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EXISTENCE THEOREMS OF BOUNDARY VALUE PROBLEMS FOR FOURTH ORDER NONLINEAR DISCRETE SYSTEMS†

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Abstract. In the manuscript, we concern with the existence of solutions of boundary value problems for fourth order nonlinear discrete systems. Some criteria for the existence of at least one nontrivial solution of the problem are obtained. The proof is mainly based upon the variational method and critical point theory. An example is presented to illustrate the main result.

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1. Introduction

We define $\mathbb{N}$ by the sets of all natural numbers, define $\mathbb{Z}$ by the sets of integers, define $\mathbb{R}$ by the sets of real numbers. For $a, b \in \mathbb{Z}$ and $a \le b$, let $\mathbb{Z}[a, b] := \mathbb{Z} \cap [a, b]$. Let u^* be the transpose of a vector u.

Consider the boundary value problem (BVP) consisting of the fourth order nonlinear discrete system

$$\Delta^2(p_{n-2}\Delta^2 u_{n-2}) + q_n u_n = f(n, u_n), \; n \in \mathbb{Z}[1, k], \tag{1.1}$$

and boundary value conditions

$$\Delta^i u_{-1} = \Delta^i u_{k-1}, i = 0, 1, 2, 3, \tag{1.2}$$

where $\Delta^j u_n = \Delta(\Delta^{j-1} u_n)$ $(j = 2, 3, 4)$, $\Delta^0 u_n = u_n$, $\Delta u_n = u_{n+1} - u_n$, $p \in C(\mathbb{R}, \mathbb{R})$, $p_{-1} = p_{k-1}$, $p_0 = p_k$, $q \in C(\mathbb{R}, \mathbb{R})$, $f(s, u) \in C(\mathbb{R}^2, \mathbb{R})$, $k \ge 1$ is an integer.

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The problem (1.1) and (1.2) can be viewed as being a discrete analogue of the fourth order boundary value problem of differential equation

$$[p(t)u''(t)]'' - q(t)u(t) = f(t, u(t)),\ t \in (0, 1), \tag{1.3}$$

and

$$u^{(i)}(0) = u^{(i)}(1), i = 0, 1, 2, 3. \tag{1.4}$$

In recent years, many authors have devoted to the study of the BVP (1.3) and (1.4) and differential equations similar in structure to (1.3) by employing various methods and obtained some interesting results, see [1, 2, 3, 6, 7, 10, 11, 12, 13, 15, 16, 19, 25, 26].

Difference equations [4, 5, 8, 9, 17, 18, 20, 21, 22, 23, 27, **?**, 29, 30, 31], the discrete analogue of differential equations, have played an important role in analysis of mathematical models of biology, physics and engineering.

Thandapani and Arockiasamy [28] in 2001 considered the fourth-order difference equation of the form,

$$\Delta^2 \left(r_n \Delta^2 u_n\right) + f(n, u_n) = 0,\ n \in \mathbb{N}(n_0),$$

where $f(n, u)$ may be classified as superlinear, sublinear, strongly superlinear, and strongly sublinear. In superlinear and sublinear cases, necessary and sufficient conditions are obtained for the existence of nonoscillatory solutions. In strongly superlinear and strongly sublinear cases, necessary and sufficient conditions are given.

Chen and Tang [8] concerned with the existence of infinitely many homoclinic orbits from 0 of the fourth-order difference system

$$\Delta^4 u_{n-2} + q_n u_n = f(n, u_{n+1}, u_n, u_{n-1}),\ n \in \mathbb{Z}.$$

By using the symmetric mountain pass theorem, they established some existence criteria to guarantee the above system have infinitely many homoclinic orbits.

In [19], the paper dealt with the periodic solutions of a class of fourth-order superlinear differential equation

$$\begin{gathered}
u^{(4)} - cu'' + a(x)u - \frac{\partial F(x, u, v)}{\partial u} = 0, 0 < x < L,\\
u^{(4)} - du'' + b(x)v - \frac{\partial F(x, u, v)}{\partial v} = 0, 0 < x < L,\\
u(0) = u''(0) = u(L) = u''(L) = 0,\\
v(0) = v''(0) = v(L) = v''(L) = 0.
\end{gathered}$$

By making use of the classical variational techniques and symmetric mountain pass lemma, the periodic solutions of a single equation in literature are extended to that of equations. What's more, the cubic growth of nonlinear term is extended to a general form of superlinear growth.

By using fixed point index theorems, He, sun and Chen [14] considered more general m-point boundary value problem with variable coefficients as follows,

$$\Delta^4 u_{n-2} + B_n \Delta^2 u_{n-1} - A_n u_n = f(n, u_n),\ n \in \mathbb{Z}[a + 1, b + 1],$$

and

$$u_a = \sum_{i=1}^{m-2} a_i u(l_i),\ u_{b+2} = \sum_{i=1}^{m-2} b_i u(l_i),$$

$$\Delta^2 u_{a-1} = \sum_{i=1}^{m-2} a_i \Delta^2 u(l_i - 1),\ \Delta^2 u_{b+1} = \sum_{i=1}^{m-2} b_i \Delta^2 u(l_i - 1),$$

and obtained the existence of positive solutions.

In 2018, Xia [29] considered the following higher order nonlinear difference equation containing both many advances and retardations

$$\sum_{i=0}^{n} r_i(X_{k-i} + X_{k+i}) + f(k, X_{k+\Gamma}, \cdots, X_k, \cdots, X_{k-\Gamma}) = 0,\ n \in \mathbb{N},\ k \in \mathbb{Z}.$$

The existence of periodic solutions are obtained by using variational techniques and the Saddle Point Theorem.

Motivated by the recent papers [5, 10], in this paper, the existence of solutions of boundary value problems fourth order nonlinear discrete systems is studied. We obtain some criteria for the existence of at least one nontrivial solution of the problem. The proof is mainly based upon the variational method and critical point theory. An example is presented to illustrate the main result.

We define $F(s,u)$ as

$$F(s,u) = \int_0^u f(s,t)dt,$$

for any $(s,u) \in \mathbb{R}^2$.

This paper is divided into six parts. Section 2, variational framework is established. In Section 3, we state the theorems obtained. Section 4 contains two basic lemmas which are useful in proof of our results. In Section 5, we finish the proofs of the theorems. Finally, in section 6, we give an example to illustrate the main result.

2. Variational framework

In this section, the necessary background needed to apply the variational methods used to prove our main results are presented.

Define the set U by

$$U := \{u : \mathbb{Z}[-1, k+2] \to \mathbb{R} | \Delta^i u_{-1} = \Delta^i u_{k-1},\ i = 0, 1, 2, 3\},$$

and define

$$(u, v) := \sum_{j=1}^{k} u_j v_j,\ \forall u, v \in U,$$

and

$$\|u\| := \left(\sum_{j=1}^{k} u_j^2\right)^{\frac{1}{2}},\ \forall u \in U.$$

For any $u \in U$ and $r > 1$, let the norm $\|\cdot\|_r$ be

$$\|u\|_r = \left(\sum_{j=1}^{k} |u_j|^r \right)^{\frac{1}{r}}.$$

By all appearance, $\|u\| = \|u\|_2$. In that $\|u\|_r$ and $\|u\|$ are equivalent, there exist two constants $d_2 \geq d_1 > 0$ such that

$$d_1\|u\| \leq \|u\|_r \leq d_2\|u\|, \ \forall u \in U. \tag{2.1}$$

Remark 2.1. By (1.2), we have

$$u_{-1} = u_{k-1}, u_0 = u_k, u_1 = u_{k+1}, u_2 = u_{k+2}, \ \forall u \in U. \tag{2.2}$$

In reality, U is isomorphic to $\mathbb{R}^k$. In a whole paper, when we say

$$u = (u_1, u_2, \cdots, u_k) \in \mathbb{R}^k$$

, we always mean that u can be extended to a vector in U so that (2.2) holds.

We define J as

$$J(u) := -\frac{1}{2}\sum_{n=1}^{k} p_{n-2}\left(\Delta^2 u_{n-2}\right)^2 - \sum_{n=1}^{k} q_n u_n^2 + \sum_{n=1}^{k} F(n, u_n), \forall u \in U. \tag{2.3}$$

Therefore, $J \in C(U, \mathbb{R})$ and

$$\frac{\partial J}{\partial u_n} = -\Delta^2(p_{n-2}\Delta^2 u_{n-2}) - q_n u_n + f(n, u_n), \ n \in \mathbb{Z}[1, k].$$

Thus, $J'(u) = 0$ when and only when

$$\Delta^2(p_{n-2}\Delta^2 u_{n-2}) + q_n u_n = f(n, u_n), \ n \in \mathbb{Z}[1, k].$$

So, we reduce the problem of finding a solution of the BVP (1.1) and (1.2) to that of seeking a critical point of the functional J on U. Denote the $k \times k$ matrix A as below.

If $k = 1$, denote $A = (0)$.

If $k = 2$, denote

$$A = \begin{pmatrix} p_{-1} + 4p_0 + 3p_1 + q_1 & -2p_0 - 4p_1 - 2p_2 \\ -2p_0 - 4p_1 - 2p_2 & p_0 + 4p_1 + 3p_2 + q_2 \end{pmatrix}.$$

If $k = 3$, denote

$$A = \begin{pmatrix} p_{-1} + 4p_0 + p_1 + q_1 & p_2 - 2(p_0 + p_1) & p_1 - 2(p_2 + p_3) \\ p_2 - 2(p_0 + p_1) & p_0 + 4p_1 + p_2 + q_2 & p_3 - 2(p_1 + p_2) \\ p_1 - 2(p_2 + p_3) & p_3 - 2(p_1 + p_2) & p_1 + 4p_2 + p_3 + q_3 \end{pmatrix}.$$

If $k = 4$, denote

$$A = \begin{pmatrix} p_{-1} + 4p_0 + p_1 + q_1 & -2(p_0 + p_1) & p_1 + p_3 & -2(p_3 + p_4) \\ -2(p_0 + p_1) & p_0 + 4p_1 + p_2 + q_2 & -2(p_1 + p_2) & p_2 + p_4 \\ p_1 + p_3 & -2(p_1 + p_2) & p_1 + 4p_2 + p_3 + q_3 & -2(p_2 + p_3) \\ -2(p_3 + p_4) & p_2 + p_4 & -2(p_2 + p_3) & p_2 + 4p_3 + p_4 + q_4 \end{pmatrix}.$$

If $k \geq 5$, denote

$$A = \begin{pmatrix} s_1 & r_1 & p_1 & 0 & \cdots & 0 & p_{k-1} & a_k \\ r_1 & s_2 & r_2 & p_2 & \cdots & 0 & 0 & p_k \\ p_1 & r_2 & s_3 & r_3 & \cdots & 0 & 0 & 0 \\ 0 & p_2 & r_3 & s_4 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & r_{k-3} & p_{k-3} & 0 \\ 0 & 0 & 0 & 0 & \cdots & b_{k-2} & r_{k-2} & p_{k-2} \\ p_{k-1} & 0 & 0 & 0 & \cdots & r_{k-2} & s_{k-1} & r_{k-1} \\ r_k & p_k & 0 & 0 & \cdots & p_{k-2} & r_{k-1} & s_k \end{pmatrix},$$

where $s_n = p_n + 4p_{n-1} + p_{n-2},\ r_n = -2(p_{n-1} + p_n),\ n = 1, 2, \cdots, k$.

Forasmuch, $J(u)$ can be rewritten by

$$J(u) = -\frac{1}{2}u^* Au + \sum_{n=1}^{k} F(n, u_n). \tag{2.4}$$

3. Theorems obtained

Now, in this section, we give theorems obtained.

Theorem 3.1. *Assume that the following assumptions are satisfied.*
(p) *For any* $n \in \mathbb{Z}[-1, k]$, $p_n \geq 0$.
(A) $b_1+s_1+p_1+p_{k-1}+s_k = 0$, $r_2+s_2+p_2+p_k+s_1 = 0$, $r_n+s_n+p_n+p_{n-2}+s_{n-1} = 0,\ n = 3, 4, \cdots, k$.
(F_1) *For any* $n \in \mathbb{Z}[1, k]$, $F(n, 0) = 0$, $f(n, u) = 0$ *when and only when* $u = 0$.
(F_2) *For any* $n \in \mathbb{Z}[1, k]$, *there exists a number* $1 < \sigma < 2$ *such that*

$$0 < uf(n, u) < \sigma F(n, u),\ \forall u \neq 0.$$

(F_3) *For any* $n \in \mathbb{Z}[1, k]$, *there exist numbers* $a_1 > 0$ *and* $1 < \varsigma \leq \sigma$ *such that*

$$F(n, u) \geq a_1 |u|^{\varsigma},\ \forall u \in \mathbb{R}.$$

Then the BVP (1.1) and (1.2) admit at least one nontrivial solution.

Theorem 3.2. *Assume that* (p), (A), (F_1) *and the following assumptions are satisfied.*
(F_4) *For any* $n \in \mathbb{Z}[1, k]$, *there exists a number* $K > 0$ *such that*

$$|f(n, u)| \leq K,\ \forall u \in \mathbb{R}.$$

(F_5) $F(n, u) \to +\infty$ *uniformly for* $n \in \mathbb{Z}[1, k]$ *as* $|u| \to +\infty$.
(F_6) *For any* $n \in \mathbb{Z}[1, k]$, *there exist numbers* $a_2 > 0$ *and* $1 < \tau < 2$ *such that*

$$F(n, u) \geq a_2 |u|^{\tau},\ \forall u \in \mathbb{R}.$$

Then the BVP (1.1) and (1.2) admit at least one nontrivial solution.

If $f(n,u_n)=\varphi(u_n)$, (1.1) reduces to the following autonomous fourth order nonlinear discrete system,

$$\Delta^2(p_{n-2}\Delta^2u_{n-2})+q_nu_n=\varphi(u_n),\ n\in\mathbb{Z}[1,k], \tag{3.1}$$

where $\varphi\in C(\mathbb{R},\mathbb{R})$.

Theorem 3.3. *Assume that* (p), (A) *and the following assumptions are satisfied.*
(Φ) *There exists a function* $\Phi(u)\in C^1(\mathbb{R},\mathbb{R})$ *such that*

$$\Phi'(u)=\varphi(u),\forall u\in\mathbb{R}.$$

(φ_1) $\varphi(0)=0$.
(φ_2) *There exists a number* $1<\hat{\sigma}<2$ *such that*

$$0<u\varphi(u)<\hat{\sigma}\Phi(u),\ \forall u\neq 0.$$

(φ_3) *There exist numbers* $a_3>0$ *and* $1<\hat{\varsigma}\leq\hat{\sigma}$ *such that*

$$\Phi(u)\geq a_3\left|u\right|^{\hat{\varsigma}},\ \forall u\in\mathbb{R}.$$

Then the BVP (3.1) and (1.2) admit at least one nontrivial solution.

Theorem 3.4. *Assume that* (p), (A), (Φ) *and the following assumptions are satisfied.*
(φ_4) *There exists a number* $\hat{K}>0$ *such that*

$$|\varphi(u)|\leq\hat{K},\ \forall u\in\mathbb{R}.$$

(φ_5) $\lim\limits_{|u|\to+\infty}\Phi(u)=+\infty$.
(φ_6) *There exist numbers* $a_4>0$ *and* $1<\hat{\tau}<2$ *such that*

$$\Phi(u)\geq a_4\left|u\right|^{\hat{\tau}},\ \forall u\in\mathbb{R}.$$

Then the BVP (3.1) and (1.2) admit at least one nontrivial solution.

It comes from (p) and (A) that 0 is an eigenvalue of A and $(1,1,\cdots,1)^*$ is an eigenvector of A corresponding to 0. Let $\lambda_1,\lambda_2,\cdots,\lambda_{k-1}$ be other eigenvalues of A. Applying matrix theory, we have $\lambda_j>0$ for all $j=1,2,\cdots,k-1$.

Let

$$\lambda_{\min}=\min\left\{\lambda_j|\lambda_j\neq 0,i=1,2,\cdots,k-1\right\}, \tag{3.2}$$

and

$$\lambda_{\max}=\max\left\{\lambda_j|\lambda_j\neq 0,i=1,2,\cdots,k-1\right\}. \tag{3.3}$$

It is obvious that the eigenspace of A associated with 0 is

$$U_2=\{u\in U:u=\{c\},c\in\mathbb{R}\}.$$

Assume that U_1 is the direct orthogonal complement of U to U_2, that is,

$$U=U_1\oplus U_2.$$

4. Two basic lemmas

Let U be a real Banach space and $J \in C^1(U, \mathbb{R})$. In general, J satisfies the Palais-Smale condition if every sequence $\left\{u^{(i)}\right\}_{i=1}^{\infty} \subset U$, such that $\left\{J\left(u^{(i)}\right)\right\}_{i=1}^{\infty}$ is bounded and $J'\left(u^{(i)}\right) \to 0$ $(i \to \infty)$, has a convergent subsequence in U.

Suppose that U is a real Banach space. Let the symbol be B_ρ the open ball in U about 0 of radius ρ, ∂B_ρ be its boundary, and $\bar{B}_\rho$ be its closure.

Lemma 4.1. *(Saddle Point Theorem [24]). Suppose that U is a real Banach space, $U = U_1 \oplus U_2$, where $U_1 \neq \{0\}$ and is finite dimensional. Suppose that $J \in C^1(U, \mathbb{R})$ satisfies the Palais-Smale condition and*
(J_1) there are constants ε, $\eta > 0$ such that $J|_{\partial B_\eta \cap U_1} \leq \varepsilon$;
(J_2) there is $\xi \in B_\eta \cap U_1$ and a constant $\chi > \varepsilon$ such that $J_{\xi + U_2} \geq \chi$.
Then J admits a critical value $c \geq \chi$, where

$$c = \inf_{g \in \Omega} \max_{u \in B_\eta \cap U_1} J(g(u)), \ \Omega = \{g \in C(\bar{B}_\eta \cap U_1, U) \mid g|_{\partial B_\eta \cap U_1} = I\}$$

and I defines as the identity operator.

Lemma 4.2. *Suppose that the suppositions (p), (A) and $(F_1)-(F_3)$ hold. Then J satisfies the Palais-Smale condition.*

Proof. Assume that $\left\{u^{(i)}\right\}_{i=1}^{\infty} \subset U$ is such that $\left\{J\left(u^{(i)}\right)\right\}_{i=1}^{\infty}$ is bounded and $J'\left(u^{(i)}\right) \to 0$ $(i \to \infty)$. Therefore, we have a positive constant a_5 such that

$$\left|J\left(u^{(i)}\right)\right| \leq a_5, \forall i \in \mathbb{N},$$

and for sufficiently large i, we have

$$\left|\left(J'\left(u^{(i)}\right), u^{(i)}\right)\right| \leq \left\|u^{(i)}\right\|.$$

From (2.4), we have

$$\left(J'\left(u_n^{(i)}\right), u_n^{(i)}\right) = -\left(u_n^{(i)}\right)^* A\left(u_n^{(i)}\right) + \sum_{n=1}^{k} f\left(n, u_n^{(i)}\right) u_n^{(i)}.$$

Hence, for sufficiently large i,

$$\begin{aligned} a_5 + \frac{1}{2}\left\|u^{(i)}\right\| &\geq J\left(u^{(i)}\right) - \frac{1}{2}\left(J'\left(u_n^{(i)}\right), u_n^{(i)}\right) \\ &= \sum_{n=1}^{k}\left[F\left(n, u_n^{(i)}\right) - \frac{1}{2} f\left(n, u_n^{(i)}\right) u_n^{(i)}\right]. \end{aligned}$$

By (F_2), (F_3) and (2.1), we have

$$\begin{aligned} a_5 + \frac{1}{2}\left\|u^{(i)}\right\| &\geq \left(1 - \frac{\sigma}{2}\right) \sum_{n=1}^{k} F\left(n, u_n^{(i)}\right) \\ &\geq \left(1 - \frac{\sigma}{2}\right) a_1 \sum_{n=1}^{k} \left|u_n^{(i)}\right|^{\varsigma} \end{aligned}$$

$$\geq \left(1-\frac{\sigma}{2}\right) a_1 d_1^{\varsigma} \left\|u^{(i)}\right\|^{\varsigma}.$$

Consequently,

$$\left(1-\frac{\sigma}{2}\right) a_1 d_1^{\varsigma} \left\|u^{(i)}\right\|^{\varsigma} - \frac{1}{2}\left\|u^{(i)}\right\| \leq a_5. \tag{4.1}$$

Since $1 < \sigma < 2$ and $1 < \varsigma \leq \sigma$, (4.1) implies that $\left\{u^{(i)}\right\}_{i=1}^{\infty}$ is bounded in U. What's more, U is finite dimensional. Therefore, there is a subsequence of $\left\{u^{(i)}\right\}_{i=1}^{\infty}$, which is convergent in U and the proof of the Palais-Smale condition is finished. □

5. Proofs of theorems

Proof of Theorem 3.1. From Lemma 4.2, J satisfies the Palais-Smale condition. We shall prove this theorem using the Saddle Point Theorem. It is sufficient to verify the suppositions (J_1) and (J_2).

Firstly, we verify the supposition (J_1). The supposition (F_2) implies that there exist numbers $a_6 > 0$ and $a_7 > 0$ such that

$$F(n, u_n) \leq a_6 |u|^{\sigma} + a_7, \forall n \in \mathbb{Z}[1,k] \times \mathbb{R}. \tag{5.1}$$

By (5.1) and (2.1), for any $u^{(1)} \in U_1$,

$$\begin{aligned}
J\left(u^{(1)}\right) &= -\frac{1}{2}\left(u^{(1)}\right)^{*} A\left(u^{(1)}\right) + \sum_{n=1}^{k} F\left(n, u_n^{(1)}\right) \\
&\leq -\frac{\lambda_{\min}}{2}\left\|u^{(1)}\right\|^{2} + a_6 \sum_{n=1}^{k}\left|u_n^{(1)}\right|^{\sigma} + a_7 k \\
&\leq -\frac{\lambda_{\min}}{2}\left\|u^{(1)}\right\|^{2} + a_6 d_2^{\sigma}\sqrt{k}\left\|u^{(1)}\right\|^{\sigma} + a_7 k.
\end{aligned}$$

Set

$$\varepsilon = a_7 k.$$

Since $1 < \sigma < 2$, there is a positive number η sufficiently large such that

$$J|_{\partial B_\eta \cap U_1} \leq \varepsilon.$$

Thus, J satisfies the supposition (J_1).

Secondly, we verify the supposition (J_2). For any given $u^{(0)} \in U_1$ and $u^{(2)} \in U_2$, let $u = u^{(0)} + u^{(2)}$. Therefore, by (F_3) and (2.1),

$$\begin{aligned}
J(u) &= -\frac{1}{2} u^{*} A u + \sum_{n=1}^{k} F(n, u_n) \\
&= -\frac{1}{2}\left(u^{(0)}\right)^{*} A\left(u^{(0)}\right) + \sum_{n=1}^{k} F\left(n, u_n^{(0)} + u_n^{(2)}\right)
\end{aligned}$$

$$
\begin{aligned}
&\geq -\frac{\lambda_{\max}}{2}\left\|u^{(0)}\right\|^{2}+a_{1}\sum_{n=1}^{k}\left|u_{n}^{(0)}+u_{n}^{(2)}\right|^{\varsigma}\\
&\geq -\frac{\lambda_{\max}}{2}\left\|u^{(0)}\right\|^{2}+a_{1}d_{1}^{\varsigma}\left[\sum_{n=1}^{k}\left|u_{n}^{(0)}+u_{n}^{(2)}\right|^{2}\right]^{\frac{\varsigma}{2}}\\
&= -\frac{\lambda_{\max}}{2}\left\|u^{(0)}\right\|^{2}+a_{1}d_{1}^{\varsigma}\left[\left\|u^{(0)}\right\|^{2}+\left\|u^{(2)}\right\|^{2}\right]^{\frac{\varsigma}{2}}\\
&\geq -\frac{\lambda_{\max}}{2}\left\|u^{(0)}\right\|^{2}+a_{1}d_{1}^{\varsigma}\left\|u^{(0)}\right\|^{\varsigma}+a_{1}d_{1}^{\varsigma}\left\|u^{(2)}\right\|^{\varsigma}.
\end{aligned}
$$

Combining with $1<\varsigma<2$, we have that there is a positive number ϱ sufficiently small such that

$$J\left(u^{(0)}+u^{(2)}\right)\geq\varrho^{\varsigma}\left(a_{1}d_{1}^{\varsigma}-\frac{\lambda_{\max}}{2}\varrho^{2-\varsigma}\right)>0,$$

for any given $u^{(0)}\in U_1$, $\left\|u^{(0)}\right\|=\varrho$ and for any $u^{(2)}\in U_2$.

Denote

$$\chi=\varrho^{\varsigma}\left(a_{1}d_{1}^{\varsigma}-\frac{\lambda_{\max}}{2}\varrho^{2-\varsigma}\right).$$

Therefore, there is $\xi=u^{(0)}\in B_{\varrho}\cap U_1$ and a constant $\chi\geq\varepsilon$ such that

$$J_{\xi+U_2}\geq\chi.$$

Hence, J satisfies the supposition (J_2).

From the Saddle Point Theorem, there is a critical point $\tilde{u}\in U$, which corresponds to a solution of (1.1) and (1.2).

Finally, we verify that $\tilde{u}$ is nontrivial, namely, $\tilde{u}\notin U_2$. Or else, $\tilde{u}\in U_2$, which means that there is a constant $d\in\mathbb{R}$ such that $\tilde{u}_n=d,\forall n\in\mathbb{Z}[1,k]$.

Since $J'(\tilde{u})=0$, then

$$-\tilde{u}^{*}A\tilde{u}+f(n,\tilde{u}_n)=0,\forall n\in\mathbb{Z}[1,k].$$

Hence, $f(n,\tilde{u}_n)=f(n,d)=0,\forall n\in\mathbb{Z}[1,k]$. On the basis of (F_1), $d=0$. For this reason, again from (F_1),

$$J(\tilde{u})=\sum_{n=1}^{k}F(n,\tilde{u})=\sum_{n=1}^{k}F(n,d)=0,$$

which is a contradiction with $J(\tilde{u})\geq\chi>0$. Consequently, the BVP (1.1) and (1.2) admit at least one nontrivial solution. □

Remark 5.1. Similar to the proof of Theorem 3.1, we can verify the result of Theorem 3.2. For simplicity, the proof is omitted.

Remark 5.2. The conclusion of Theorem 3.3 is obtained from Theorem 3.1 and the conclusion of Theorem 3.4 is obtained from Theorem 3.2.

6. An Example

As an application of Theorem 3.1, in this section, we give an example to illustrate the main result.

Example 6.1. Consider the equation

$$\Delta^2(n\Delta^2 u_{n-2}) = \varsigma u_n|u_n|^{\varsigma-2} + \sigma u_n|u_n|^{\sigma-2},\ n \in \mathbb{Z}[1,4], \tag{6.1}$$

and boundary value conditions

$$u_{-1} = u_3,\ \Delta u_{-1} = \Delta u_3,\ \Delta^2 u_{-1} = \Delta^2 u_3,\ \Delta^3 u_{-1} = \Delta^3 u_3. \tag{6.2}$$

Here $1 < \varsigma \leq \sigma < 2$. We have

$$p_n = n+2,\ q_n \equiv 0, n \in \mathbb{Z}[1,4],$$

with

$$p_0 = 5,\ p_1 = 6,$$

and

$$f(n,u) = \varsigma u|u|^{\varsigma-2} + \sigma u|u|^{\sigma-2}, F(n,u) = |u|^{\varsigma} + |u|^{\sigma}.$$

Also,

$$A = \begin{pmatrix} 32 & -18 & 8 & -22 \\ -18 & 22 & -14 & 10 \\ 8 & -14 & 24 & -18 \\ -22 & 10 & -18 & 30 \end{pmatrix},$$

and the eigenvalues of A are $\lambda_1 = 0,\ \lambda_2 \approx 15.3568,\ \lambda_3 \approx 19.4814$ and $\lambda_4 \approx 73.1618$. It is easy to verify that all the suppositions of Theorem 3.1 are satisfied. Hence, the BVP (6.1) and (6.2) admit at least one nontrivial solution.

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