J. Appl. Math. & Informatics Vol. **37***(2019), No. 5 - 6, pp. 383 - 397*
https://doi.org/10.14317/jami.2019.383

δ-FUZZY IDEALS IN PSEUDO-COMPLEMENTED DISTRIBUTIVE LATTICES

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ABSTRACT. In this paper, we introduce δ-fuzzy ideals in a pseudo complemented distributive lattice in terms of fuzzy filters. It is proved that the set of all δ-fuzzy ideals forms a complete distributive lattice. The set of equivalent conditions are given for the class of all δ-fuzzy ideals to be a sublattice of the fuzzy ideals of L. Moreover, δ-fuzzy ideals are characterized in terms of fuzzy congruences.

AMS Mathematics Subject Classification : 06D72, 08A72, 03E72.
Key words and phrases : Pseudo-complemented distributive lattice, Stone lattice, δ-ideal, fuzzy ideal, fuzzy filter, δ-fuzzy ideal.

1. Introduction

The theory of pseudo-complementation was introduced and extensively studied in semi-lattices and particularly in distributive lattices by Frink [9] and Birkhoff [8]. Later, pseudo-complements in Stone algebras have been studied by several authors like Balbes [7], Frink [9], Grätzer [10], etc. In 2012, Rao [13], introduced the concept of δ-ideal in a distributive lattice in terms of pseudo-complementation and filters.

On the other hand, the notion of a fuzzy set initiated by Zadeh in [17]. Rosenfeld [14] has developed the concept of fuzzy subgroups. Since then, several authors have developed interesting results on fuzzy theory, like ([1],[2],[3],[4],[5],[6],[11],[14],[15],[16])

In this paper, the concept of δ-fuzzy ideals is introduced in a distributive lattice in terms of pseudo-complementation and fuzzy filters. Some properties of these δ-fuzzy ideals are studied and then proved that the set of all δ-fuzzy ideals can be made into a complete distributive lattice. We derive a set of equivalent conditions for the class of all δ-fuzzy ideals to become a sublattice to the lattice of all fuzzy ideals, which leads to a characterization of Stone lattices. We also prove

Received April 4, 2019. Revised August 11, 2019. Accepted August 17, 2019 *Corresponding author.

that the homomorphic image of a δ-fuzzy ideal is again a δ-fuzzy ideal. Finally, δ-fuzzy ideal of a pseudo-complemented distributive lattice is characterized in terms of fuzzy congruences.

2. Preliminaries

We refer to Grätzer [10] for the elementary properties of lattices. An algebra $L = (L; \wedge, \vee, *, 0, 1)$ is of type $(2, 2, 1, 0, 0)$ is a psueo-complemented distributive lattice, if the following conditions hold:

(1) $(L; \wedge, \vee, 0, 1)$ is a bounded distributive lattice, and
(2) for all $a, b \in L,\ a \wedge b = 0 \Leftrightarrow a \wedge b^* = a$.

Remark 2.1. The pseudo-complement a^* of an element a is the greatest element disjoint from a, if such an element exists.

A distributive lattice L in which every element has a pseudo-complement is called a pseudo-complemented lattice.

Theorem 2.1 ([10])**.** *For any two elements a, b of a pseudo-complemented lattice, we have the following:*

(1) $0^{**} = 0$,
(2) $a \wedge a^* = 0$,
(3) $a \leq b \Rightarrow b^* \leq a^*$,
(4) $a \leq a^{**}$,
(5) $a^{***} = a^*$,
(6) $(a \vee b)^* = a^* \wedge b^*$,
(7) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

An element x of a a pseudo-complemented lattice is called closed, if $x = x^{**}$.

Definition 2.2 ([7])**.** A pseudo-complemented distributive lattice L is called a Stone lattice, if for all $x \in L$, it satisfies the property:

$$x^* \vee x^{**} = 1.$$

Definition 2.3 ([13])**.** Let L be a pseudo-complemented distributive lattice. Then for any filter F, the set define

$$\delta(F) = \{x \in L : x^* \in F\}$$

is an ideal of L.

Definition 2.4 ([13])**.** Let L be a pseudo-complemented distributive lattice. An ideal I of L is called a δ-ideal, if $I = \delta(F)$, for some filter F of L.

Definition 2.5 ([17])**.** Let X be any nonempty set. A mapping $\mu : X \longrightarrow [0, 1]$ is called a fuzzy subset of X.

The unit interval $[0, 1]$ together the the operations min and max form a complete distributive lattice. We often write $\wedge$ for minimum or infimum and $\vee$ for maximum or supremum. That is, for all $\alpha, \beta \in [0, 1]$ we have, $\alpha \wedge \beta = min\{\alpha, \beta\}$ and $\alpha \vee \beta = max\{\alpha, \beta\}$.

The characteristics function of any set A is defined as:

$$\chi_A(x) = \begin{cases} 1 \text{ , if } x \in A \\ 0 \text{ , if } x \notin A. \end{cases}$$

Definition 2.6 ([14])**.** Let μ and θ be fuzzy subsets of a set A. Define the fuzzy subsets $\mu \cup \theta$ and $\mu \cap \theta$ of A as follows: for each $x \in A$,

$$(\mu \cup \theta)(x) = \mu(x) \vee \theta(x) \text{ and } (\mu \cap \theta)(x) = \mu(x) \wedge \theta(x).$$

Then $\mu \cup \theta$ and $\mu \cap \theta$ are called the union and intersection of μ and θ, respectively.

For any collection, $\{\mu_i : i \in I\}$ of fuzzy subsets of X, where I is a nonempty index set, the least upper bound $\bigcup_{i \in I} \mu_i$ and the greatest lower bound $\bigcap_{i \in I} \mu_i$ of the μ_i's are given by for each $x \in X$,

$$(\bigcup_{i \in I} \mu_i)(x) = \bigvee_{i \in I} \mu_i(x) \text{ and } (\bigcap_{i \in I} \mu_i)(x) = \bigwedge_{i \in I} \mu_i(x),$$

respectively.

For each $t \in [0, 1]$, the set

$$\mu_t = \{x \in A : \mu(x) \geq t\}$$

is called the level subset of μ at t [17].

Definition 2.7 ([14])**.** Let f be a function from X into Y; μ be a fuzzy subset of X; and θ be a fuzzy subset of Y. The image of μ under f, denoted by $f(\mu)$, is a fuzzy subset of Y defined as follows: for each $y \in Y$,

$$f(\mu)(y) = \begin{cases} Sup\{\mu(x) : x \in f^{-1}(y)\} \text{ , if } f^{-1}(y) \neq \phi \\ 0, \quad \text{otherwise} \end{cases}.$$

The preimage of θ under f, symbolized by $f^{-1}(\theta)$, is a fuzzy subset of X defined as follows: for each $x \in X$,

$$f^{-1}(\theta)(x) = \theta(f(x)).$$

Definition 2.8 ([15])**.** A fuzzy subset μ of a bounded lattice L is called a fuzzy ideal of L, if for all $x, y \in L$ the following conditions are satisfied:

(1) $\mu(0) = 1$,
(2) $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$,
(3) $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$.

Definition 2.9 ([15])**.** A fuzzy subset μ of a bounded lattice L is called a fuzzy filter of L, if for all $x, y \in L$ the following conditions are satisfied:

(1) $\mu(1) = 1$,
(2) $\mu(x \vee y) \geq \mu(x) \vee \mu(y)$,
(3) $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$.

We define the binary operations "+" and "$\cdot$" on the set of all fuzzy subsets of L as:

$$(\mu + \theta)(x) = Sup\{\mu(y) \wedge \theta(z) : y, z \in L,\ y \vee z = x\} \text{ and}$$
$$(\mu \cdot \theta)(x) = Sup\{\mu(y) \wedge \theta(z) : y, z \in L,\ y \wedge z = x\}.$$

If μ and θ are fuzzy ideals of L, then $\mu \cdot \theta = \mu \wedge \theta = \mu \cap \theta$ and $\mu + \theta = \mu \vee \theta$ is a fuzzy ideal generated by $\mu \cup \theta$.

If μ and θ are fuzzy filters of L, then $\mu + \theta = \mu \wedge \theta$ (the pointwise infimum of μ and θ) and $\mu \cdot \theta = \mu \vee \theta$ (the supremum of μ and θ).

Definition 2.10 ([12]). Let L be a lattice, $x \in L$ and $\alpha \in [0,1]$. Define a fuzzy subset α_x of L as:

$$\alpha_x(y) = \begin{cases} 1 \text{ , if } \ y \leq x \\ \alpha \text{ , if } \ y \nleq x \end{cases}$$

is a fuzzy ideal of L.

Remark 2.2 ([12]). α_x is called the α-level principal fuzzy ideal corresponding to x.
Similarly, a fuzzy subset α^x of L defined

$$\alpha^x(y) = \begin{cases} 1 \text{ , if } \ x \leq y \\ \alpha \text{ , if } \ x \nleq y \end{cases}$$

is the α-level principal fuzzy filter corresponding to x.

Definition 2.11 ([15]). A proper fuzzy ideal μ of L is called prime fuzzy ideal of L, if for any two fuzzy ideals θ, η of L, $\theta \cap \eta \subseteq \mu \Rightarrow \theta \subseteq \mu$ or $\eta \subseteq \mu$.

Definition 2.12 ([15]). A fuzzy subset θ of $L \times L$ is said to be a fuzzy congruence on L, if for any $x, y, z \in L$, the following hold:

(1) $\theta(x,x) = 1$,
(2) $\theta(x,y) = \theta(y,x)$,
(3) $\theta(x,y) \wedge \theta(y,z) \leq \theta(x,z)$,
(4) $\theta(x,y) \leq \theta(x \vee z, y \vee z) \wedge \theta(x \wedge z, y \wedge z)$.

Note that a fuzzy subset μ of L is nonempty if there exists $x \in L$ such that $\mu(x) \neq 0$. The set of all fuzzy ideals and fuzzy filters of L are denoted by $FI(L)$ and $FF(L)$ respectively.

3. δ-Fuzzy Ideals

In this section, we study δ-fuzzy ideals in a pseudo-complemented distributive lattice and its property. Throughout the rest of this paper L stands for a pseudo-complemented distributive lattice $(L, \vee, \wedge, *, 0, 1)$.

Definition 3.1. For any fuzzy filter μ of L, define the fuzzy subset $\delta(\mu)$ as follows:

$$\delta(\mu)(x) = \mu(x^*) \text{ for each } x \in L.$$

Lemma 3.2. *For any fuzzy filter μ of L, $\delta(\mu)$ is a fuzzy ideal of L.*

Proof. For any fuzzy filter μ of L. Since $0^* = 1$, we get $\delta(\mu)(0) = \mu(0^*) = 1$. Let $x, y \in L$. Then $\delta(\mu)(x \vee y) = \mu((x \vee y)^*) = \mu(x^* \wedge y^*) = \mu(x^*) \wedge \mu(y^*) = \delta(\mu)(x) \wedge \delta(\mu)(y)$. Thus $\delta(\mu)$ is a fuzzy ideal of L. □

The proof of the following lemma is quite routine and will be omitted.

Lemma 3.3. *For any fuzzy filters μ and θ of L, we have the following.*

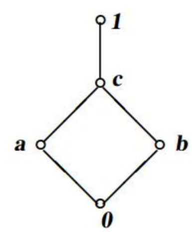
(1) $(\mu \cap \delta(\mu))(x) = \mu(0)$ *for each* $x \in L$,
(2) $\delta(\mu)(x) = \delta(\mu)(x^{**})$ *for each* $x \in L$,
(3) $\mu(x) \leq \delta(\mu)(x^*)$ *for each* $x \in L$,
(4) $\mu \subseteq \theta \Rightarrow \delta(\mu) \subseteq \delta(\theta)$,
(5) $\delta(\mu \cap \theta) = \delta(\mu) \cap \delta(\theta)$.

Lemma 3.4. *For any fuzzy filter μ of L, $\delta(\mu) = \chi_L$ if and only if $\mu = \chi_L$.*

Proof. Let $\delta(\mu) = \chi_L$. Then for each $x \in L$, $\delta(\mu)(x) = 1$. Since $0 = 1^* \in L$, we get $\mu(0) = \mu(1^*) = \delta(\mu)(1) = 1$. Since μ is a fuzzy filter, we have $\mu(0) \leq \mu(x)$ for each $x \in L$. This implies that $\mu(x) = 1$, for each $x \in L$. The converse part is trivial. □

Definition 3.5. A fuzzy ideal μ of L is a δ-fuzzy ideal, if $\mu = \delta(\theta)$ for some fuzzy filter θ of L.

Example 3.6. Consider the distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given below.



Define fuzzy subsets μ and θ of L as follows: $\mu(0) = 1 = \mu(a)$, $\mu(b) = \mu(c) = \mu(1) = 0$ and $\theta(1) = 1 = \theta(c) = \theta(b)$, $\theta(a) = \theta(0) = 0$. Then it can be easily verified that μ and θ are fuzzy ideal and fuzzy filter of L respectively in which $\mu = \delta(\theta)$. Thus μ is a δ-fuzzy ideal of L.

Every δ-fuzzy ideal is a fuzzy ideal but the converse may not be true. For this, we have the following example.

Example 3.7. If we define a fuzzy subset η of L given in the above example as $\eta(0) = 1$, $\eta(a) = \eta(b) = \eta(c) = 0.5$ and $\eta(1) = 0$, then η is a fuzzy ideal but not a δ-fuzzy ideal of L. Now we proceed to show η is not a δ-fuzzy ideal. Assume that $\eta = \delta(\lambda)$ for some fuzzy filter λ. Since $c^* = 1^* = 0$, we get $\eta(1) = \delta(\lambda)(1) = \lambda(0) = \eta(c)$. Which is a contradiction. This shows that we can not find any fuzzy filter λ of L such that $\eta = \delta(\lambda)$. Then η is a fuzzy ideal but not a δ-fuzzy ideal.

Lemma 3.8. *If F is a filter of L, then $\delta(\chi_F) = \chi_{\delta(F)}$.*

Proof. Let $x \in L$. If $x^* \in F$, then $x \in \delta(F)$ and $\delta(\chi_F)(x) = 1 = \chi_{\delta(F)}(x)$. If $x^* \notin F$, then $x \notin \delta(F)$ and $\delta(\chi_F)(x) = 0 = \chi_{\delta(F)}(x)$. Thus $\delta(\chi_F) = \chi_{\delta(F)}$. □

Corollary 3.9. *For any nonempty subset I of L. I is a δ-ideal of L if and only if χ_I is a δ-fuzzy ideal of L.*

Proof. Let I be a δ-ideal of L. Then there is a filter F of L such that $I = \delta(F)$. Thus by the above lemma, we have $\chi_I = \delta(\chi_F)$. So χ_I is a δ-fuzzy ideal of L. Conversely, suppose χ_I is a δ-fuzzy ideal of L. Then there is a fuzzy filter θ of L such that $\chi_I = \delta(\theta)$. Since θ is a fuzzy filter of L, every level subset of θ is a filter of L. To show I is a δ-ideal, it is enough to show that $I = \delta(\theta_1)$. Now, let $x \in I$. Then $\chi_I(x) = 1$ and $x^* \in \theta_1$. Thus $x \in \delta(\theta_1)$. Again, let $x \in \delta(\theta_1)$. Then $x^* \in \theta_1$ and $\delta(\theta)(x) = 1 = \chi_I(x)$. Thus $x \in I$. So I is a δ-ideal of L. □

Lemma 3.10. *For each $x \in L$, α_{x^*} is a δ-fuzzy ideal of L, $\alpha \in [0, 1]$.*

Proof. For any $x \in L$, α_{x^*} is a fuzzy ideal of L. To prove α_{x^*} is a δ-fuzzy ideal of L, it is enough to show that $\alpha_{x^*} = \delta(\alpha^x)$. Let $a \in L$. If $a \leq x^*$, then $\alpha_{x^*}(a) = 1$ and $x \leq a^*$. Which implies $\delta(\alpha^x)(a) = 1$. If $a \nleq x^*$, then $\alpha_{x^*}(a) = \alpha$ and $x \nleq a^*$. Thus $\delta(\alpha^x)(a) = \alpha$. So $\alpha_{x^*} = \delta(\alpha^x)$. Hence α_{x^*} is a δ-fuzzy ideal. □

Let us recall a dense element of a pseudo-complemented distributive lattice. An element x of a pseudo-complemented lattice is called dense if $x^* = 0$. The set D of all dense elements of L is a filter of L.

Lemma 3.11. *Let μ be a proper δ-fuzzy ideal. Then $\mu(x) = \mu(1)$ for each $x \in D$.*

Proof. Let μ be a proper δ-fuzzy ideal. Then there is a proper fuzzy filter θ of L such that $\mu = \delta(\theta)$. Since $1 \in D$, we get $\mu(1) = \theta(0)$. Let $x \in D$. Then $x^* = 0$ and $\mu(x) = \delta(\theta)(x) = \theta(x^*) = \theta(0) = \mu(1)$. □

Let us denote the set of all δ-fuzzy ideals of L by $FI^\delta(L)$. Then by Example 3.6, we can easily verified that $FI^\delta(L)$ is not a sublattice of the class $FI(L)$ of all fuzzy ideals of L. If we define the fuzzy subsets θ and λ of L as follows:

$$\theta(b) = \theta(c) = \theta(1) = 1,\ \theta(a) = \theta(0) = 0 \text{ and}$$
$$\lambda(a) = \lambda(c) = \lambda(1) = 1,\ \lambda(b) = \lambda(0) = 0.$$

Then clearly θ and λ are fuzzy filters of L. But $\delta(\theta) \vee \delta(\lambda)$ is not a δ-fuzzy ideal of L. We thus have the following theorem.

Theorem 3.12. *The set $FI^\delta(L)$ forms a complete distributive lattice with respect to inclusion ordering of fuzzy sets.*

Proof. Clearly $(FI^\delta(L), \subseteq)$ is a partially ordered set. For any two fuzzy filters μ, θ of L, define the binary operations $\cap$ and $\underline{\vee}$ as follows:

$$\delta(\mu) \cap \delta(\theta) = \delta(\mu \cap \theta) \text{ and } \delta(\mu) \underline{\vee} \delta(\theta) = \delta(\mu \vee \theta).$$

It is clear that $\delta(\mu\cap\theta)$ is the infimum of $\delta(\mu)$ and $\delta(\theta)$ in $FI^{\delta}(L)$. Also $\delta(\mu)\underline{\vee}\delta(\theta)$ is a δ-fuzzy ideal of L. Now we prove $\delta(\mu)\underline{\vee}\delta(\theta)$ is the supremum of $\{\delta(\mu),\ \delta(\theta)\}$ in $FI^{\delta}(L)$. Since $\mu \subseteq \mu \vee \theta$ and $\theta \subseteq \mu \vee \theta$, we get $\delta(\mu) \subseteq \delta(\mu \vee \theta)$ and $\delta(\theta) \subseteq \delta(\mu \vee \theta)$. This implies that $\delta(\mu \vee \theta)$ is an upper bound of $\{\delta(\mu),\ \delta(\theta)\}$. Let η be any δ-fuzzy ideal containing $\delta(\mu)$ and $\delta(\theta)$. Then there exists a fuzzy filter λ such that $\eta = \delta(\lambda)$ and $\delta(\mu) \subseteq \delta(\lambda),\ \delta(\theta) \subseteq \delta(\lambda)$. Now we proceed to show $\delta(\mu \vee \theta) \subseteq \delta(\lambda)$. For any $x \in L$, we have

$$\begin{aligned}
\delta(\mu \vee \theta)(x) &= (\mu \vee \theta)(x^*) \\
&= Sup\{\mu(a) \wedge \theta(b) : a \wedge b = x^*\} \\
&\leq Sup\{\mu(a^{**}) \wedge \theta(b^{**}) : a \wedge b = x^*\} \\
&\leq Sup\{\lambda(a^{**}) \wedge \lambda(b^{**}) : a^{**} \wedge b^{**} = x^*\} \\
&\leq Sup\{\lambda(y) \wedge \lambda(z) : y \wedge z = x^*\} \\
&= \lambda(x^*) \\
&= \delta(\lambda)(x).
\end{aligned}$$

Thus $\delta(\mu)\underline{\vee}\delta(\theta)$ is the supremum of $\{\delta(\mu),\ \delta(\theta)\}$ in $FI^{\delta}(L)$. So $(FI^{\delta}(L), \cap, \underline{\vee})$ is a lattice. Now we prove the distributivity. For any $\delta(\mu),\ \delta(\theta),\ \delta(\eta) \in FI^{\delta}(L)$,

$$\begin{aligned}
\delta(\mu)\underline{\vee}(\delta(\theta) \cap \delta(\eta)) &= \delta(\mu \vee (\theta \cap \eta)) \\
&= \delta(\mu \vee \theta) \cap \delta(\mu \vee \eta) \\
&= (\delta(\mu)\underline{\vee}\delta(\theta)) \cap (\delta(\mu)\underline{\vee}\delta(\eta)).
\end{aligned}$$

Then $FI^{\delta}(L)$ is a distributive lattice. Next, we prove the completeness. Since $\{0\}$ and L are δ-ideals, $\chi_{\{0\}}$ and χ_L are least and greatest elements of $FI^{\delta}(L)$. Let $\{\delta(\mu_i) : i \in I\}$ be a subfamily of $FI^{\delta}(L)$. Then $\bigcap_{i\in I} \delta(\mu_i)$ is a fuzzy ideal of L. Now,

$$\begin{aligned}
(\bigcap_{i\in I} \delta(\mu_i))(x) &= Inf\{\mu_i(x^*) : i \in I\} \\
&= (\bigcap_{i\in I} \mu_i)(x^*) \\
&= \delta(\bigcap_{i\in I} \mu_i)(x).
\end{aligned}$$

This shows that $(\bigcap_{i\in I} \delta(\mu_i)) \in FI^{\delta}(L)$. Thus $(FI^{\delta}(L), \cap, \underline{\vee})$ is a complete distributive lattice. □

Lemma 3.13. *Every proper δ-fuzzy ideal is contained in a minimal prime fuzzy ideal.*

Proof. Let μ be a proper δ-fuzzy ideal of L. Then $\mu = \delta(\theta)$ for some proper fuzzy filter θ of L. Since D is a filter of L, we have χ_D is a fuzzy filter and $\mu \cap \chi_D \leq \alpha$, where $\alpha = \mu(1)$. By corollary 1.6 [15], there exists a minimal prime fuzzy ideal η of L such that $\mu \subseteq \eta$ and $\eta \cap \chi_D \leq \alpha$. □

In the following theorem, we established set of equivalent conditions for the class of δ-fuzzy ideals to be a sublattice of the set of fuzzy ideals. We also characterize Stone lattices in terms of δ-fuzzy ideals.

Theorem 3.14. *In L the following conditions are equivalent:*

(1) *L is a Stone lattice,*
(2) *For any $x, y \in L$, $(x \wedge y)^* = x^* \vee y^*$,*
(3) *For any two fuzzy filters μ, θ of L, $\delta(\mu) \vee \delta(\theta) = \delta(\mu \vee \theta)$,*
(4) *$FI^{\delta}(L)$ is a sublattice of $FI(L)$.*

Proof. The proof of $1 \Rightarrow 2$ and $3 \Rightarrow 4$ is straightforward. Now we proceed to prove the following.

($2 \Rightarrow 3$): Assume the condition (2). Let μ and θ be fuzzy filters of L. We have always $\delta(\mu) \vee \delta(\theta) \subseteq \delta(\mu \vee \theta)$. We know that $\delta(\mu) \vee \delta(\theta)$ is the smallest fuzzy ideal containing $\delta(\mu)$ and $\delta(\theta)$. To prove our claim, it is enough to show $\delta(\mu \vee \theta)$ is the smallest fuzzy ideal containing $\delta(\mu)$ and $\delta(\theta)$. Let λ be any fuzzy ideal containing $\delta(\mu)$ and $\delta(\theta)$. Now, we proceed to show $\delta(\mu \vee \theta) \subseteq \lambda$. For any $x \in L$, we have

$$\begin{aligned} \delta(\mu \vee \theta)(x) &= (\mu \vee \theta)(x^*) \\ &= Sup\{\mu(a) \wedge \theta(b) : a \wedge b = x^*\} \\ &\leq Sup\{\mu(a^{**}) \wedge \theta(b^{**}) : a \wedge b = x^*\} \\ &= Sup\{\delta(\mu)(a^*) \wedge \delta(\theta)(b^*) : a \wedge b = x^*\} \\ &\leq Sup\{\lambda(a^*) \wedge \lambda(b^*) : a \wedge b = x^*\} \\ &\leq Sup\{\lambda(a^*) \wedge \lambda(b^*) : a^* \vee b^* = x^{**}\} \\ &\leq Sup\{\lambda(y) \wedge \lambda(z) : y \vee z = x^{**}\} \\ &= \lambda(x^{**}) \\ &\leq \lambda(x). \end{aligned}$$

Thus $\delta(\mu \vee \theta)$ is the smallest fuzzy ideal containing $\delta(\mu)$ and $\delta(\theta)$. So $\delta(\mu) \vee \delta(\theta) = \delta(\mu \vee \theta)$.

($4 \Rightarrow 1$): Assume that $FI^{\delta}(L)$ is a sublattice of $FI(L)$. Let $\alpha \in [0, 1)$. Then by Lemma 3.10, α_{x^*} and $\alpha_{x^{**}}$ are both δ-fuzzy ideals of L. Suppose that $x^* \vee x^{**} \neq 1$. Then $\alpha_{x^*} \vee \alpha_{x^{**}}$ is a proper δ-fuzzy ideal of L. Hence there exists a minimal prime fuzzy ideal θ such that $\alpha_{x^*} \vee \alpha_{x^{**}} \subseteq \theta$ and $\theta \cap \chi_D \leq \alpha$. Now we need to find $(\theta \cap \chi_D)(x^* \vee x^{**})$. Now $(\alpha_{x^*} \vee \alpha_{x^{**}})(x^* \vee x^{**}) \geq \alpha_{x^*}(x^*) \wedge \alpha_{x^{**}}(x^{**}) = 1$. Since $\alpha_{x^*} \vee \alpha_{x^{**}} \subseteq \theta$, we get that $\theta(x^* \vee x^{**}) = 1$. We know that $x^* \vee x^{**}$ is a dense element, so we have $\chi_D(x^* \vee x^{**}) = 1$. This implies that $(\theta \cap \chi_D)(x^* \vee x^{**}) = 1$. This is a contradiction. Thus $x^* \vee x^{**} = 1$ for each $x \in L$. So L is a Stone lattice. □

Theorem 3.15. *In L the following conditions are equivalent:*

(1) *L is a Boolean algebra,*
(2) *Every α-level principal fuzzy ideal is a δ-fuzzy ideal,*

(3) *For any fuzzy ideal* μ *of* L*,* $\mu(x) = \mu(x^{**})$ *for all* $x \in L$*,*
(4) D *is a singleton set.*

Proof. $(1 \Rightarrow 2)$: Suppose that L is a Boolean algebra. Then every element of L is closed. This implies $\alpha_x = \alpha_{x^{**}}$ for all $x \in L$. By lemma (3.10), $\alpha_{x^{**}} = \delta(\alpha^{x^*})$. Thus every α-level principal fuzzy ideal is a δ-fuzzy ideal.

$(2 \Rightarrow 3)$: Assume that every α-level principal fuzzy ideal is a δ-fuzzy ideal. Let μ be any fuzzy ideal of L. Since $x \leq x^{**}$, we get $\mu(x^{**}) \leq \mu(x)$. For each $x \in L$, α_x is a δ-fuzzy ideal of L. Then there exists a fuzzy filter θ of L such that $\alpha_x = \delta(\theta)$ and $\alpha_x(x) = \alpha_x(x^{**})$. This shows that $x^{**} \leq x$ and $\mu(x) \leq \mu(x^{**})$. Hence $\mu(x) = \mu(x^{**})$ for all $x \in L$.

$(3 \Rightarrow 4)$: Suppose that condition 3 is true. For each $x \in L$, $x \leq x^{**}$. Now we proceed to show $x^{**} \leq x$. For each $x \in L$, α_x is a fuzzy ideal of L. By the assumption, we have $\alpha_x(x) = \alpha_x(x^{**})$. This implies $x^{**} \leq x$. This shows that $x = x^{**}$ for all $x \in L$. Thus every element of L is a closed element. Assume that D is not a singleton set. Then there exists an element $x \in D$ such that $x \neq 1$. This implies $x^* = 0$ and $x^{**} = 1$. Since every element is closed, we get $x = 1$. This is a contradiction. Thus D is a singleton set.

$(4 \Rightarrow 1)$: Suppose that $D = \{d\}$. For any $x \in L$, $x \vee x^* \in D$. Then $x \wedge x^* = 0$ and $x \vee x^* = d$. This implies $0 \leq x \leq x \vee x^* = d$ for all $x \in L$. This shows that L is a bounded distributive lattice in which each elements is complemented. Thus L is a Boolean algebra. □

We now characterize δ-fuzzy ideal in terms of fuzzy congruence relations.

Theorem 3.16. *For any fuzzy filter* μ *of* L*, define a fuzzy relation* $\theta(\mu)$ *as:*

$$\theta(\mu)(x, y) = Sup\{\mu(a) : x \wedge a = y \wedge a,\ a \in L\} \text{ for each } x, y \in L.$$

Then $\theta(\mu)$ *is a fuzzy congruence relation on* L*.*

Proof. Let μ be a fuzzy filter of L. We prove that $\theta(\mu)$ is a fuzzy congruence on L. For any $x, y \in L$, clearly $\theta(\mu)(x, x) = 1$ and $\theta(\mu)(x, y) = \theta(\mu)(y, x)$.

(1) If $x \wedge a = z \wedge a$ and $z \wedge b = y \wedge b$, then we get that $x \wedge (a \wedge b) = y \wedge (a \wedge b)$. Thus

$$\begin{aligned} \theta(\mu)(x, z) \wedge \theta(\mu)(z, y) &= Sup\{\mu(a) : x \wedge a = z \wedge a,\ a \in L\} \\ &\quad \wedge Sup\{\mu(b) : z \wedge b = y \wedge b,\ b \in L\} \\ &= Sup\{\mu(a) \wedge \mu(b) : x \wedge a = z \wedge a,\ z \wedge b = y \wedge b\} \\ &\leq Sup\{\mu(a \wedge b) : x \wedge (a \wedge b) = y \wedge (a \wedge b)\} \\ &\leq Sup\{\mu(c) : x \wedge c = y \wedge c,\ a \in L\} \\ &= \theta(\mu)(x, y). \end{aligned}$$

(2) For all $x_1, x_2, y_1, y_2 \in L$,

$$\begin{aligned} &\theta(\mu)(x_1, y_1) \wedge \theta(\mu)(x_2, y_2) \\ = \;& Sup\{\mu(a) : x_1 \wedge a = y_1 \wedge a,\ a \in L\} \end{aligned}$$

$$\begin{aligned} & \wedge Sup\{\mu(b) : x_2 \wedge b = y_2 \wedge b,\ b \in L\} \\ = & \ Sup\{\mu(a) \wedge \mu(b) : x_1 \wedge a = y_1 \wedge a,\ x_2 \wedge b = y_2 \wedge b\} \\ \leq & \ Sup\{\mu(a \wedge b) : (x_1 \wedge x_2) \wedge (a \wedge b) = (y_1 \wedge y_2) \wedge (a \wedge b)\} \\ \leq & \ Sup\{\mu(c) : (x_1 \wedge x_2) \wedge c = (y_1 \wedge y_2) \wedge c\} \\ = & \ \theta(\mu)(x_1 \wedge x_2, y_1 \wedge y_2). \end{aligned}$$

(3) If $x_1 \wedge a = y_1 \wedge a$ and $x_2 \wedge b = y_2 \wedge b$, then $(x_1 \wedge a) \vee (x_2 \wedge b) = (y_1 \wedge a) \vee (y_2 \wedge b)$. Thus

$$\begin{aligned} & (x_1 \vee x_2) \wedge ((a \vee x_2) \wedge (x_1 \vee b) \wedge (a \vee b)) \\ & = (y_1 \vee y_2) \wedge ((a \vee y_2) \wedge (y_1 \vee b) \wedge (a \vee b)). \end{aligned}$$

Since $x_1 \wedge a = y_1 \wedge a$ and $x_2 \wedge b = y_2 \wedge b$, we have $(x_1 \vee b) \wedge (x_2 \vee a) \wedge (a \vee b) = (y_1 \vee b) \wedge (y_2 \vee a) \wedge (a \vee b)$. Since μ is a fuzzy filter of L,

$$\begin{aligned} \mu((x_1 \vee b) \wedge (x_2 \vee a) \wedge (a \vee b)) & = \mu(x_1 \vee b) \wedge \mu(x_2 \vee a) \wedge \mu(a \vee b) \\ & \geq \mu(a) \wedge \mu(b). \end{aligned}$$

Then

$$\begin{aligned} & \theta(\mu)(x_1, y_1) \wedge \theta(\mu)(x_2, y_2) \\ = & \ Sup\{\mu(a) : x_1 \wedge a = y_1 \wedge a,\ a \in L\} \wedge \ Sup\{\mu(b) : x_2 \wedge b = y_2 \wedge b,\ b \in L\} \\ = & \ Sup\{\mu(a) \wedge \mu(b) : x_1 \wedge a = y_1 \wedge a,\ x_2 \wedge b = y_2 \wedge b\} \\ = & \ Sup\{\mu(a \wedge b) : x_1 \wedge a = y_1 \wedge a,\ x_2 \wedge b = y_2 \wedge b\} \\ \leq & \ Sup\{\mu((x_1 \vee b) \wedge (x_2 \vee a) \wedge (a \vee b)) : (x_1 \vee x_2) \\ & \wedge((x_1 \vee b) \wedge (x_2 \vee a) \wedge (a \vee b)) = (y_1 \vee y_2) \wedge ((y_1 \vee b) \wedge (y_2 \vee a) \wedge (a \vee b))\} \\ \leq & \ Sup\{\mu(c) : (x_1 \vee x_2) \wedge c = (y_1 \vee y_1) \wedge c,\ c \in L\} \\ = & \ \theta(\mu)(x_1 \vee x_2, y_1 \vee y_2). \end{aligned}$$

Thus $\theta(\mu)$ is a fuzzy congruence on L. □

Theorem 3.17. *For any fuzzy ideal μ of L, the fuzzy subset η_μ of L defined as:*

$$\eta_\mu(x) = Sup\{\mu(a) : x^* \wedge a^* = 0,\ a \in L\}$$

is a fuzzy filter of L.

Proof. Let μ be a fuzzy ideal of L. Since $1^* = 0$, we get that $1^* \wedge a^* = 0$ for all $a \in L$. Thus $\eta_\mu(1) \geq \mu(0) = 1$. So $\eta_\mu(1) = 1$. For any $x, y \in L$,

$$\begin{aligned} & \eta_\mu(x) \wedge \eta_\mu(y) \\ = & \ Sup\{\mu(a) : x^* \wedge a^* = 0,\ a \in L\} \wedge Sup\{\mu(b) : y^* \wedge b^* = 0,\ b \in L\} \\ = & \ Sup\{\mu(a) \wedge \mu(b) : x^* \wedge a^* = 0,\ y^* \wedge b^* = 0\} \\ = & \ Sup\{\mu(a \vee b) : x^* \wedge a^* = 0,\ y^* \wedge b^* = 0\}. \end{aligned}$$

Since $x^* \wedge a^* = 0$ and $y^* \wedge b^* = 0$, we get that $x^{**} \wedge a^* = a^*$ and $y^{**} \wedge b^* = b^*$. This shows that $(x \wedge y)^{**} \wedge (a \vee b)^* = (a \vee b)^*$. Since L is a pseudo-complemented lattice, we get $(x \wedge y)^* \wedge (a \vee b)^* = 0$. Using this fact, we have

$$\begin{aligned}\eta_\mu(x) \wedge \eta_\mu(y) &\leq Sup\{\mu(a \vee b) : (x \wedge y)^* \wedge (a \vee b)^* = 0\} \\ &\leq Sup\{\mu(c) : (x \wedge y)^* \wedge c^* = 0\} \\ &= \eta_\mu(x \wedge y).\end{aligned}$$

Thus $\eta_\mu(x \wedge y) \geq \eta_\mu(x) \wedge \eta_\mu(y)$. On the other hand,

$$\begin{aligned}\eta_\mu(x) &= Sup\{\mu(a) : x^* \wedge a^* = 0,\ a \in L\} \\ &\leq Sup\{\mu(a) : (x \vee y)^* \wedge a^* = 0,\ a \in L\} \\ &= \eta_\mu(x \vee y).\end{aligned}$$

Similarly, $\eta_\mu(x \vee y) \geq \eta_\mu(y)$. So $\eta_\mu(x \vee y) \geq \eta_\mu(x) \vee \eta_\mu(y)$. Hence η_μ is a fuzzy filter of L. □

Let θ be a fuzzy congruence on L and $x \in L$ the fuzzy subset θ_x of L is defined by

$$\theta_x(y) = \theta(x, y) \text{ for all } y \in L$$

is called a fuzzy congruence class of L determined by θ and x. In [16], B. Yuan and W. Wu observed that, the fuzzy congruence class θ_0 of L determined by 0 is a fuzzy ideal of L. In the following theorem, we characterize δ-fuzzy ideal in terms of fuzzy congruence.

Theorem 3.18. *For any fuzzy ideal μ of L, the following conditions are equivalent:*

(1) *μ is a δ-fuzzy ideal,*
(2) $\mu = \theta_0(\eta_\mu)$,
(3) *$\mu = \theta_0(\eta)$ for some fuzzy filter η of L.*

Proof. The proof of $2 \Rightarrow 3$ is straightforward. Now we prove the following.

($1 \Rightarrow 2$): Assume that μ is a δ-fuzzy ideal of L. Then $\mu = \delta(\eta)$ for some fuzzy filter η of L. For any $x \in L$,

$$\begin{aligned}\theta_0(\eta_\mu)(x) &= \theta(\eta_\mu)(x, 0) \\ &= Sup\{\eta_\mu(a) : x \wedge a = 0,\ a \in L\} \\ &\geq \eta_\mu(x^*) \\ &= Sup\{\mu(b) : x^{**} \wedge b^* = 0,\ b \in L\} \\ &\geq \mu(x).\end{aligned}$$

Conversely, let $x \in L$. Then

$$\begin{aligned}\theta_0(\eta_\mu)(x) &= Sup\{\eta_\mu(a) : x \wedge a = 0,\ a \in L\} \\ &= Sup\{Sup\{\mu(b) : a^* \wedge b^* = 0,\ b \in L\} : x \wedge a = 0\}.\end{aligned}$$

Now we need to show $\mu(x) \geq \eta_\mu(a)$ for each $a \in L$ such that $x \wedge a = 0$. Fix an element b in L satisfying $x \wedge a = 0$ and $a^* \wedge b^* = 0$. Then $x \leq a^*$ and $a^* \leq b^{**}$. This implies $b^* \leq x^*$. Since $\mu = \delta(\eta)$ and η is a fuzzy filter, we have $\mu(x) = \eta(x^*) \geq \eta(b^*) = \mu(b)$. Thus $\mu(x) \geq \mu(b)$ and

$$\mu(x) \geq Sup\{\mu(b) : a^* \wedge b^* = 0,\ b \in L\} = \eta_\mu(a).$$

This shows that $\mu(x) \geq \eta_\mu(a)$ for each $a \in L$ such that $x \wedge a = 0$. So $\mu(x) \geq \theta_0(\eta_\mu)(x)$. So $\mu = \theta_0(\eta_\mu)$.

$(3 \Rightarrow 1)$: Assume that $\mu = \theta_0(\eta)$ for some fuzzy filter η of L. For any $x \in L$,

$$\mu(x) = Sup\{\eta(a) : x \wedge a = 0\} \geq \eta(x^*) = \delta(\eta)(x).$$

Thus $\delta(\eta) \subseteq \mu$.

Conversely, let $x, a \in L$ such that $x \wedge a = 0$. Then $a \leq x^*$. Since η is a fuzzy filter, we get $\eta(a) \leq \eta(x^*)$. This implies $\delta(\eta)(x) \geq \eta(a)$ for each $a \in L$ such that $x \wedge a = 0$. This shows that $\delta(\eta)(x)$ is an upper bound of $\{\eta(a) : x \wedge a = 0\}$. Thus $\delta(\eta)(x) \geq \theta_0(\eta)(x) = \mu(x)$ for each $x \in L$. So $\mu \subseteq \delta(\eta)$. Hence μ is a δ-fuzzy ideal of L. □

4. δ-Fuzzy Ideals and Homomorphism

In this section, some properties of the homomorphic images and the inverse images of δ-fuzzy ideals are studied.

Throughout this section L and $L^{'}$ denote distributive pseudo-complemented lattices with least elements 0 and $0^{'}$ respectively and $f : L \longrightarrow L^{'}$ denotes an onto homomorphism and $Kerf = \{0\}$.

In [13], M. S. Rao obsered that, for any two pseudo-complemented distributive lattices L and $L^{'}$ with pseudo-complementation $*$. If $f : L \longrightarrow L^{'}$ an onto homomorphism and $Kerf = \{0\}$. Then $f(x^*) = (f(x))^*$ for all $x \in L$. In the following theorem we prove that the homomorphic image of a δ-fuzzy ideal is again a δ-fuzzy ideal.

Theorem 4.1. *Let μ be a δ-fuzzy ideal of L. Then $f(\mu)$ is a δ fuzzy ideal of $L^{'}$.*

Proof. Let μ be a δ-fuzzy ideal of L. Then $\mu = \delta(\theta)$, for some fuzzy filter θ of L. Since μ is a fuzzy ideal and θ is a fuzzy filter, $f(\mu)$ and $f(\theta)$ are fuzzy ideal and fuzzy filter, respectively. Now we prove $f(\delta(\theta)) = \delta(f(\theta))$. For any $y \in L^{'}$,

$$\begin{aligned} f(\delta(\theta))(y) &= Sup\{\delta(\theta)(a) : a \in L,\ a \in f^{-1}(y)\} \\ &= Sup\{\theta(a^*) : a \in f^{-1}(y)\}. \end{aligned}$$

Since $f(a^*) = (f(a))^*$ and $f(a) = y$, we get $f(a^*) = y^*$. This implies $a^* \in f^{-1}(y^*)$. Based on this fact we have the following,

$$\begin{aligned} f(\delta(\theta))(y) &\leq Sup\{\theta(b) : b \in f^{-1}(y^*)\} \\ &= f(\theta)(y^*) \\ &= \delta(f(\theta))(y). \end{aligned}$$

Thus $f(\delta(\theta)) \subseteq \delta(f(\theta))$.

Conversely, for any $y \in L^{'}$,

$$\begin{aligned} \delta(f(\theta))(y) &= f(\theta)(y^*) \\ &= Sup\{\theta(a) : a \in L,\ a \in f^{-1}(y^*)\} \end{aligned}$$

$$
\begin{aligned}
&\leq Sup\{\theta(a^{**}) : a \in f^{-1}(y^*)\}\\
&= Sup\{\delta(\theta)(a^*) : a \in f^{-1}(y^*)\}.
\end{aligned}
$$

Since $f(a) = y^*$, we get $a^* \in f^{-1}(y^{**})$. Using this fact, we have the following

$$
\begin{aligned}
\delta(f(\theta))(y) &\leq Sup\{\delta(\theta)(b) : b \in L,\ b \in f^{-1}(y^{**})\}\\
&= f(\delta(\theta))(y^{**})\\
&\leq f(\delta(\theta))(y).
\end{aligned}
$$

Thus $\delta(f(\theta)) \subseteq f(\delta(\theta))$. So $\delta(f(\theta)) = f(\delta(\theta))$. Hence the homomorphic image of a δ-fuzzy ideal is a δ-fuzzy ideal. □

Corollary 4.2. *For any $x \in L$, $f(\alpha_{x^*}) = \delta(f(\alpha^x))$.*

Lemma 4.3. *If μ is a δ-fuzzy ideal of $L^{'}$, then $f^{-1}(\mu)$ is a δ fuzzy ideal of L.*

Proof. Let μ be a δ-fuzzy ideal of $L^{'}$. Then there is a fuzzy filter θ of $L^{'}$ such that $\mu = \delta(\theta)$. Since θ is a fuzzy filter of L, $f^{-1}(\theta)$ is a fuzzy filter of L. Now for any $x \in L$, $f^{-1}(\delta(\theta))(x) = \delta(\theta)(f(x)) = \theta(f(x^*) = f^{-1}(\theta)(x^*) = \delta(f^{-1}(\theta))(x)$. Thus $f^{-1}(\mu)$ is a δ-fuzzy ideal of L. □

Theorem 4.4. *The class $FI^{\delta}(L)$ of δ-fuzzy ideals of L is homomorphic to the class $FI^{\delta}(L^{'})$ of δ-fuzzy ideals of $L^{'}$.*

Proof. Define $g : FI^{\delta}(L) \longrightarrow FI^{\delta}(L^{'})$ by $g(\mu) = \delta(f(\theta))$, where $\mu = \delta(\theta)$ for some fuzzy filter θ of L. It can be easily verified that $g(\chi_{\{0\}}) = \chi_{\{0'\}}$ and $g(\chi_L) = \chi_M$. Let $\eta,\ \lambda \in FI^{\delta}(L)$. Then there are fuzzy filters μ and θ of L such that $\eta = \delta(\mu)$ and $\lambda = \delta(\theta)$. Thus $\eta \cap \lambda = \delta(\mu \cap \theta)$ and $\eta \underline{\vee} \lambda = \delta(\mu \vee \theta)$ are δ-fuzzy ideals. So $\delta(f(\mu \cap \theta))$ and $\delta(f(\mu \vee \theta))$ are δ-fuzzy ideals of $L^{'}$. Since $\mu \cap \theta \subseteq \mu$ and $\mu \cap \theta \subseteq \theta$, we have $\delta(f(\mu \cap \theta)) \subseteq \delta(f(\theta)) \cap \delta(f(\mu))$. For any $y \in L^{'}$,

$$
\begin{aligned}
(\delta(f(\mu)) \cap \delta(f(\theta)))(y) = Sup\{\mu(a) : a \in f^{-1}(y^*),\ a \in L\}\\
\wedge\, Sup\{\theta(b) : b \in f^{-1}(y^*),\ b \in L\}
\end{aligned}
$$

Since f is a homomorphism and $f(a) = y^*,\ f(b) = y^*$, we get $f(a \vee b) = y^*$. Using this fact, we have

$$
\begin{aligned}
(\delta(f(\mu)) \cap \delta(f(\theta)))(y) &\leq Sup\{\mu(a \vee b) : a \vee b \in f^{-1}(y^*)\}\\
&\quad \wedge\, Sup\{\theta(a \vee b) : a \vee b \in f^{-1}(y^*)\}\\
&= Sup\{\mu(a \vee b) \wedge \theta(a \vee b) : a \vee b \in f^{-1}(y^*)\}\\
&= Sup\{(\mu \cap \theta)(a \vee b) : a \vee b \in f^{-1}(y^*)\}\\
&\leq Sup\{(\mu \cap \theta)(c) : c \in f^{-1}(y^*)\}\\
&= f(\mu \cap \theta)(y^*)\\
&= \delta(f(\mu \cap \theta))(y)
\end{aligned}
$$

Hence $\delta(f(\mu)) \cap \delta(f(\theta)) = \delta(f(\mu \cap \theta))$. Therefore $g(\eta \cap \lambda) = g(\eta) \cap g(\lambda)$.

On the other hand, $g(\eta \underline{\vee} \lambda) = g(\delta(\mu)\underline{\vee}\delta(\theta)) = g(\delta(\mu \vee \theta)) = \delta(f(\mu \vee \theta))$. Since f is an onto homomorphism, we have $f(\mu \vee \theta) = f(\mu) \vee f(\theta)$. Then $\delta(f(\mu \vee \theta)) = \delta(f(\mu) \vee f(\theta))$. Thus

$$\begin{aligned} g(\eta \underline{\vee} \lambda) &= \delta(f(\mu) \vee f(\theta)) \\ &= \delta(f(\mu))\underline{\vee}\delta(f(\theta)) \\ &= g(\eta)\underline{\vee}g(\lambda). \end{aligned}$$

So $g(\eta \underline{\vee} \lambda) = g(\eta)\underline{\vee}g(\lambda)$. Hence g is a homomorphism. □

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