

$$\lim_{k\to\infty} p(x_{n(k)}, x_{m(k)}) = \lim_{k\to\infty} p(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon. \tag{3.14}$$

In light of (3.1), (3.14), $(\varphi, \psi) \in \Phi_1 \times \Phi_3$ and Lemma 2.3, we conclude that

$$\begin{aligned} \int_0^{\varepsilon} \varphi(t)dt &= \limsup_{k\to\infty} \int_0^{p(x_{n(k)}, x_{m(k)})} \varphi(t)dt = \limsup_{k\to\infty} \int_0^{p(fx_{n(k)-1}, fx_{m(k)-1})} \varphi(t)dt \\ &\le \limsup_{k\to\infty} \left(\int_0^{p(x_{n(k)-1}, x_{m(k)-1})} \varphi(t)dt - \psi(p(x_{n(k)-1}, x_{m(k)-1})) \right) \\ &\le \limsup_{k\to\infty} \int_0^{p(x_{n(k)-1}, x_{m(k)-1})} \varphi(t)dt - \liminf_{k\to\infty} \psi(p(x_{n(k)-1}, x_{m(k)-1})) \\ &\le \int_0^{\varepsilon} \varphi(t)dt - \psi(\varepsilon) < \int_0^{\varepsilon} \varphi(t)dt, \end{aligned}$$

which is absurd. Hence (3.10) holds.

Given $\varepsilon > 0$ and δ denotes the number appearing in (p_3). It follows from (3.10) that there exists $N \in \mathbb{N}$ satisfying

$$p(x_N, x_n) < \delta, \quad p(x_N, x_m) < \delta, \quad \forall n, m > N,$$

which together with (p_3) ensures that

$$d(x_n, x_m) \le \varepsilon, \quad \forall n, m > N.$$

Hence $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. By completeness of X, there exists a point $u \in X$ satisfying

$$\lim_{n\to\infty} x_n = u. \tag{3.15}$$

It follows from (3.10) that for each $\varepsilon > 0$ there exists $M \in \mathbb{N}$ satisfying

$$p(x_n, x_m) < \varepsilon, \quad \forall m > n > M,$$

which together with (p_2) and (3.15) yields that

$$0 \le p(x_n, u) \le \liminf_{m\to\infty} p(x_n, x_m) \le \varepsilon, \quad \forall n \ge M,$$

which means that

$$\lim_{n\to\infty} p(x_n, u) = 0. \tag{3.16}$$

By means of (3.1), (3.16), $(\varphi, \psi) \in \Phi_1 \times \Phi_3$ and Lemma 2.3, we infer that

$$0 \le \int_0^{p(fx_n, fu)} \varphi(t)dt \le \int_0^{p(x_n,u)} \varphi(t)dt - \psi(p(x_n, u)) \to 0 \quad \text{as } n \to \infty,$$

that is,

$$\lim_{n\to\infty} \int_0^{p(fx_n, fu)} \varphi(t)dt = 0,$$

which together with Lemma 2.4 gives that

$$\lim_{n\to\infty} p(fx_n, fu) = \lim_{n\to\infty} p(x_{n+1}, fu) = 0,$$

In light of (p_1) and (3.8), we conclude that

$$0 \le p(x_n, fu) \le p(x_n, x_{n+1}) + p(x_{n+1}, fu) \to 0 \quad \text{as } n \to \infty,$$

that is,

$$\lim_{n\to\infty} p(x_n, fu) = 0. \tag{3.17}$$

Combining (3.16) and (3.17) and using Lemma 2.2, we have $u = fu$.

Secondly we show that $p(u, u) = 0$. Suppose that $p(u, u) > 0$. In view of (3.1) and $(\varphi, \psi) \in \Phi_1 \times \Phi_3$, we deduce that

$$\begin{aligned} 0 &< \int_0^{p(u,u)} \varphi(t)dt = \int_0^{p(fu,fu)} \varphi(t)dt \\ &\le \int_0^{p(u,u)} \varphi(t)dt - \psi(p(u, u)) < \int_0^{p(u,u)} \varphi(t)dt, \end{aligned} \tag{3.18}$$

which is a contradiction. Hence $p(u, u) = 0$.

Thirdly we show that f has a unique fixed point in X. Suppose that u and v are two fixed points of f in X. Similar to the proof of (3.18), we conclude that $p(u, u) = p(v, v) = 0$. Suppose that $p(u, v) > 0$. On account of (3.1) and $(\varphi, \psi) \in \Phi_1 \times \Phi_3$, we get that

$$\begin{aligned} 0 &< \int_0^{p(u,v)} \varphi(t)dt = \int_0^{p(fu,fv)} \varphi(t)dt \\ &\le \int_0^{p(u,v)} \varphi(t)dt - \psi(p(u, v)) < \int_0^{p(u,v)} \varphi(t)dt, \end{aligned}$$

which is ridiculous. Consequently $p(u, v) = 0$, which together with $p(u, u) = 0$ and Lemma 2.2 that $u = v$. This completes the proof. □

Theorem 3.2. *Let (X, d) be a complete metric space and let p be a w-distance in X. Assume that $f : X \to X$ satisfies that*

$$\int_0^{p(fx,fy)} \varphi(t)dt \le \int_0^{p(x,y)} \varphi(t)dt - \psi(p(fx, y)), \quad \forall x, y \in X, \tag{3.19}$$

where $(\varphi,\psi)\in\Phi_2\times\Phi_3$. *Then* f *has a unique fixed point* $u\in X$ *such that* $p(u,u)=0$, $\lim_{n\to\infty}p(f^nx_0,u)=0$ *and* $\lim_{n\to\infty}f^nx_0=u$ *for each* $x_0\in X$.

Proof. Firstly we show that f has a fixed point $u\in X$. Pick an arbitrary point x_0 in X and put $x_n=f^nx_0$ for each $n\in\mathbb{N}_0$. Now we consider two cases as follows:

Case 1. $x_{n_0}=x_{n_0-1}$ for some $n_0\in\mathbb{N}$. It follows that x_{n_0-1} is a fixed point of f and $\lim_{n\to\infty}f^nx_0=x_{n_0-1}$. Suppose that $p(x_{n_0-1},x_{n_0-1})>0$. Making use of (3.19) and $(\varphi,\psi)\in\Phi_2\times\Phi_3$, we conclude immediately that

$$\begin{aligned}&\int_0^{p(x_{n_0-1},x_{n_0-1})}\varphi(t)dt=\int_0^{p(fx_{n_0-1},fx_{n_0-1})}\varphi(t)dt\\&\le\int_0^{p(x_{n_0-1},x_{n_0-1})}\varphi(t)dt-\psi(p(fx_{n_0-1},x_{n_0-1}))\\&=\int_0^{p(x_{n_0-1},x_{n_0-1})}\varphi(t)dt-\psi(p(x_{n_0-1},x_{n_0-1}))<\int_0^{p(x_{n_0-1},x_{n_0-1})}\varphi(t)dt,\end{aligned}$$

which is a contradiction. Hence $p(x_{n_0-1},x_{n_0-1})=0$, which means that

$$\lim_{n\to\infty}p(f^nx_0,x_{n_0-1})=p(x_{n_0-1},x_{n_0-1})=0;$$

Case 2. $x_n\neq x_{n-1}$ for all $n\in\mathbb{N}$. Suppose that (3.2) holds. It follows from (3.2), (3.19) and $(\varphi,\psi)\in\Phi_2\times\Phi_3$ that

$$\begin{aligned}0&\le\int_0^{p(x_{n_0},x_{n_0+1})}\varphi(t)dt=\int_0^{p(fx_{n_0-1},fx_{n_0})}\varphi(t)dt\\&\le\int_0^{p(x_{n_0-1},x_{n_0})}\varphi(t)dt-\psi(p(x_{n_0},x_{n_0}))\le 0,\end{aligned}$$

which yields that

$$\int_0^{p(x_{n_0},x_{n_0+1})}\varphi(t)dt=0,$$

which together with $\varphi\in\Phi_2$ gives (3.3). Combining (3.2), (3.3) and (p_1), we know that

$$0\le p(x_{n_0-1},x_{n_0+1})\le p(x_{n_0-1},x_{n_0})+p(x_{n_0},x_{n_0+1})=0,$$

that is, (3.4) holds. By virtue of (3.2), (3.4) and Lemma 2.2, we deduce that $x_{n_0}=x_{n_0+1}$, which is a contradiction and hence (3.5) holds. Suppose that there exists $q\in\mathbb{N}$ with

$$p(x_q,x_{q+1})>p(x_{q-1},x_q).\tag{3.20}$$

In light of (3.5), (3.19), (3.20) and $(\varphi,\psi)\in\Phi_2\times\Phi_3$, we conclude that

$$\begin{aligned}0&<\int_0^{p(x_{q-1},x_q)}\varphi(t)dt<\int_0^{p(x_q,x_{q+1})}\varphi(t)dt=\int_0^{p(fx_{q-1},fx_q)}\varphi(t)dt\\&\le\int_0^{p(x_{q-1},x_q)}\varphi(t)dt-\psi(p(x_q,x_q))\le\int_0^{p(x_{q-1},x_q)}\varphi(t)dt,\end{aligned}$$

which is a contradiction. By means of (3.5), we get that

$$0 < p(x_n, x_{n+1}) \le p(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}. \tag{3.21}$$

It follows from (3.21) that the sequence $\{p(x_n, x_{n+1})\}_{n\in\mathbb{N}_0}$ is positive and decreasing. Hence there exists a constant $v \ge 0$ with (3.7). Suppose that there exists $j \in \mathbb{N}$ with

$$p(x_j, x_{j+2}) > p(x_{j-1}, x_{j+1}). \tag{3.22}$$

In light of (3.19), (3.22) and $(\varphi, \psi) \in \Phi_2 \times \Phi_3$, we conclude that

$$\begin{aligned} 0 &\le \int_0^{p(x_{j-1},x_{j+1})} \varphi(t)dt < \int_0^{p(x_j,x_{j+2})} \varphi(t)dt = \int_0^{p(fx_{j-1},fx_{j+1})} \varphi(t)dt \\ &\le \int_0^{p(x_{j-1},x_{j+1})} \varphi(t)dt - \psi(p(x_j, x_{j+1})) \le \int_0^{p(x_{j-1},x_{j+1})} \varphi(t)dt, \end{aligned}$$

which is a contradiction. Hence

$$0 \le p(x_n, x_{n+2}) \le p(x_{n-1}, x_{n+1}), \quad \forall n \in \mathbb{N}. \tag{3.23}$$

It follows from (3.23) that the sequence $\{p(x_n, x_{n+2})\}_{n\in\mathbb{N}_0}$ is nonnegative and nonincreasing. Hence there exists a constant $b \ge 0$ with

$$\lim_{n\to\infty} p(x_n, x_{n+2}) = b. \tag{3.24}$$

Suppose that $v > 0$. By virtue of (3.7), (3.19), (3.24), $(\varphi, \psi) \in \Phi_2 \times \Phi_3$ and Lemma 2.3, we gain that

$$\begin{aligned} 0 &\le \int_0^b \varphi(t)dt = \limsup_{n\to\infty} \int_0^{p(x_n,x_{n+2})} \varphi(t)dt = \limsup_{n\to\infty} \int_0^{p(fx_{n-1},fx_{n+1})} \varphi(t)dt \\ &\le \limsup_{n\to\infty} \left(\int_0^{p(x_{n-1},x_{n+1})} \varphi(t)dt - \psi(p(x_n, x_{n+1})) \right) \\ &\le \limsup_{n\to\infty} \int_0^{p(x_{n-1},x_{n+1})} \varphi(t)dt - \liminf_{n\to\infty} \psi(p(x_n, x_{n+1})) \\ &\le \int_0^b \varphi(t)dt - \psi(v) < \int_0^b \varphi(t)dt, \end{aligned}$$

which is absurd and hence (3.8) holds. In a similar manner, we find that there exists a constant $c \ge 0$ with

$$\lim_{n\to\infty} p(x_{n+1}, x_n) = c. \tag{3.25}$$

Suppose that $c > 0$. Put $\limsup_{n\to\infty} p(x_n, x_n) = w$. It follows that there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}_0}$ satisfying

$$\lim_{k\to\infty} p(x_{n_k}, x_{n_k}) = w. \tag{3.26}$$

Using (3.19) and $(\varphi, \psi) \in \Phi_2 \times \Phi_3$, we deduce that

$$\begin{aligned} 0 &\le \int_0^{p(x_n, x_{n+1})} \varphi(t)dt = \int_0^{p(fx_{n-1}, fx_n)} \varphi(t)dt \\ &\le \int_0^{p(x_{n-1}, x_n)} \varphi(t)dt - \psi(p(x_n, x_n)) \le \int_0^{p(x_{n-1}, x_n)} \varphi(t)dt, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{3.27}$$

Letting $n \to \infty$ in (3.27) and using (3.8) and Lemma 2.4, we know that

$$\lim_{n\to\infty} \left(\int_0^{p(x_{n-1}, x_n)} \varphi(t)dt - \psi(p(x_n, x_n)) \right) = 0,$$

which gives that

$$\begin{aligned} &\lim_{n\to\infty} \psi(p(x_n, x_n)) \\ &= \lim_{n\to\infty} \int_0^{p(x_{n-1}, x_n)} \varphi(t)dt - \lim_{n\to\infty} \left(\int_0^{p(x_{n-1}, x_n)} \varphi(t)dt - \psi(p(x_n, x_n)) \right) \\ &= 0. \end{aligned} \tag{3.28}$$

In light of (3.8), (3.19), (3.26)-(3.28), $(\varphi, \psi) \in \Phi_2 \times \Phi_3$ and Lemma 2.4, we deduce that

$$\begin{aligned} 0 &= \limsup_{k\to\infty} \int_0^{p(x_{n_k}, x_{n_k+1})} \varphi(t)dt = \limsup_{k\to\infty} \int_0^{p(fx_{n_k-1}, fx_{n_k})} \varphi(t)dt \\ &\le \limsup_{k\to\infty} \left(\int_0^{p(x_{n_k-1}, x_{n_k})} \varphi(t)dt - \psi(p(x_{n_k}, x_{n_k})) \right) \\ &\le \limsup_{k\to\infty} \int_0^{p(x_{n_k-1}, x_{n_k})} \varphi(t)dt - \liminf_{k\to\infty} \psi(p(x_{n_k}, x_{n_k})) \le 0 - \psi(w), \end{aligned}$$

which together with $\psi \in \Phi_3$ yields that $\psi(w) = 0$, that is, $w = 0$. Note that p is nonnegative. It follows that

$$\lim_{n\to\infty} p(x_n, x_n) = 0. \tag{3.29}$$

In view of (3.19), (3.25), (3.29), $(\varphi, \psi) \in \Phi_2 \times \Phi_3$ and Lemma 2.4, we get that

$$\begin{aligned} 0 &= \limsup_{n\to\infty} \int_0^{p(x_{n+1}, x_{n+1})} \varphi(t)dt = \limsup_{n\to\infty} \int_0^{p(fx_n, fx_n)} \varphi(t)dt \\ &\le \limsup_{n\to\infty} \left(\int_0^{p(x_n, x_n)} \varphi(t)dt - \psi(p(x_{n+1}, x_n)) \right) \\ &\le \limsup_{n\to\infty} \int_0^{p(x_n, x_n)} \varphi(t)dt - \liminf_{n\to\infty} \psi(p(x_{n+1}, x_n)) \le 0 - \psi(c), \end{aligned}$$

which means that $\psi(c) \le 0$. It follows from $\psi \in \Phi_3$ that $\psi(c) = 0$ and $c = 0$.

Now we assert that (3.10) holds. Otherwise there exists a constant $\varepsilon > 0$ such that for each positive integer k, there are subsequences $\{m(k)\}_{k\in\mathbb{N}}$, $\{n(k)\}_{k\in\mathbb{N}}$

satisfying (3.11)-(3.13). Letting $k \to \infty$ in (3.13) and using (3.8), (3.9) and (3.12), we know that

$$\begin{aligned}\lim_{k\to\infty} p(x_{n(k)}, x_{m(k)}) &= \lim_{k\to\infty} p(x_{n(k)-1}, x_{m(k)-1}) \\ &= \lim_{k\to\infty} p(x_{n(k)}, x_{m(k)-1}) = \varepsilon.\end{aligned} \tag{3.30}$$

In light of (3.19), (3.30), $(\varphi, \psi) \in \Phi_2 \times \Phi_3$ and Lemma 2.3, we conclude that

$$\begin{aligned}\int_0^{\varepsilon} \varphi(t)dt &= \limsup_{k\to\infty} \int_0^{p(x_{n(k)},x_{m(k)})} \varphi(t)dt = \limsup_{k\to\infty} \int_0^{p(fx_{n(k)-1},fx_{m(k)-1})} \varphi(t)dt \\ &\le \limsup_{k\to\infty} \left(\int_0^{p(x_{n(k)-1},x_{m(k)-1})} \varphi(t)dt - \psi(p(x_{n(k)}, x_{m(k)-1})) \right) \\ &\le \limsup_{k\to\infty} \int_0^{p(x_{n(k)-1},x_{m(k)-1})} \varphi(t)dt - \liminf_{k\to\infty} \psi(p(x_{n(k)}, x_{m(k)-1})) \\ &\le \int_0^{\varepsilon} \varphi(t)dt - \psi(\varepsilon) < \int_0^{\varepsilon} \varphi(t)dt,\end{aligned}$$

which is absurd. Hence (3.10) holds.

Similar to the proof of Theorem 3.1, we deduce that f has a fixed point $u \in X$.

Secondly we show that $p(u, u) = 0$. Suppose that $p(u, u) > 0$. In view of (3.19) and $(\varphi, \psi) \in \Phi_2 \times \Phi_3$, we deduce that

$$\begin{aligned}0 &< \int_0^{p(u,u)} \varphi(t)dt = \int_0^{p(fu,fu)} \varphi(t)dt \\ &\le \int_0^{p(u,u)} \varphi(t)dt - \psi(p(fu, u)) < \int_0^{p(u,u)} \varphi(t)dt,\end{aligned} \tag{3.31}$$

which is a contradiction. Hence $p(u, u) = 0$.

Thirdly we show that f has a unique fixed point in X. Suppose that u and v are two fixed points of f in X. Similar to the proof of (3.31), we conclude that $p(u, u) = p(v, v) = 0$. Suppose that $p(u, v) > 0$. On account of (3.19) and $(\varphi, \psi) \in \Phi_2 \times \Phi_3$, we get that

$$\begin{aligned}0 &< \int_0^{p(u,v)} \varphi(t)dt = \int_0^{p(fu,fv)} \varphi(t)dt \le \int_0^{p(u,v)} \varphi(t)dt - \psi(p(fu, v)) \\ &= \int_0^{p(u,v)} \varphi(t)dt - \psi(p(u, v)) < \int_0^{p(u,v)} \varphi(t)dt,\end{aligned}$$

which is impossible. Consequently $p(u, v) = 0$, which together with $p(u, u) = 0$ and Lemma 2.2 that $u = v$. This completes the proof. □

Similar to the proofs of Theorems 3.1 and 3.2, we get the following results and omit their proofs.

Theorem 3.3. *Let (X, d) be a complete metric space and let p be a w-distance in X. Assume that $f : X \to X$ is an orbitally continuous mapping and satisfies that*

$$\int_0^{p(fx,fy)} \varphi(t)dt \le \int_0^{\frac{1}{2}[p(x,fx)+p(y,fy)]} \varphi(t)dt - \psi\left(\frac{1}{2}[p(x, fx) + p(y, fy)]\right), \quad \forall x, y \in X, \tag{3.32}$$

where $(\varphi, \psi) \in \Phi_2 \times \Phi_3$. Then f has a unique fixed point $u \in X$ such that $p(u, u) = 0$, $\lim_{n\to\infty} p(f^n x_0, u) = 0$ and $\lim_{n\to\infty} f^n x_0 = u$ for each $x_0 \in X$.

Theorem 3.4. *Let (X, d) be a complete metric space and let p be a w-distance in X. Assume that $f : X \to X$ satisfies that*

$$\int_0^{p(fx,fy)} \varphi(t)dt \le \int_0^{p(x,y)} \varphi(t)dt - \psi(p(fx, fy)), \quad \forall x, y \in X, \tag{3.33}$$

where $(\varphi, \psi) \in \Phi_1 \times \Phi_3$. Then f has a unique fixed point $u \in X$ such that $p(u, u) = 0$, $\lim_{n\to\infty} p(f^n x_0, u) = 0$ and $\lim_{n\to\infty} f^n x_0 = u$ for each $x_0 \in X$.

4. Remarks and illustrative examples

In this section, we construct four nontrivial examples to compare the fixed point theorems obtained in Section 3 with the known results in Section 1.

Remark 4.1. In case $p(x, y) = d(x, y)$ for all $x, y \in X$, then Theorem 3.1 reduces to Theorem 1.3, which extends Theorems 1.1 and 1.2. The following example proves that Theorem 3.1 generalizes indeed Theorems 1.1-1.3.

Example 4.1. Let $X = [0, \sqrt{5}]$ be endowed with the Euclidean metric $d = |\cdot|$, $p : X \times X \to \mathbb{R}^+$, $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ and $f : X \to X$ be defined by

$$p(x, y) = y, \quad \forall x, y \in X, \quad \varphi(t) = 1, \quad \psi(t) = \frac{1}{3}t^2, \quad \forall t \in \mathbb{R}^+$$

and

$$fx = \begin{cases} x - \frac{1}{2}x^2, & \forall x \in [0, 1), \\ 0, & \forall x \in [1, \sqrt{5}]. \end{cases}$$

It is clear that p is a w-distance in X and $(\varphi, \psi) \in \Phi_1 \times \Phi_3$. Let $x, y \in X$. In order to verify (3.1), we have to consider two possible cases as follows:

Case 1. $(x, y) \in X \times [0, 1)$. It is clear that

$$\begin{aligned}\int_0^{p(fx,fy)} \varphi(t)dt &= \int_0^{y-\frac{1}{2}y^2} 1dt = y - \frac{1}{2}y^2 \le y - \frac{1}{3}y^2 \\ &= \int_0^{y} 1dt - \frac{1}{3}y^2 = \int_0^{p(x,y)} \varphi(t)dt - \psi(p(x, y));\end{aligned}$$

Case 2. $(x,y) \in X \times [1,\sqrt{5}]$. Note that

$$\int_0^{p(fx,fy)} \varphi(t)dt = 0 \le y - \frac{1}{3}y^2 = \int_0^{p(x,y)} \varphi(t)dt - \psi(p(x,y)).$$

That is, (3.1) holds. Hence the conditions of Theorem 3.1 are satisfied. It follows from Theorem 3.1 that f has a unique fixed point in X.

However we cannot use Theorem 1.3 to prove the existence of fixed points of the mapping f in X. Otherwise, there exists $(\varphi,\psi) \in \Phi_1 \times \Phi_3$ satisfying (1.3). It follows that

$$\begin{aligned}
0 &< \int_0^{\frac{1}{2}} \varphi(t)dt = \limsup_{y\to 1^-} \int_0^{|0-(y-\frac{1}{2}y^2)|} \varphi(t)dt = \limsup_{y\to 1^-} \int_0^{d(f1,fy)} \varphi(t)dt \\
&\le \limsup_{y\to 1^-} \left(\int_0^{d(1,y)} \varphi(t)dt - \psi\big(d(1,y)\big) \right) \\
&\le \limsup_{y\to 1^-} \int_0^{d(1,y)} \varphi(t)dt - \liminf_{y\to 1^-} \psi(d(1,y)) \le 0 - \psi(0) = 0,
\end{aligned}$$

which is impossible. It follows from Remarks 2.1 and 2.2 that Theorems 1.1 and 1.2 are useless in proving the existence of fixed points of f.

Remark 4.2. The following example shows that Theorem 3.2 differs from Theorem 1.3.

Example 4.2. Let $X = [0,3]$ be endowed with the Euclidean metric $d = |\cdot|$, $p : X \times X \to \mathbb{R}^+$, $\varphi,\psi : \mathbb{R}^+ \to \mathbb{R}^+$ and $f : X \to X$ be defined by

$$p(x,y) = x+y, \quad \forall x,y \in X, \quad \varphi(t) = 2t, \quad \psi(t) = \frac{1}{4}t^2, \quad \forall t \in \mathbb{R}^+$$

and

$$fx = \begin{cases} \frac{x}{2}, & \forall x \in [0,2], \\ \frac{x}{3}, & \forall x \in (2,3]. \end{cases}$$

It is clear that p is a w-distance in X and $(\varphi,\psi) \in \Phi_2 \times \Phi_3$. Let $x,y \in X$. In order to verify (3.19), we have to consider four possible cases as follows:

Case 1. $(x,y) \in [0,2] \times [0,2]$. It is clear that

$$\begin{aligned}
\int_0^{p(fx,fy)} \varphi(t)dt &= \int_0^{\frac{x+y}{2}} 2tdt = \frac{(x+y)^2}{4} \le (x+y)^2 - \frac{1}{4}\left(\frac{x}{2}+y\right)^2 \\
&= \int_0^{p(x,y)} \varphi(t)dt - \psi(p(fx,y));
\end{aligned}$$

Case 2. $(x,y) \in [0,2] \times (2,3]$. Note that

$$\begin{aligned}
\int_0^{p(fx,fy)} \varphi(t)dt &= \int_0^{\frac{3x+2y}{6}} 2tdt = \left(\frac{3x+2y}{6}\right)^2 \le (x+y)^2 - \frac{1}{4}\left(\frac{x}{2}+y\right)^2 \\
&= \int_0^{x+y} 2tdt - \psi\left(\frac{x}{2}+y\right) = \int_0^{p(x,y)} \varphi(t)dt - \psi(p(fx,y));
\end{aligned}$$

Case 3. $(x,y) \in (2,3] \times [0,2]$. It follows that

$$\begin{aligned}\int_0^{p(fx,fy)} \varphi(t)dt &= \int_0^{\frac{2x+3y}{6}} 2tdt = \left(\frac{2x+3y}{6}\right)^2 \le (x+y)^2 - \frac{1}{4}\left(\frac{x}{3}+y\right)^2 \\ &= \int_0^{x+y} 2tdt - \psi\left(\frac{x}{3}+y\right) = \int_0^{p(x,y)} \varphi(t)dt - \psi(p(fx,y));\end{aligned}$$

Case 4. $(x,y) \in (2,3] \times (2,3]$. It is easy to verify that

$$\begin{aligned}\int_0^{p(fx,fy)} \varphi(t)dt &= \int_0^{\frac{x+y}{3}} 2tdt = \left(\frac{x+y}{3}\right)^2 \le (x+y)^2 - \frac{1}{4}\left(\frac{x}{3}+y\right)^2 \\ &= \int_0^{x+y} 2tdt - \psi\left(\frac{x}{3}+y\right) = \int_0^{p(x,y)} \varphi(t)dt - \psi(p(fx,y)).\end{aligned}$$

That is, (3.19) holds. Hence the conditions of Theorem 3.2 are satisfied. It follows from Theorem 3.2 that f has a unique fixed point in X.

However we cannot use Theorem 1.3 to prove the existence of fixed points of the mapping f in X. Otherwise, there exists $(\varphi,\psi) \in \Phi_1 \times \Phi_3$ satisfying (1.3). It follows that

$$\begin{aligned}0 &< \int_0^{\frac{1}{3}} \varphi(t)dt = \limsup_{y\to 2^+} \int_0^{|1-\frac{y}{3}|} \varphi(t)dt = \limsup_{y\to 2^+} \int_0^{d(f2,fy)} \varphi(t)dt \\ &\le \limsup_{y\to 2^+} \left(\int_0^{d(2,y)} \varphi(t)dt - \psi\big(d(2,y)\big)\right) \\ &\le \limsup_{y\to 2^+} \int_0^{d(2,y)} \varphi(t)dt - \liminf_{y\to 2^+} \psi(d(2,y)) \le 0 - \psi(0) = 0,\end{aligned}$$

which is impossible.

Remark 4.3. The example below is an application of Theorem 3.3.

Example 4.3. Let $X = \mathbb{R}^+$ be endowed with the Euclidean metric $d = |\cdot|$, $p : X \times X \to \mathbb{R}^+$, $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ and $f : X \to X$ be defined by

$$p(x,y) = x+y, \quad \forall x,y \in X, \quad \varphi(t) = 4t, \quad \psi(t) = \frac{\sqrt{3}}{8}t^2, \quad \forall t \in \mathbb{R}^+$$

and

$$fx = \frac{\sqrt{3}}{2}x, \quad \forall x \in \mathbb{R}^+.$$

It is easy to see that p is a w-distance in X, $(\varphi, \psi) \in \Phi_2 \times \Phi_3$ and

$$\begin{aligned}\int_0^{p(fx,fy)} \varphi(t)dt &= \int_0^{\frac{\sqrt{3}}{2}(x+y)} 4tdt = \frac{3}{2}(x+y)^2 \le \frac{100+57\sqrt{3}}{128}(x+y)^2 \\ &= \int_0^{\frac{1}{2}(1+\frac{\sqrt{3}}{2})(x+y)} 4tdt - \psi\left(\frac{1}{2}\left(1+\frac{\sqrt{3}}{2}\right)(x+y)\right) \\ &= \int_0^{\frac{1}{2}[p(x,fx)+p(y,fy)]} \varphi(t)dt - \psi\left(\frac{1}{2}[p(x,fx)+p(y,fy)]\right), \quad \forall x,y \in X.\end{aligned}$$

That is, the conditions of Theorem 3.3 are fulfilled. It follows from Theorem 3.3 that f has a unique fixed point in X.

Remark 4.4. In case $p(x,y) = d(x,y)$ for all $x, y \in X$, then Theorem 3.4 reduces to Theorem 1.4. The following example proves that Theorem 3.4 generalizes indeed Theorem 1.4.

Example 4.4. Let $X = \mathbb{R}^+$ be endowed with the Euclidean metric $d = |\cdot|$, $p : X \times X \to \mathbb{R}^+$, $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ and $f : X \to X$ be defined by

$$p(x,y) = \sqrt{y}, \quad \forall x,y \in X, \quad \varphi(t) = t, \quad \psi(t) = \frac{1}{4}t^2, \quad \forall t \in \mathbb{R}^+$$

and

$$fx = \begin{cases} 0, & \forall x \in [0,4], \\ \frac{\sqrt{x}}{4}, & \forall x \in (4,+\infty). \end{cases}$$

It is clear that p is a w-distance in X and $(\varphi, \psi) \in \Phi_1 \times \Phi_3$. Let $x, y \in X$. In order to verify (3.33), we have to consider two possible cases as follows:

Case 1. $(x,y) \in X \times [0,4]$. It is clear that

$$\int_0^{p(fx,fy)} \varphi(t)dt = 0 \le \frac{1}{2}y = \int_0^{p(x,y)} \varphi(t)dt - \psi(p(fx,fy));$$

Case 2. $(x,y) \in X \times (4,+\infty)$. Note that

$$\begin{aligned}\int_0^{p(fx,fy)} \varphi(t)dt &= \int_0^{\frac{1}{2}y^{\frac{1}{4}}} tdt = \frac{\sqrt{y}}{8} \le \frac{1}{2}y - \frac{1}{8}\sqrt{y} \\ &= \int_0^{\sqrt{y}} tdt - \psi\left(\frac{1}{2}y^{\frac{1}{4}}\right) = \int_0^{p(x,y)} \varphi(t)dt - \psi(p(fx,fy)).\end{aligned}$$

That is, (3.33) holds. Hence the condition of Theorem 3.4 are satisfied. It follows from Theorem 3.4 that f has a unique fixed point in X.

However we cannot use Theorem 1.4 to prove the existence of fixed points of the mapping f in X. Otherwise, there exists $(\varphi, \psi) \in \Phi_1 \times \Phi_3$ satisfying (1.4).

It follows that

$$
\begin{aligned}
0 &< \int_0^{\frac{1}{2}} \varphi(t)dt = \limsup_{y\to 4^+} \int_0^{|0-\frac{\sqrt{y}}{4}|} \varphi(t)dt = \limsup_{y\to 4^+} \int_0^{d(f4,fy)} \varphi(t)dt \\
&\le \limsup_{y\to 4^+} \left(\int_0^{d(4,y)} \varphi(t)dt - \psi\big(d(f4,fy)\big) \right) \\
&\le \limsup_{y\to 4^+} \int_0^{d(4,y)} \varphi(t)dt - \liminf_{y\to 4^+} \psi\left(\frac{\sqrt{y}}{4}\right) \le 0 - \psi\left(\frac{1}{2}\right) < 0,
\end{aligned}
$$

which is absurd.

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