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A WEAKLY COUPLED SYSTEM OF SINGULARLY PERTURBED CONVECTION-DIFFUSION EQUATIONS WITH DISCONTINUOUS SOURCE TERM[†]

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Abstract. In this paper, we consider boundary value problem for a weakly coupled system of two singularly perturbed differential equations of convection diffusion type with discontinuous source term. In general, solution of this type of problems exhibits interior and boundary layers. A numerical method based on streamline diffusiom finite element and Shishkin meshes is presented. We derive an error estimate of order $O(N^{-2}\ln^2 N)$ in the maximum norm with respect to the perturbation parameters. Numerical experiments are also presented to support our theoritical results.

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1. Introduction

Differential equations with a small parameter $(0 < \varepsilon \ll 1)$ multiplying the highest order derivatives, termed as Singularly Perturbed Differential Equations (SPDEs), arise in diverse areas of applied mathematics, including linearized Navier - Stokes equations of high Reynolds number, heat transfer problem with large Peclet number, drift diffusion equations of semiconductor device modelling, chemical reactor theory, etc.,

In general, this type of equations exhibit boundary and/or interior layers. Standard numerical methods like finite difference and finite element methods on uniform mesh for solving this type of euations fail to produce good approximations to exact solutions. Many authors [2, 3, 12, 13, 14] have developed efficient numerical methods to resolve boundary and interior layers. A good number of

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articles have been appearing in the past three decades on non-classical methods which cover mostly single second order equation. But, a few authors only have considered system of SPDEs [6, 7, 8, 9, 11].

In this paper, we consider the following boundary value problem for coupled system of singularly perturbed second order ordinary differential equations of convection-diffusion type with discontinuous source term:

$$\begin{aligned} L_1\bar{u} &:= -\varepsilon u_1''(x) + b_1(x)u_1'(x) + a_{11}(x)u_1(x) + a_{12}(x)u_2(x) = f_1(x), \\ L_2\bar{u} &:= -\mu u_2''(x) + b_2(x)u_2'(x) + a_{21}(x)u_1(x) + a_{22}(x)u_2(x) = f_2(x), \end{aligned} \tag{1}$$

$x \in (\Omega^- \cup \Omega^+)$,

$$u_1(0) = 0 = u_1(1), \quad u_2(0) = 0 = u_2(1), \tag{2}$$

with conditions on coefficients

$$b_k(x) \geq \beta_k > 0, \quad \text{for} \quad k = 1, 2, \tag{3}$$

$$a_{ij}(x) \leq 0, \quad \text{for} \quad i, j = 1, 2 \quad \text{and} \quad i \neq j, \tag{4}$$

$$a_{11}(x) > |a_{12}(x)|, \quad a_{22}(x) > |a_{21}(x)|, \quad \forall x \in \bar{\Omega}. \tag{5}$$

$$\xi A \xi^T \geq \alpha \xi \xi^T \quad \text{for every} \quad \xi = (\xi_1, \xi_2) \in \Re^2, \quad A = [a_{ij}], \tag{6}$$

$$\text{and} \quad \alpha - \frac{1}{2} b_k' \geq \sigma_k, \quad \text{for some} \quad \alpha, \sigma_k > 0, \quad k = 1, 2, \tag{7}$$

where $0 < \varepsilon < \mu << 0$ are small parameters, $\Omega = (0, 1)$, $\Omega^- = (0, d)$, $\Omega^+ = (d, 1)$, $d \in \Omega$, and $u_1, u_2 \in U \equiv C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$, $\bar{u} = (u_1, u_2)^T$. Further it is assumed that the source terms f_1, f_2 are sufficiently smooth on $\bar{\Omega} \setminus \{d\}$; both the functions $f_1(x)$ and $f_2(x)$ are assumed to have a single discontinuity at the point $d \in \Omega$. That is $f_i(d-) \neq f_i(d+), i = 1, 2$. In general this discontinuity gives rise to interior layers in the solution of the problem. Because $f_i, i = 1, 2$ are discontinuous at d the solution $\bar{u}$ of (1) - (2) does not necessarily have a continuous second derivative at the point d. That is $u_1, u_2 \notin C^2(\Omega)$. But the first derivative of the solution exists and is continuous.

Systems of this kind have applications in electro analytic chemistry when investigating diffusion processes complicated by chemical reactions. The parameters multiplying the highest derivatives characterize the diffusion coefficient of the substances. Other applications include equations of prey-predator population dynamics. As mentioned above, in general, classical numerical methods fail to produce good approximations to singularly perturbed equations. Hence various methods are proposed in the literature in order to obtain numerical solution to singularly perturbed system of second order differential equations subject to Dirichlet type boundary conditions when the source terms are smooth [9]. In-general the Galerkin FEM even on layer adapted mesh for convection-diffusion type equations does not yield satisfactory result because of the convergence of the stability problem of this method. Hence in [4], authors suggested a SD-FEM to overcome this stability problem. This SDFEM was first introduced in [1] for a convection dominated convection-diffusion equation. The authors proposed a modification of the standard Galerkin finite element method that actually represents a Petrov-Galerkin FEM with the test functions adapted in such a way as to produce a small amount of artificial diffusion in the streamline direction, thereby enhancing stability. Therefore, this method is also known as the streamline-diffusion Petrov-Galerkin method. It can be also considered as the finite element method that adds weighted residuals to the standard Galerkin FEM. The SDFEM has been successively applied to numerical solving of single convection diffusion problem with a smooth source function. Motivated by the works of H-G. Roos et al.[4], in the present paper we suggest a numerical method for the above BVP. This method is based on Streamline - Diffusion Finite Element Method (SDFEM) with layer adapted meshes like Shishkin meshes. For this method we derive an error estimate of order $O(N^{-2}\ln^2 N)$ for Shishkin mesh, in the maximum norm. In order to capture a boundary layer with a numerical method, it is essential that the approximate solutions generated by the numerical method are defined globally at each point of the domain of the exact solution. The numerical solution obtained from a finite element method defined only at the mesh points, is extended it to the whole domain by a simple interpolation process such as piecewise linear interpolation. Because we want our technique to be capable of extension to complex problems in higher dimensions, we only consider the finite element subspaces by piecewise polynomial basis functions.

In this connection, we wish to state that the authors [7] considered the same type of problem (1) - (2) with $\varepsilon = \mu$ and proved almost first order convergence with respect to ε on a Shishkin mesh of the finite difference method with special discretization at the point d. When we compute numerical solutions, it is not desirable to obtain error estimates in L^1,L^2 or energy norm, as they do not detect the local phenomena such as boundary or interior layer. Therefore the most appropriate norm for the study of singular perturbation problem is the maximum norm [2]. The main significance of this paper is that the error estimate for numerical solution is given in terms of the maximum norm. Now we define the maximum norm of $\bar{u} = (u_1, u_2)$ as

$$\begin{gathered}
\| \bar{u} \|_\infty = \max_{i=1,2}\{\| u_i \|_\infty\}, \quad \| u_i \|_\infty = \max_{x\in[0,1]} | u_i(x) |, i = 1,2 \\
\| \bar{u} \|_{\infty[x_{i-1},x_i]} = \max\{\| u_1 \|_{\infty[x_{i-1},x_i]}, \| u_2 \|_{\infty[x_{i-1},x_i]}\}, \\
\| u_i \|_{\infty[x_{i-1},x_i]} = \max_{x\in[x_{i-1},x_i]} | u_i(x) |, i = 1,2.
\end{gathered}$$

Further we define

$$| \bar{u}(x) | = | (u_1(x), u_2(x)) | = \max(| u_1(x) |, | u_2(x) |).$$

Remark 1.1. Through out this paper, C denotes generic constants that are independent of the parameters ε, μ and N, the dimension of the discrete problem. We also assume $\varepsilon \leq CN^{-1}$ and $\mu \leq CN^{-1}$ as is generally the case in practice for convection-diffusion type equations.

For our later analysis it is useful to have a decomposition of $\bar{u}$ in the smooth part $\bar{v}$ and the layer part $\bar{w}$. That is

$$\bar{u} = \bar{v} + \bar{w}, \quad \text{where} \quad \bar{v} = (v_1, v_2), \quad \bar{w} = (w_1, w_2).$$

Theorem 1.1. *[Derivative Estimates] With the decomposition of the above, we have the sharper bounds of the solution and its derivatives of the problem (1)–(2).*

$$\textit{For} \quad j = 1, 2, \ |v_j^{(k)}(x)| \leq C, \quad k = 0(1)3 \ x \in \bar{\Omega},$$

$$|w_j(x)| \leq C \begin{cases} Ce_{1,\mu}(x), \ x \in \Omega^-, \\ Ce_{2,\mu}(x), \ x \in \Omega^+, \end{cases}$$

$$|w_1^{(k)}(x)| \leq \begin{cases} C(\varepsilon^{-k} e_{1,\varepsilon}(x) + \mu^{-k} e_{1,\mu}(x)), \ x \in \Omega^-, \\ C(\varepsilon^{-k} e_{2,\varepsilon}(x) + \mu^{-k} e_{2,\mu}(x)), \ x \in \Omega^+, \ k = 1(1)3, \end{cases}$$

$$|w_2^{(k)}(x)| \leq \begin{cases} C\mu^{-k} e_{1,\mu}(x), \ x \in \Omega^-, \\ C\mu^{-k} e_{2,\mu}(x), \ x \in \Omega^+, \ k = 1, 2, \end{cases}$$

$$|w_2^{'''}(x)| \leq \begin{cases} C\mu^{-1}(\varepsilon^{-2} e_{1,\varepsilon}(x) + \mu^{-2} e_{1,\mu}(x)), \ x \in \Omega^-, \\ C\mu^{-1}(\varepsilon^{-2} e_{2,\varepsilon}(x) + \mu^{-2} e_{2,\mu}(x)), \ x \in \Omega^+, \end{cases}$$

where

$$e_{1,\omega}(x) = e^{-\frac{\beta(d-x)}{\omega}}, \quad e_{2,\omega}(x) = e^{-\frac{\beta(1-x)}{\omega}}, \quad \omega = \varepsilon, \mu, \ \textit{and} \ \beta = \min\{\beta_1, \beta_2\}.$$

This paper is organized as follows. Section 2 presents a weak formulation of the BVP (1) -(2) and describes a finite element discretization of the problem. Section 3 presents a role of projection operator on approximation space and error representation. It also includes an analysis of the corresponding scheme on Shishkin meshes and an interpolation error in the maximum norm. In Section 4 we present a detailed error analysis of the projection operator and other error terms. The paper concludes with numerical examples.

2. Analytical results

A standard weak formulation of (1)–(2) is: Find $\bar{u} = (u_1, u_2) \in (H_0^1(\Omega))^2$ such that

$$B(\bar{u}, \bar{v}) = f^*(\bar{v}), \quad \forall \quad \bar{v} \in (H_0^1(\Omega))^2 \tag{8}$$

with

$$B(\bar{u}, \bar{v}) := (B_1(\bar{u}, \bar{v}), B_2(\bar{u}, \bar{v})) \quad \text{and} \quad f^*(\bar{v}) := (f_1^*(\bar{v}), f_2^*(\bar{v})),$$

where

$$B_1(u_1, v_1) := \varepsilon(u_1', v_1') + (b_1 u_1', v_1) + (a_{11} u_1 + a_{12} u_2, v_1), \tag{9}$$

$$B_2(u_2, v_2) := \mu(u_2', v_2') + (b_2 u_2', v_2) + (a_{21} u_1 + a_{22} u_2, v_2) \tag{10}$$

and

$$f_1^*(v_1) = (f_1, v_1), \quad f_2^*(v_2) = (f_2, v_2).$$

Here $H_0^1(\Omega)$ denotes the usual Sobolev space and $(.,.)$ is the inner product on $L^2(\Omega)$. Now we define a norm on $(H_0^1(\Omega))^2$ associated with the bilinear form $B(.,.)$, called energy norm as

$$|||\bar{u}|||_{H_0^1} = [\varepsilon|u_1|_1^2 + \mu|u_2|_1^2 + \sigma(\|u_1\|_0^2 + \|u_2\|_0^2)]^{1/2}, \tag{11}$$

where $\|u\|_0 := (u,u)^{1/2}$ is the standard norm on $L_2(\Omega)$, $\sigma = \min\{\sigma_1, \sigma_2\}$ while $|u|_1 := \|u'\|_0$ is the usual semi-norm on $H_0^1(\Omega)$. We also use the notation $\|\bar{u}\|_0 = (\|u_1\|_0^2 + \|u_2\|_0^2)^{1/2}$ for the norm in $(L_2(\Omega))^2$. It is obvious that B is a bilinear functional defined on $(H_0^1(\Omega))^2$. We now prove that it is coercive with respect to $|||.|||_{H_0^1}$, that is $|\ B(\bar{u},\bar{u})\ | \geq \frac{1}{2}|||\bar{u}|||^2_{H_0^1}$, where $|\ B(\bar{u},\bar{v})\ | = \sqrt{B_1(\bar{u},\bar{v})^2 + B_2(\bar{u},\bar{v})^2}$.

Lemma 2.1. *A bilinear functional B satisfies the coercive property with respect to $|||.|||_{H_0^1}$.*

Proof. Let $\bar{u} = (u_1, u_2) \in (H_0^1(\Omega))^2$. Then

$$\begin{aligned}
|B(\bar{u},\bar{u})| &= \sqrt{B_1(\bar{u},\bar{u})^2 + B_2(\bar{u},\bar{u})^2} \\
&\geq \frac{1}{2}[|B_1(\bar{u},\bar{u})| + |B_2(\bar{u},\bar{u})|] \\
&= \frac{1}{2}[\varepsilon(u_1', u_1') + (b_1 u_1', u_1) + (a_{11}u_1 + a_{12}u_2, u_1) + \mu(u_2', u_2') \\
&\quad +(b_2 u_2', u_2) + (a_{21}u_1 + a_{22}u_2, u_2)] \\
&\geq \frac{1}{2}[\varepsilon|u_1|_1^2 + \mu|u_2|_1^2 + \int_0^1 b_1(x)u_1'u_1dx + \int_0^1 b_2(x)u_2'u_2dx \\
&\quad +(\alpha u_1, u_1) + (\alpha u_2, u_2)] \\
&= \frac{1}{2}[\varepsilon|u_1|_1^2 + \mu|u_2|_1^2 + \int_0^1 \frac{b_1(x)}{2}\frac{d}{dx}(u_1^2) + \int_0^1 \alpha u_1^2 dx \\
&\quad + \int_0^1 \frac{b_2(x)}{2}\frac{d}{dx}(u_2^2)] + \int_0^1 \alpha u_2^2 dx \\
&= \frac{1}{2}[\varepsilon|u_1|_1^2 + \mu|u_2|_1^2 - \frac{1}{2}\int_0^1 u_1^2 d(b_1(x)) + \int_0^1 \alpha u_1^2 dx \\
&\quad -\frac{1}{2}\int_0^1 u_2^2 d(b_2(x)) + \int_0^1 \alpha u_2^2 dx] \\
&= \frac{1}{2}[\varepsilon|u_1|_1^2 + \mu|u_2|_1^2 + \int_0^1 (\alpha - \frac{1}{2}b_1'(x))u_1^2 dx \\
&\quad + \int_0^1 (\alpha - \frac{1}{2}b_2'(x))u_2^2 dx] \\
&\geq \frac{1}{2}[\varepsilon|u_1|_1^2 + \mu|u_2|_1^2 + \min\{\sigma_1,\sigma_2\}[\int_0^1 u_1^2 dx + \int_0^1 u_1^2 dx]] \\
|B(\bar{u},\bar{u})| &\geq \frac{1}{2}[\varepsilon|u_1|_1^2 + \mu|u_2|_1^2 + \sigma(\|u_1\|_0^2 + \|u_2\|_0^2)].
\end{aligned}$$

Therefore we have

$$|B(\bar{u},\bar{u})| \geq \frac{1}{2}|||\bar{u}|||^2_{H^1_0}.$$

Hence B is coercive with respect to $|||.|||_{H^1_0}$. □

Also we observe that B is continuous in the energy norm, that is, $|B(\bar{u},\bar{v})| \leq \beta^{'}|||\bar{u}|||_{H^1_0} \cdot |||\bar{v}|||_{H^1_0}$ for some $\beta^{'} > 0$. Further f^* is a bounded linear functional on $(H^1_0(\Omega))^2$. By Lax-Milgram Theorem [12], we conclude that the problem (8) has a unique solution.

2.1. Discretization of weak problem. Let $\Omega^N_\varepsilon = \{x_0, x_1, \cdots, x_N\}$ be the set of mesh points x_i, for some positive integer N. For $i \in \{1,2,\cdots,N\}$ we set $h_i = x_i - x_{i-1}$ to be the local mesh step size and for $i \in \{1,2,\cdots,N\}$ let $\bar{h_i} = (h_i + h_{i+1})/2$. For the discretization of (9)-(10) we use linear finite elements with a lumping for both B and f^*[5, 4]. We form the discrete problem as

$$\begin{aligned}
B_{1h}(\bar{u},\bar{v}) \quad &:= \quad \varepsilon(u_1', v_1') + (b_1 u_1', v_1) + \sum_{i=1}^{N-1} \bar{h_i} a_{11,i} u_{1,i} v_{1,i} + \sum_{i=1}^{N-1} \bar{h_i} a_{12,i} u_{2,i} v_{1,i} \\
&\quad + \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} \delta_{1,k}(-\varepsilon u_1'' + b_1 u_1' + a_{11}u_1 + a_{12}u_2) b_1 v_1' dx, \\
B_{2h}(\bar{u},\bar{v}) \quad &:= \quad \mu(u_2', v_2') + (b_2 u_2', v_2) + \sum_{i=1}^{N-1} \bar{h_i} a_{21,i} u_{1,i} v_{2,i} + \sum_{i=1}^{N-1} \bar{h_i} a_{22,i} u_{2,i} v_{2,i} \\
&\quad + \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} \delta_{2,k}(-\mu u_2'' + b_2 u_2' + a_{21}u_1 + a_{22}u_2) b_2 v_2^{'} dx, \\
f^*_{1h}(\bar{v}) \quad &:= \quad (f_1, v_1) + \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} \delta_{1,k} f_1 b_1 v_1^{'} dx, \\
\text{and} \quad f^*_{2h}(\bar{v}) \quad &:= \quad (f_2, v_2) + \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} \delta_{2,k} f_2 b_2 v_2^{'} dx.
\end{aligned}$$

Then we have

$$\begin{aligned}
&B_h(\bar{u},\bar{v}) := (B_{1h}(\bar{u},\bar{v}), B_{2h}(\bar{u},\bar{v})) \\
\text{and} \quad &f^*_h(\bar{v}) := (f^*_{1h}(\bar{v}), f^*_{2h}(\bar{v})).
\end{aligned}$$

Now the discrete problem of (8) is: Find $\bar{u}_h \in V_h^2$ such that

$$B_h(\bar{u}_h, \bar{v}_h) = f^*_h(\bar{v}_h), \quad \forall \quad \bar{v}_h \in V_h^2, \tag{12}$$

where $V_h^2 = V_h \times V_h$, V_h is a finite dimensional subspace of $H_0^1(\Omega)$ and the basis functions of V_h are given by

$$\phi_i(x) = \begin{cases} \frac{x-x_{i-1}}{h_i}, & x \in [x_{i-1}, x_i] \\ \frac{x_{i+1}-x}{h_{i+1}}, & x \in [x_i, x_{i+1}] \\ 0, & x \notin [x_{i-1}, x_{i+1}]. \end{cases}$$

Then $\{\bar{\Phi}_i\}_{i=1}^{2N-2}$ where $\bar{\Phi}_i = (\phi_i, 0)$ for $i = 1, 2, ..., N-1$ and $\bar{\Phi}_i = (0, \phi_{N-i+1})$ for $i = N, N+1, ..., 2N-2$, is a basis function of V_h^2. Here we define a discrete energy norm on V_h^2 associated with the bilinear form $B_h(.,.)$ as

$$\begin{aligned} |||\bar{u}_h|||_{V_h} &= [\varepsilon |u_{1h}|_1^2 + \mu |u_{2h}|_1^2 + \sigma(\|u_{1h}\|_0^2 + \|u_{2h}\|_0^2) \\ &\quad + \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \delta_{1,i} b_1^2(x_i)(u'_{1h}(x))^2 dx \\ &\quad + \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \delta_{2,i} b_2^2(x_i)(u'_{2h}(x))^2 dx]^{1/2}. \end{aligned}$$

B_h is a bilinear functional defined on V_h^2. Further we have to prove that it is coercive with respect to $|||.|||_{V_h}$, that is $|B_h(\bar{u}_h, \bar{u}_h)| \geq \varsigma |||\bar{u}_h|||_{V_h}^2$, for some $\varsigma > 0$. Also B_h is continuous in the discrete energy norm and f_h^* is a bounded linear functional on V_h. By Lax-Milgram Theorem, we conclude that the discrete problem (12) has a unique solution and it is also stable [12]. The difference scheme corresponding to the discrete problem (12) is

$$\bar{L}^N \bar{U}_i := (L_1^N \bar{U}_i, L_2^N \bar{U}_i) = (f_{1h}(\phi_i, 0), f_{2h}(0, \phi_i)), \tag{13}$$

$$U_{1,0} = 0, \quad U_{1,N} = 0, \quad U_{2,0} = 0, \quad U_{2,N} = 0, \tag{14}$$

where

$$L_1^N \bar{U}_i = -\varepsilon(D^+ U_{1,i} - D^- U_{1,i}) + \alpha_{1,i} D^+ U_{1,i} + \beta_{1,i} D^- U_{1,i} + \gamma_{1,i} U_{1,i} + \gamma'_{1,i} U_{2,i},$$

$$L_2^N \bar{U}_i = -\mu(D^+ U_{2,i} - D^- U_{2,i}) + \alpha_{2,i} D^+ U_{2,i} + \beta_{2,i} D^- U_{2,i} + \gamma_{2,i} U_{2,i} + \gamma'_{2,i} U_{1,i}.$$

Here $\bar{U}_i = (U_{1,i}, U_{2,i})$, $U_{1,i} = U_1(x_i)$, $a_{11,i} = a_{11}(x_i)$, and similarly for $U_{2,i}$, $a_{12,i}$, $a_{21,i}$, $a_{22,i}$ and $i = 1(1)N-1$,

$$\alpha_{1,i} = h_{i+1} \int_{x_i}^{x_{i+1}} (b_1 \phi'_{i+1} \phi_i + \delta_{1,i+1} b_1^2 \phi'_{i+1} \phi'_i + \delta_{1,i+1} b_1 a_{11} \phi_{i+1} \phi'_i) dx,$$

$$\beta_{1,i} = -h_i \int_{x_{i-1}}^{x_i} (b_1 \phi'_{i-1} \phi_i + \delta_{1,i} b_1^2 \phi'_{i-1} \phi'_i + \delta_{1,i} b_1 a_{11} \phi_{i-1} \phi'_i) dx,$$

$$\gamma_{1,i} = \bar{h}_i \widehat{a_{11,i}} + \int_{x_{i-1}}^{x_i} \delta_{1,i} b_1 a_{11} \phi'_i dx + \int_{x_i}^{x_{i+1}} \delta_{1,i+1} b_1 a_{11} \phi'_i dx,$$

$$\gamma'_{1,i} = \bar{h}_i a_{12,i} + \int_{x_{i-1}}^{x_i} \delta_{1,i} b_1 a_{12} \phi'_i dx + \int_{x_i}^{x_{i+1}} \delta_{1,i+1} b_1 a_{12} \phi'_i dx,$$

$$\alpha_{2,i} = h_{i+1}\int_{x_i}^{x_{i+1}} (b_2\phi'_{i+1}\phi_i + \delta_{2,i+1}b_2^2\phi'_{i+1}\phi'_i + \delta_{2,i+1}b_2a_{22}\phi_{i+1}\phi'_i)dx,$$

$$\beta_{2,i} = -h_i\int_{x_{i-1}}^{x_i} (b_2\phi'_{i-1}\phi_i + \delta_{2,i}b_2^2\phi'_{i-1}\phi'_i + \delta_{2,i}b_2a_{22}\phi_{i-1}\phi'_i)dx,$$

$$\gamma_{2,i} = \bar{h}_i\widehat{a_{22,i}} + \int_{x_{i-1}}^{x_i} \delta_{2,i}b_2a_{22}\phi'_idx + \int_{x_i}^{x_{i+1}} \delta_{2,i+1}b_2a_{22}\phi'_idx,$$

and $$\gamma'_{2,i} = \bar{h}_ia_{21,i} + \int_{x_{i-1}}^{x_i} \delta_{2,i}b_2a_{21}\phi'_idx + \int_{x_i}^{x_{i+1}} \delta_{2,i+1}b_2a_{21}\phi'_idx.$$

To preserve an $M-$ matrix of the corresponding coefficient matrix, we take

$$\widehat{a_{11,i}} = \frac{\bar{b}_1^2}{\beta_1}\|a_{11}\|_{L^\infty[x_i,x_{i+1}]}, \quad \widehat{a_{22,i}} = \frac{\bar{b}_2^2}{\beta_2}\|a_{22}\|_{L^\infty[x_i,x_{i+1}]}, \quad i = 1(1)N-1,$$

and if the local mesh step is small enough, then it is possible to choose $\delta_{k,i} = 0, k = 1,2$. In other case, we shall choose $\delta_{k,i}$ from the condition, $\alpha_{k,i}$, $i = 1(1)N-1$ of the difference scheme (13)–(14) equal to zero. Since $\delta_{k,i}$ is positive we have

$$\delta_{1,i} = \begin{cases} 0, & h_i \le \dfrac{2\varepsilon}{\bar{b}_1}, \\ |\int_{x_{i-1}}^{x_i} b_1\phi'_i\phi_{i-1}dx[\int_{x_{i-1}}^{x_i}(b_1^2\phi'_i\phi'_{i-1} + b_1a_{11}\phi_i\phi'_{i-1})dx]^{-1}|, & h_i > \dfrac{2\varepsilon}{\bar{b}_1}. \end{cases}$$

and also

$$\delta_{2,i} = \begin{cases} 0, & h_i \le \dfrac{2\mu}{\bar{b}_2}, \\ |\int_{x_{i-1}}^{x_i} b_2\phi'_i\phi_{i-1}dx[\int_{x_{i-1}}^{x_i}(b_2^2\phi'_i\phi'_{i-1} + b_2a_{22}\phi_i\phi'_{i-1})dx]^{-1}|, & h_i > \dfrac{2\mu}{\bar{b}_2}, \end{cases}$$

where $\bar{b}_1 = \|b_1\|_{L^\infty(\Omega)}$ and $\bar{b}_2 = \|b_2\|_{L^\infty(\Omega)}$. Now, the scheme is stable because the coefficient matrix is $M-$ matrix [12]. We derive the following estimates of $\delta_{1,i}$ and $\delta_{2,i}$

$$\delta_{k,i} \le \begin{cases} Ch_i & \text{for} \quad i = 1(1)\frac{N}{4} \quad \text{and} \quad i = \frac{N}{2}+1(1)\frac{3N}{4}, \\ 0 & \text{for} \quad i = \frac{N}{4}+1(1)\frac{N}{2} \quad \text{and} \quad i = \frac{3N}{4}(1)N-1, \end{cases}$$

where $k = 1,2$.

3. Error Analysis - I

Now the given discrete problem is: Find $\bar{u}_h \in V_h^2 \subset (H_0^1(\Omega))^2$ such that

$$B_h(\bar{u}_h, \bar{v}_h) = f_h^*(\bar{v}_h), \quad \forall \bar{v}_h \in V_h^2. \tag{15}$$

Since the above discrete problem has a unique solution and some interpolant $\bar{u}^I \in V_h^2$ of $\bar{u}$ is well defined. We define a biorthogonal basis of V_h^2 with respect to B_h to be the set of functions $\{\bar{\Lambda}^j\}_{j=1}^{2N-2}$ where $\bar{\Lambda}^j = (\lambda_1^j, \lambda_2^j)$ for $j = 1.2, ..., 2N-2$, which satisfies

$$B_h(\bar{\Phi}_i, \bar{\Lambda}^j) = (\delta_{ij}, \delta_{ij}) \quad \text{for} \quad i,j = 1,2,...,2N-2. \tag{16}$$

In otherwords

$$\begin{aligned} B_{1h}(\bar{\Phi}_i, \bar{\Lambda}^j) &= \delta_{ij} \quad \text{for} \quad i,j = 1,2,...,2N-2, \\ B_{2h}(\bar{\Phi}_i, \bar{\Lambda}^j) &= \delta_{ij}, \end{aligned}$$

where δ_{ij} is the Kronecker symbol. Then the components u_{1h} and u_{2h} can be uniquely represented as

$$\begin{aligned} u_{1h} &= \sum_{i=1}^{N-1} B_{1h}((u_{1h}, u_{2h}), (\lambda_1^i, \lambda_2^i))\phi_i \\ \text{and} \quad u_{2h} &= \sum_{i=1}^{N-1} B_{2h}((u_{1h}, u_{2h}), (\lambda_1^{N+i-1}, \lambda_2^{N+i-1}))\phi_i. \end{aligned}$$

Define linear transformations $P_1, P_2 : (H_0^1(\Omega))^2 \longrightarrow V_h$ such that

$$\begin{aligned} P_1\bar{u} &:= \sum_{i=1}^{N-1} B_{1h}((u_1, u_2), (\lambda_1^i, \lambda_2^i))\phi_i \\ \text{and} \quad P_2\bar{u} &:= \sum_{i=1}^{N-1} B_{2h}((u_1, u_2), (\lambda_1^{N+i-1}, \lambda_2^{N+i-1}))\phi_i. \end{aligned}$$

Let $\bar{P} = (P_1, P_2)$ and $\bar{u}_h \in V_h^2$. Then

$$\begin{aligned} \bar{P}\bar{u}_h &= (P_1\bar{u}_h, P_2\bar{u}_h) \\ &= (\sum_{i=1}^{N-1} B_{1h}((u_{1h}, u_{2h}), (\lambda_1^i, \lambda_2^i))\phi_i, \sum_{i=1}^{N-1} B_{2h}((u_{1h}, u_{2h}), (\lambda_1^{N+i-1}, \lambda_2^{N+i-1}))\phi_i) \\ &= (u_{1h}, u_{2h}). \end{aligned}$$

That is, $\bar{P}\bar{u}_h = \bar{u}_h$, $\forall \bar{u}_h \in V_h^2$.

Hence $\bar{P}$ is a projection operator on V_h^2. Now, the error $\bar{u} - \bar{u}_h$ can be written as,

$$\bar{u} - \bar{u}_h = \bar{u} - \bar{u}^I + \bar{P}(\bar{u}^I - \bar{u}) + \bar{P}\bar{u} - \bar{u}_h. \tag{17}$$

We estimate this error in the rest of this section.

3.1. Shishkin mesh. For the discretization described above we shall use a mesh of the general type introduced in [10], but here adapted for the layers at $x = d$. When $0 < \varepsilon < \mu << 1$, the solution of the problem (1)–(2) has a overlapping boundary layers at $x = d$ and $x = 1$. This necessitates the construction of layer adapted meshes at these points. Let $N > 8$ be a positive even integer and

$$\sigma_\mu = \min\{\frac{d}{2}, \frac{\mu}{\beta}\tau_0 \ln N\}, \quad \sigma_\varepsilon = \min\{\frac{1-d}{4}, \frac{\sigma_\mu}{2}, \frac{\varepsilon}{\beta}\tau_0 \ln N\}, \qquad \tau_0 \geq 2.$$

When $\sigma_\varepsilon = \frac{\sigma_\mu}{2}$, then $\mu = O(\varepsilon)$, and the result can be easily obtained. Therefore, we only consider the case $\sigma_\varepsilon < \frac{\sigma_\mu}{2}$. Let $\Omega_1 = (0, d - \sigma_\mu), \Omega_2 = (d - \sigma_\mu, d -$

$\sigma_\varepsilon), \Omega_3 = (\sigma_\varepsilon, d), \Omega_4 = (d, 1-\sigma_\mu), \Omega_5 = (1-\sigma_\mu, 1-\sigma_\varepsilon), \Omega_6 = (1-\sigma_\varepsilon, 1)$. Our mesh will be equidistant on $\bar{\Omega}_S$, where

$$\Omega_S = \Omega_1 \cup \Omega_4,$$

and graded on $\bar{\Omega}_0$ where

$$\Omega_0 = \Omega_2 \cup \Omega_3 \cup \Omega_5 \cup \Omega_6.$$

We choose the transition points to be

$$x_{N/4} = d - \sigma_\mu, \quad x_{3N/8} = d - \sigma_\varepsilon, \quad x_{3N/4} = 1 - \sigma_\mu, \quad x_{7N/8} = 1 - \sigma_\varepsilon.$$

Because of the specific layers, here we have to use four mesh generating functions φ_1, φ_2, φ_3 and φ_4 which are all continuous and piecewise continuously differentiable, with the following properties: φ_1 and φ_3 are monotonically increasing and φ_2 and φ_4 are monotonically decreasing functions and

$$\begin{aligned} \varphi_1(1/4) &= 0, & \varphi_1(3/8) &= \ln N \\ \varphi_2(3/8) &= \ln N, & \varphi_2(1/2) &= 0 \\ \varphi_3(3/4) &= 0, & \varphi_3(7/8) &= \ln N \\ \varphi_4(7/8) &= \ln N, & \varphi_4(1) &= 0. \end{aligned}$$

The mesh points are

$$x_i = \begin{cases} \frac{4i}{N}(d-\sigma_\mu), & i = 0(1)N/4 \\ d - \sigma_\mu + \frac{\tau_0}{\beta}(\mu-\varepsilon)\varphi_1(t_i), & i = N/4+1(1)3N/8 \\ d - \frac{\tau_0}{\beta}\varepsilon\varphi_2(t_i), & i = 3N/8+1(1)N/2 \\ d + \frac{4}{N}(1-d-2\sigma_\mu)(i-N/2), & i = N/2+1(1)3N/4 \\ 1 - \sigma_\mu + \frac{\tau_0}{\beta}(\mu-\varepsilon)\varphi_3(t_i), & i = 3N/4+1(1)7N/8 \\ 1 - \frac{\tau_0}{\beta}\varepsilon\varphi_4(t_i), & i = 7N/8+1(1)N, \end{cases}$$

where $t_i = i/N$. We define new functions ψ_1, ψ_2, ψ_3 and ψ_4 by

$$\varphi_i = -\ln\psi_i, \quad i = 1(1)4.$$

There are several mesh-characterizing functions ψ in the literature, but we shall use only those which correspond to Shishkin mesh with the following properties

$$\max|\psi'| \quad = \quad C\ln N,$$

and

$$\psi_1(t) = e^{-4(2t-1/2)lnN}, \qquad \psi_2(t) = e^{-4(1-2t)lnN},$$
$$\psi_3(t) = e^{-8(t-3/4)lnN}, \qquad \psi_4(t) = e^{-8(1-t)lnN}.$$

Also, on the coarse part Ω_S we have

$$h_i \le CN^{-1}.$$

It can be easily seen that on the layer part Ω_0 of the Shishkin mesh

$$h_i \le C(\mu-\varepsilon)N^{-1}\ln N, \quad x_i \in \Omega_2 \cup \Omega_5,$$

$$h_i \leq C\varepsilon N^{-1} \ln N, \quad x_i \in \Omega_3 \cup \Omega_6.$$

4. Interpolation Error

Initially we consider the interpolation error in the maximum norm and we compute the interpolation error for the components u_1 and u_2.

Lemma 4.1. *For the Shishkin mesh and $i = 1, 2$, we have*

$$|u_i(x) - u_i^I(x)| \leq \begin{cases} CN^{-2} \ln^2 N, x \in \bar{\Omega}_0 \\ CN^{-2}, x \in \bar{\Omega}_S \end{cases}$$

Proof. First we consider the case $i = 1$ for the Shishkin mesh. Let $x \in \Omega^-$. To prove the estimates, we use the decomposition of solution as smooth and layer components and triangle inequality

$$|(u_1 - u_1^I)(x)| \leq |(v_1 - v_1^I)(x)| + |(w_1 - w_1^I)(x)|. \tag{18}$$

Then the first term of (18) will be

$$\begin{aligned} |(v_1 - v_1^I)(x)| &\leq 2 \int_{x_{i-1}}^{x_i} |v_1''(t)|(t - x_{i-1})dt \\ &\leq 2C \int_{x_{i-1}}^{x_i} (t - x_{i-1})dt \\ &\leq Ch_i^2, \quad x \in [x_{i-1}, x_i]. \end{aligned}$$

If $x \in \Omega_1 \cap \Omega^-$ then $|(v_1 - v_1^I)(x)| \leq CN^{-2}$. In case $x \in \Omega_2 \cap \Omega^-$ we have

$$|(v_1 - v_1^I)(x)| \leq C(\mu - \varepsilon)^2 N^{-2} \ln^2 N.$$

When $x \in \Omega_3 \cap \Omega^-$ we get

$$|(v_1 - v_1^I)(x)| \leq C\varepsilon^2 N^{-2} \ln^2 N.$$

Again if $x \in \Omega_1 \cap \Omega^-$, the second term of (18) will be

$$\begin{aligned} |(w_1 - w_1^I)(x)| &\leq 2\|w_1(x)\|_{L_\infty[x_{i-1}, x_i]} \\ &\leq C \max_{1 \leq i \leq \frac{N}{4}} e^{\frac{-\beta(d - x_i)}{\mu}} \\ |(w_1 - w_1^I)(x)| &\leq CN^{-\tau_0}. \end{aligned}$$

Now let $x \in \Omega_2 \cap \Omega^-$ we have

$$\begin{aligned} |(w_1 - w_1^I)(x)| &\leq Ch_i^2 \|w_1(x)\|, \quad x \in [x_{i-1}, x_i] \\ &\leq Ch_i^2 \max_i e^{\frac{-\beta(d - x_i)}{\mu}} \\ |(w_1 - w_1^I)(x)| &\leq C(\mu - \varepsilon)^2 N^{-2} \ln^2 N. \end{aligned}$$

and if $x \in \Omega_3 \cap \Omega^-$ then we have

$$|(w_1 - w_1^I)(x)| \leq 2 \int_{x_{i-1}}^{x_i} |w_1^{''}(t)|(t - x_{i-1})dt, \quad x \in [x_{i-1}, x_i]$$

$$
\begin{aligned}
&\leq Ch_i^2 \max_i [\varepsilon^{-2} e^{-\frac{\beta(d-x_i)}{\varepsilon}} + \mu^{-2} e^{-\frac{\beta(d-x_i)}{\mu}}] \\
&\leq C(\varepsilon N^{-1} \ln N)^2 (\varepsilon^{-2} + \mu^{-2}) \\
&\leq CN^{-2} \ln^2 N (1 + (\varepsilon/\mu)^2).
\end{aligned}
$$

Similarly we will also obtain the same estimate on $x \in \Omega^+$. From equation (18), the result is proved for $i = 1$. When $i = 2$, the bounds of the interpolation error are same except the singular part w_2 on $\Omega_3 \cap \Omega^-$. If $x \in [x_{i-1}, x_i]$ then

$$
\begin{aligned}
|(w_2 - w_2^I)(x)| &\leq 2 \int_{x_{i-1}}^{x_i} |w_2^{''}(t)|(t - x_{i-1}) dt \\
&\leq Ch_i^2 \max_i [\mu^{-2} e^{-\frac{\beta(d-x_i)}{\mu}}] \\
&\leq C(\varepsilon N^{-1} \ln N)^2 \mu^{-2} \\
&\leq CN^{-2} \ln^2 N (\varepsilon/\mu)^2.
\end{aligned}
$$

Hence the lemma is proved. □

In the later analysis, the following estimates will be used

$$
e_{1,\omega}(x) \leq \begin{cases} C, & x \in \Omega^- \cap \Omega_0 \\ CN^{-\tau_0}, & x \in \Omega^- \cap \Omega_S, \end{cases} \quad e_{2,\omega}(x) \leq \begin{cases} C, & x \in \Omega^+ \cap \Omega_0 \\ CN^{-\tau_0}, & x \in \Omega^+ \cap \Omega_S, \end{cases} \tag{19}
$$

where $\omega = \varepsilon, \mu$.

5. Error Analysis - II

The difference scheme (12) can be rewritten as,

$$
\begin{cases} \frac{-\varepsilon}{\hbar_i}(p_{1,i+1}(\frac{U_{1,i+1}-U_{1,i}}{h_{i+1}}) - p_{1,i}(\frac{U_{1,i}-U_{1,i-1}}{h_i})) + r_{1,i}(\frac{U_{1,i}-U_{1,i-1}}{\hbar_i}) + q_{1,i}U_{1,i} \\ +q'_{1,i}U_{2,i} = f^*_{1h,i}, \quad i = 1(1)N-1, \end{cases}
$$

$$
\begin{cases} \frac{-\mu}{\hbar_i}(p_{2,i+1}(\frac{U_{2,i+1}-U_{2,i}}{h_{i+1}}) - p_{2,i}(\frac{U_{2,i}-U_{2,i-1}}{h_i})) + r_{2,i}(\frac{U_{2,i}-U_{2,i-1}}{\hbar_i}) + q_{2,i}U_{2,i} \\ +q'_{2,i}U_{1,i} = f^*_{2h,i}, \quad i = 1(1)N-1, \end{cases}
$$

where $p_{1,i} = 1 - \frac{\alpha_{1,i-1}}{\varepsilon}$, $q_{1,i} = \frac{\gamma_{1,i}}{\hbar_i}$, $r_{1,i} = \frac{\alpha_{1,i-1}+\beta_{1,i}}{h_i}$, $q'_{1,i} = \frac{\gamma'_{1,i}}{\hbar_i}$ and $p_{2,i} = 1 - \frac{\alpha_{2,i-1}}{\mu}$, $q_{2,i} = \frac{\gamma_{2,i}}{\hbar_i}$, $r_{2,i} = \frac{\alpha_{2,i-1}+\beta_{2,i}}{h_i}$, $q'_{2,i} = \frac{\gamma'_{2,i}}{\hbar_i}$, $i = 1(1)N-1$. For further analysis of (17), we shall need additional assumptions for the mesh.

Remark 5.1. If $\tau_0 N^{-1} \max |\varphi'| \leq \dfrac{2(1-p)\beta}{\max\{\|b_1\|_{L^\infty(\Omega)}, \|b_2\|_{L^\infty(\Omega)}\}}$, $\varepsilon < d\|b_1\|_{L^\infty(\Omega)} N^{-1}$ and $\mu < d\|b_2\|_{L^\infty(\Omega)} N^{-1}$ for some $0 < p < 1$ then

$$
p_{k,i} \geq p > 0, \quad r_{k,i} \geq \frac{\beta_k}{2} > 0, \quad q_{k,i} + q'_{k,i} > 0, \quad k = 1, 2.
$$

The above remark gives raise to estimate the discrete Green's function $\bar{\Lambda}^j = (\lambda_1^j, \lambda_2^j)$ and now we can apply Lemma 5.3, from [16] to estimate of this discrete Green's function. For this we define $\|\lambda^j\|_{L^1(\Omega)} = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \mid \lambda^j \mid dx$.

Lemma 5.1. *On an arbitrary mesh, the discrete Green's function $\bar{\Lambda}^j = (\lambda_1^j, \lambda_2^j)$, defined as the solution of the discrete problem (12), satisfies $\|\lambda_k^j\|_{L^\infty(\bar{\Omega})} \le C$ and $\|(\lambda_k^j)'\|_{L^1(\Omega)} \le C$ for $k = 1, 2$.*

Proof. Following the procedure adapted in [5], we can prove this lemma. □

Remark 5.2. Since $\lambda_k^j \in V_h$, $k = 1, 2$, the corresponding matrix of the difference scheme derived from (16) is an $M-$ matrix and the discrete maximum principle for that scheme we have $\lambda_{k,i}^j > 0$, it follows that

$$\begin{aligned}
\|\lambda_k^j\|_{L^1(\Omega)} &= \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \lambda_k^j dx \\
&= \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (\lambda_{k,i-1}^j \phi_{i-1} + \lambda_{k,i}^j \phi_i) dx \\
&= \sum_{i=1}^{N} \frac{h_i}{2} (\lambda_{k,i-1}^j + \lambda_{k,i}^j) = \sum_{i=1}^{N} \bar{h}_i \lambda_{k,i}^j.
\end{aligned}$$

If $\|\lambda_k^j\|_{L^\infty(\bar{\Omega})} \le C$, then from the previous estimate we have $\|\lambda_k^j\|_{L^1(\Omega)} \le C$.

We now proceed to estimate the remaining parts of the equation (17). Then we estimate $|u_j(x) - u_{jh}(x)|$ for each interval $[x_{i-1}, x_i]$.

5.1. Projection Error. Let $x_i \in \bar{\Omega}_\varepsilon^N$ be a mesh point. From equation (17), the projection error at the points of the mesh is

$$\bar{P}(\bar{u}^I - \bar{u})(x_i) = (P_1(\bar{u}^I - \bar{u})(x_i), P_2(\bar{u}^I - \bar{u})(x_i)).$$

Each of the components of the above will be estimated seperately. We have

$$\begin{aligned}
P_1(\bar{u}^I - \bar{u})(x_i) &= B_{1h}(((u_1^I - u_1), (u_2^I - u_2)), (\lambda_1^i, \lambda_2^i)) \\
&= \varepsilon((u_1^I - u_1)', \lambda_1^{i'}) + (b_1(u_1^I - u_1)', \lambda_1^i) \\
&+ \sum_{j=1}^{N-1} \bar{h}_j a_{11}(x_j)(u_{1,j}^I - u_{1,j}) \lambda_{1,j}^i \\
&+ \sum_{j=1}^{N-1} \bar{h}_j a_{12}(x_j)(u_{2,j}^I - u_{2,j}) \lambda_{1,j}^i + \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} \delta_{1j}(-\varepsilon(u_1^I - u_1)'' \\
&+ b_1(u_1^I - u_1)' + a_{11}(u_1^I - u_1) + a_{12}(u_2^I - u_2)) b_1(\lambda_1^i)' dx.
\end{aligned}$$

We seperately consider each integral in the above equation. Applying the integration by parts to the first integral in $P_1(\bar{u}^I - \bar{u})(x_i)$ and using the properties

$u_1^I(x_i) = u_1(x_i), \quad u_2^I(x_i) = u_2(x_i)$ for $i = 1(1)N-1$ and $(\lambda_1^i)'' = 0$, we obtain

$$\int_0^1 \varepsilon (u_1^I - u_1)^{'} (\lambda_1^i)' dx = 0.$$

The second integral of the above equation will be transformed in to

$$\begin{aligned} \mid \int_0^1 b_1 (u_1^I - u_1)^{'} \lambda_1^i dx \mid &= \mid -\int_0^1 (u_1^I - u_1)(b_1^{'} \lambda_1^i + b_1 (\lambda_1^i)^{'}) dx \mid \\ &= \mid -\sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} (u_1^I - u_1)(b_1^{'} \lambda_1^i + b_1 (\lambda_1^i)^{'}) dx \mid \\ &\le C \| u_1^I - u_1 \|_{L^\infty(\bar{\Omega})} (\| \lambda_1^k \|_{L^\infty(\bar{\Omega})} + \| (\lambda_1^i)^{'} \|_{L^1(\bar{\Omega})}) \\ &\le C \| u_1^I - u_1 \|_{L^\infty(\bar{\Omega})}. \end{aligned}$$

For the analysis of the remaining part of $P_1(\bar{u}^I - \bar{u})(x_i)$, we use the decomposition of $\bar{u}$ and Theorem 1.1. For the first component v_1 of the smooth part

$$\begin{aligned} \mid \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} \varepsilon \delta_{1,j} v_1'' b_1 (\lambda_1^i)' dx \mid &\le C\varepsilon \mid \delta_{1,j} \mid \| v_1^{''} \|_{L^\infty(\bar{\Omega})} \| (\lambda_1^i)' \|_{L^1(\bar{\Omega})} \\ &\le C\varepsilon N^{-1}. \end{aligned}$$

For the layer part w_1 of the first component of the solution, we need the following estimate of $|(\lambda_1^i)'(x)|$, $x \in [x_{j-1}, x_j]$. Then we have

$$|(\lambda_1^i)'(x)| = \frac{1}{h_j} \mid \lambda_1^i(x_j) - \lambda_1^i(x_{j-1}) \mid \le CN \| \lambda_1^i \|_{L^\infty(\bar{\Omega})} \le CN.$$

On $x_i \in \bar{\Omega}_S$, we have $h_j \ge 2\max\{d, 1-d\} N^{-1}$ and hence

$$\begin{aligned} &\mid \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} \varepsilon \delta_{1,j} w_1^{''} b_1 (\lambda_1^i)^{'} dx \mid \\ \le\ & C\varepsilon N^{-1} \Big(\sum_{j=1}^{\frac{N}{4}} \int_{x_{j-1}}^{x_j} \mid w_1^{''} \mid\mid (\lambda_1^i)^{'} \mid dx + \sum_{j=\frac{N}{2}+1}^{\frac{3N}{4}} \int_{x_{j-1}}^{x_j} \mid w_1^{''} \mid\mid (\lambda_1^i)^{'} \mid dx \Big) \\ \le\ & C\varepsilon N^{-1} |(\lambda_1^i)^{'}| \Big([\varepsilon^{-2} \sum_{j=1}^{\frac{N}{4}} \int_{x_{j-1}}^{x_j} e^{\frac{-\beta(d-x)}{\varepsilon}} dx + \mu^{-2} \sum_{j=1}^{\frac{N}{4}} \int_{x_{j-1}}^{x_j} e^{\frac{-\beta(d-x)}{\mu}} dx] \\ & + [\varepsilon^{-2} \sum_{\frac{N}{2}+1}^{\frac{3N}{4}} \int_{x_{j-1}}^{x_j} e^{\frac{-\beta(1-x)}{\varepsilon}} dx + \mu^{-2} \sum_{j=\frac{N}{2}+1}^{\frac{3N}{4}} \int_{x_{j-1}}^{x_j} e^{\frac{-\beta(1-x)}{\mu}} dx] \Big) \\ \le\ & C \Big(\sum_{j=1}^{\frac{N}{4}} [e^{\frac{-\beta(d-x)}{\varepsilon}}]_{x_{j-1}}^{x_j} + \frac{\varepsilon}{\mu} \sum_{j=1}^{\frac{N}{4}} [e^{\frac{-\beta(d-x)}{\mu}}]_{x_{j-1}}^{x_j}] + \sum_{j=\frac{N}{2}+1}^{\frac{3N}{4}} [e^{\frac{-\beta(1-x)}{\varepsilon}}]_{x_{j-1}}^{x_j} \end{aligned}$$

$$
\begin{aligned}
&+\frac{\varepsilon}{\mu}\sum_{\frac{N}{2}+1}^{\frac{3N}{4}}[e^{\frac{-\beta(1-x)}{\mu}}]_{x_{j-1}}^{x_j}])\\
\leq\;& C(1+\frac{\varepsilon}{\mu})N^{1-\tau_0}, \quad [\text{since}, \quad \varepsilon\leq\mu],\\
\leq\;& CN^{1-\tau_0}.
\end{aligned}
$$

$$
\begin{aligned}
|\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}\delta_{1,j}b_1^2(u_1^I-u_1)'(\lambda_1^i)'dx\,| &= |-\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}\varepsilon\delta_{1,j}(u_1^I-u_1)(b_1^2(\lambda_1^i)')'dx|\\
&\leq CN^{-1}\|u_1^I-u_1\|_{L^\infty(\bar\Omega)},
\end{aligned}
$$

$$
\begin{aligned}
|\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}\delta_{1,j}a_{11}b_1(u_1^I-u_1)(\lambda_1^i)'dx| &\leq CN^{-1}\|u_1^I-u_1\|_{L^\infty(\bar\Omega)}\|\lambda_1^i\|_{L^1(\bar\Omega)}\\
&\leq CN^{-1}\|u_1^I-u_1\|_{L^\infty(\bar\Omega)},
\end{aligned}
$$

and

$$
|\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}\delta_{1,j}a_{12}b_1(u_2^I-u_2)(\lambda_1^i)'dx\,| \leq CN^{-1}\|u_2^I-u_2\|_{L^\infty(\bar\Omega)}.
$$

Using the results of Lemma 4.1 and combining the above estimates, we get

$$
P_1(\bar u^I-\bar u)(x_i)\leq CN^{-2}\ln^2 N + C\varepsilon N^{-1}+CN^{1-\tau_0}.
$$

And now we have to estimate the second component $P_2(\bar u^I-\bar u)(x_i)$ as follows

$$
\begin{aligned}
P_2(\bar u^I-\bar u)(x_i) &= B_{2h}(((u_1^I-u_1),(u_2^I-u_2)),(\lambda_1^i,\lambda_2^i))\\
&= \mu((u_2^I-u_2)',\lambda_2^{i'})+(b_2(u_2^I-u_2)',\lambda_2^i)\\
&\quad+\sum_{j=1}^{N-1}\bar h_j a_{21,j}(u_{1,j}^I-u_{1,j})\lambda_{2,j}^i\\
&\quad+\sum_{j=1}^{N-1}\bar h_j a_{22,j}(u_{2,j}^I-u_{2,j})\lambda_{2,j}^i+\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}\delta_{2,j}(-\mu(u_2^I-u_2)''\\
&\quad+b_2(u_2^I-u_2)'+a_{21}(u_1^I-u_1)+a_{22}(u_2^I-u_2))b_2(\lambda_2^i)'dx.
\end{aligned}
$$

For estimating the above terms, we follow the similar procedure adapted for the first component of projection operator P_1. Using the properties of λ_2^i, we get the same bounds for all the terms except $|\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}\mu\delta_{2j}w_2''b_2(\lambda_2^i)'dx|$. For the layer part w_2 of the solution component u_2, we have

$$
|\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}\mu\delta_{2,j}w_2''b_2(\lambda_2^i)'dx|
$$

$$\begin{aligned}
&\leq C\mu N^{-1}(\sum_{j=1}^{\frac{N}{4}}\int_{x_{j-1}}^{x_j}|w_2''||(\lambda_2^i)'|dx+\sum_{j=\frac{N}{2}+1}^{\frac{3N}{4}}\int_{x_{j-1}}^{x_j}|\,w_2''||(\lambda_2^i)'|dx)\\
&\leq C\mu N^{-1}|(\lambda_2^i)'|([\mu^{-2}\sum_{j=1}^{\frac{N}{4}}\int_{x_{j-1}}^{x_j}e^{\frac{-\beta(d-x)}{\mu}}dx]+[\mu^{-2}\sum_{j=\frac{N}{2}+1}^{\frac{3N}{4}}\int_{x_{j-1}}^{x_j}e^{\frac{-\beta(1-x)}{\mu}}dx])\\
&\leq C(\sum_{j=1}^{\frac{N}{4}}[e^{\frac{-\beta(d-x)}{\mu}}]_{x_{j-1}}^{x_j}]+\sum_{\frac{N}{2}+1}^{\frac{3N}{4}}[e^{\frac{-\beta(1-x)}{\mu}}]_{x_{j-1}}^{x_j}])\\
&\leq CN^{1-\tau_0}.
\end{aligned}$$

Finally, we arrive at the following estimate for P_2

$$P_2(\bar{u}^I-\bar{u})(x_i)\leq CN^{-2}\ln^2 N+C\mu N^{-1}+CN^{1-\tau_0}.$$

Since $|\,\bar{P}(\bar{u}^I-\bar{u})(x_i)\,|=\max(|\,P_1(\bar{u}^I-\bar{u})(x_i)\,|,|\,P_2(\bar{u}^I-\bar{u})(x_i)\,|)$, we have

$$|\,\bar{P}(\bar{u}^I-\bar{u})(x_i)\,|\leq CN^{-2}\ln^2 N+C(\varepsilon+\mu)N^{-1}+CN^{1-\tau_0}. \tag{20}$$

The remaining part in the error representation in (17) is consistency error.

5.2. Consistency Error. Let $\bar{K}=(K_1,K_2)=\bar{P}\bar{u}-\bar{u}_h=((P_1\bar{u}-u_{1h}),(P_2\bar{u}-u_{2h}))$. That is, $K_1=P_1\bar{u}-u_{1h}$ and $K_2=P_2\bar{u}-u_{2h}$. Now,

$$\begin{aligned}
K_1 &= P_1\bar{u}-u_{1h}\\
&= \sum_{i=1}^{N-1}B_{1h}((u_1,u_2),(\lambda_1^i,\lambda_2^i))\phi_i-\sum_{i=1}^{N-1}B_{1h}((u_{1h},u_{2h}),(\lambda_1^i,\lambda_2^i))\phi_i\\
&= \sum_{i=1}^{N-1}B_{1h}((u_1,u_2),(\lambda_1^i,\lambda_2^i))\phi_i+\sum_{i=1}^{N-1}f_1((\lambda_1^i,\lambda_2^i))\phi_i\\
&\quad -\sum_{i=1}^{N-1}B_1((u_1,u_2),(\lambda_1^i,\lambda_2^i))\phi_i-\sum_{i=1}^{N-1}B_{1h}((u_{1h},u_{2h}),(\lambda_1^i,\lambda_2^i))\phi_i\\
&= \sum_{i=1}^{N-1}(B_{1h}-B_1)((u_1,u_2),(\lambda_1^i,\lambda_2^i))-\sum_{i=1}^{N-1}f_{1h}((\lambda_1^i,\lambda_2^i))\phi_i\\
&\quad +\sum_{i=1}^{N-1}f_1((\lambda_1^i,\lambda_2^i))\phi_i.\\
K_1 &= \sum_{i=1}^{N-1}(B_{1h}-B_1)((u_1,u_2),(\lambda_1^i,\lambda_2^i)\phi_i+\sum_{i=1}^{N-1}(f_1-f_{1h})((\lambda_1^i,\lambda_2^i))\phi_i.
\end{aligned}$$

Then we have

$$K_1(x_i) = (B_{1h}-B_1)((u_1,u_2),(\lambda_1^i,\lambda_2^i))+(f_1-f_{1h})((\lambda_1^i,\lambda_2^i))$$

$$\begin{aligned} &= B_{1h}((u_1,u_2),(\lambda_1^i,\lambda_2^i)) \\ &\quad -B_1((u_1,u_2),(\lambda_1^i,\lambda_2^i)) + f_1((\lambda_1^i,\lambda_2^i)) - f_{1h}((\lambda_1^i,\lambda_2^i)) \\ K_1(x_i) &= (\sum_{j=1}^{N-1} \bar{h}_j a_{11,j} u_{1,j} \lambda_{1,j}^i - \int_0^1 a_{11} u_1 \lambda_1^i dx) + (\sum_{j=1}^{N-1} \bar{h}_j a_{12,j} u_{2,j} \lambda_{1,j}^i \\ &\quad - \int_0^1 a_{12} u_2 \lambda_1^i dx) + (\int_0^1 f_1 \lambda_1^i dx - \sum_{j=1}^{N-1} \bar{h}_j f_{1,j} \lambda_{1,j}^i), \end{aligned}$$

where $u_{1,i} = u_1(x_i)$, $u_{2,i} = u_2(x_i)$ and $a_{12,i} = a_{12}(x_i)$. Similarly we get

$$\begin{aligned} K_2(x_i) &= (\sum_{j=1}^{N-1} \bar{h}_j a_{21,j} u_{1,j} \lambda_{2,j}^{N+i-1} - \int_0^1 a_{21} u_1 \lambda_2^{N+i-1} dx) \\ &\quad +(\sum_{j=1}^{N-1} \bar{h}_j a_{22,j} u_{2,j} \lambda_{2,j}^{N+i-1} - \int_0^1 a_{22} u_2 \lambda_2^{N+i-1} dx) \\ &\quad +(\int_0^1 f_2 \lambda_2^{N+i-1} dx - \sum_{j=1}^{N-1} \bar{h}_j f_{2,j} \lambda_{2,j}^{N+i-1}). \end{aligned}$$

Now we define

$$\begin{aligned} K_1^*(x_i) &= \sum_{j=1}^{N-1} \bar{h}_j a_{11,j} u_{1,j} \lambda_{1,j}^i - \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} (a_{11}u_1)^I \lambda_1^i dx + \sum_{j=1}^{N-1} \bar{h}_j a_{12,j} u_{2,j} \lambda_{1,j}^i \\ &\quad - \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} (a_{12}u_2)^I \lambda_1^i dx + \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} f_1^I \lambda_1^i dx - \sum_{j=1}^{N-1} \bar{h}_j f_{1,j} \lambda_{1,j}^i. \end{aligned}$$

Then we can write $K_1(x_i)$ as

$$K_1(x_i) = \begin{cases} K_1^*(x_i) + \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} ((a_{11}u_1)^I - (a_{11}u_1))\lambda_1^i dx + \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} ((a_{12}u_2)^I \\ -(a_{12}u_2))\lambda_1^i dx - \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} ((f_1)^I - f_1)\lambda_1^i dx. \end{cases} \tag{21}$$

The later sums of $K_1(x_i)$ can be bounded by

$$\begin{aligned} &|\sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} ((a_{11}u_1)^I - (a_{11}u_1))\lambda_1^i dx| \\ \leq\ & C(\|u_1 - u_1^I\|_{L^\infty(\bar{\Omega})} \|a_{11}\|_{L^\infty(\bar{\Omega})} + \|a_{11}^I - a_{11}\|_{L^\infty(\bar{\Omega})} \|u_1\|_{L^\infty(\bar{\Omega})}) \|\lambda_1^i\|_{L^1(\Omega)} \\ \leq\ & C(\|u_1 - u_1^I\|_{L^\infty(\bar{\Omega})} + N^{-2} \|u_1\|_{L^\infty(\bar{\Omega})}) \|\lambda_1^i\|_{L^1(\Omega)} \\ \leq\ & C(\|u_1 - u_1^I\|_{L^\infty(\bar{\Omega})} + N^{-2}) \\ \leq\ & CN^{-2} \max |\ \psi'\ |^2 \\ \leq\ & CN^{-2} \ln^2 N, \end{aligned}$$

$$|\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}((a_{12}u_2)^I-(a_{12}u_2))\lambda_1^i dx| \leq CN^{-2}\ln^2 N,$$

and

$$|\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}(f_1-f_1^I)\lambda_1^i dx| \quad \leq CN^{-2}\|\lambda_1^i\|_{L^1(\Omega)} \leq CN^{-2}.$$

If we define $K_2^*(x_i)$ similar to $K_1^*(x_i)$, then we can write

$$K_2(x_i)=\begin{cases} K_2^*(x_i)+\sum_{j=1}^N\int_{x_{j-1}}^{x_j}((a_{21}u_1)^I-(a_{21}u_1))\lambda_2^{N+i-1}dx \\ \qquad +\sum_{j=1}^N\int_{x_{j-1}}^{x_j}((a_{22}u_2)^I \\ -(a_{22}u_2))\lambda_2^{N+i-1}dx-\sum_{j=1}^N\int_{x_{j-1}}^{x_j}((f_2)^I-f_2)\lambda_2^{N+i-1}dx. \end{cases} \tag{22}$$

We can also estimate the later sums of $K_2(x_i)$ as done for $K_1(x_i)$. In the pointwise errors, $K_1(x_i)$ and $K_2(x_i)$ it remains only to estimate the expressions $K_1^*(x_i)$ and $K_2^*(x_i)$. First we write $K_1^*(x_i)$ and $K_2^*(x_i)$ in the form

$$K_1^*(x_i) =< (a_{11}u_1)^I, \lambda_1^i >_h + < (a_{12}u_2)^I, \lambda_1^i >_h - < f_1^I, \lambda_1^i >_h, \tag{23}$$

$$K_2^*(x_i) =< (a_{21}u_1)^I, \lambda_2^{N+i-1} >_h + < (a_{22}u_2)^I, \lambda_2^{N+i-1} >_h - < f_2^I, \lambda_2^{N+i-1} >_h, \tag{24}$$

where

$$< g, \omega^i >_h = \sum_{j=1}^{N-1}\bar{h}_j g(x_j)\omega_j^i - \sum_{j=1}^{N}\int_{x_{j-1}}^{x_j} g(x)\omega^i(x)dx,$$

for a piecewise linear function g, not neccessarily continuous. For integrals in the previous formula, we use Simpson's rule

$$< g, \omega^k >_h = \frac{1}{6}\sum_{i=1}^{N-1}(h_i(g_i^- - g_{i-1}^+) - h_{i+1}(g_{i+1}^- - g_i^+))\omega_i^k. \tag{25}$$

In order to estimate $K_1^*(x_k)$ and $K_2^*(x_k)$, we start with the decomposition of the solution $\bar{u}$. Hence we seperately analyze smooth part $\bar{v}$ and the layer part $\bar{w}$. Now the equation (23) can be rewritten as

$$K_1^*(x_k) =< (a_{11}v_1+a_{12}v_2)^I, \lambda_1^k >_h + < (a_{11}w_1+a_{12}w_2)^I, \lambda_1^k >_h - < f_1^I, \lambda_1^i >_h . \tag{26}$$

First, we estimate the third term of the equation (26)

$$\begin{aligned} |\langle f_1^I, \lambda_1^k\rangle_h| &\leq C\sum_{i=1}^{N-1}|h_i(f_{1,i}^- - f_{1,i-1}^+) - h_{i+1}(f_{1,i+1}^- f_{1,i}^+)|\lambda_{1,i}^k \\ &\leq C\sum_{i=1}^{N-1}|h_i^2 f_1'(\xi_i) - h_{i+1}^2 f_1'(\xi_{i+1})|\lambda_{1,i}^k, \quad \xi_i\in[x_{i-1},x_i] \\ &\leq CN^{-2}\|f_1''\|_{L^\infty(\Omega)}\sum_{i=1}^{N-1}(\xi_{i+1}-\xi_i)\lambda_{1,i}^k \end{aligned}$$

$$\leq \quad CN^{-2}\|f_1''\|_{L^\infty(\Omega)}\|\lambda_1^k\|_{L^\infty(\Omega)},$$

since $h_i \leq CN^{-1}$, $\|\lambda_1^k\|_{L^\infty(\Omega)} \leq C$ and $\|f_1''\|_{L^\infty(\Omega)} \leq C$. Finally, we get

$$|\langle f_1^I, \lambda^i\rangle_h| \leq CN^{-2}. \tag{27}$$

Now, the first term in the above expression containing the regular component v_1 and v_2, that can be easily estimated. In fact,

$$\begin{aligned}
&| < (a_{11}v_1 + a_{12}v_2)^I, \lambda_1^k >_h | \\
\leq\quad & C[\|a_{11}\|_{L^\infty(\bar\Omega)} \sum_{i=1}^{N-1} |h_i(v_{1,i}^- - v_{1,i-1}^+) - h_{i+1}(v_{1,i+1}^- - v_{1,i}^+)|\lambda_{1,i}^k \\
& +\|a_{12}\|_{L^\infty(\bar\Omega)} \sum_{i=1}^{N-1} |h_i(v_{2,i}^- - v_{2,i-1}^+) - h_{i+1}(v_{2,i+1}^- - v_{2,i}^+)|\lambda_{1,i}^k] \\
\leq\quad & C[\sum_{i=1}^{N-1} |h_i^2 v_1'(\xi_i) - h_{i+1}^2 v_1'(\xi_{i+1})|\lambda_{1,i}^k + \sum_{i=1}^{N-1} |h_i^2 v_2'(\xi_i) - h_{i+1}^2 v_2'(\xi_{i+1})|\lambda_{1,i}^k] \\
& ,\xi_i \in [x_{i-1}, x_i] \\
\leq\quad & C[N^{-2}\|v_1''\|_{L^\infty(\bar\Omega)}] \sum_{i=1}^{N-1} (\xi_{i+1} - \xi_i)\lambda_{1,i}^k \\
\leq\quad & CN^{-2}\|\lambda_1^k\|_{L^\infty(\bar\Omega)},
\end{aligned}$$

by using Theorem 1.1, $h_i \leq CN^{-1}, i = 1(1)N-1$ and $\|\lambda_1^k\|_{L^\infty(\bar\Omega)} \leq C$. Finally, we get

$$| < (a_{11}v_1 + a_{12}v_2)^I, \lambda_1^k >_h | \leq CN^{-2}. \tag{28}$$

Now, Let us denote the coefficient in $< (a_{11}w_1 + a_{12}w_2)^I, \lambda_1^k >_h$ corresponding to $\lambda_{1,i}^k$ by m_i. Depending on the values of index i, we consider different cases. In general, $g_i^\pm$ denotes right-limit and left-limit of a function g at a mesh point x_i.

Case 1: When $1 \leq i \leq \frac{N}{4}-1$ or $\frac{N}{2}+1 \leq i \leq \frac{3N}{4}-1$. That is, $[x_{i-1}, x_{i+1}] \subset \Omega_S$. The coefficient m_i can be estimated by

$$\begin{aligned}
|m_i| \quad &= \quad | h_i(a_{11,i}^- w_{1,i}^- - a_{11,i-1}^+ w_{1,i-1}^+) - h_{i+1}(a_{11,i+1}^- w_{1,i+1}^- - a_{11,i}^+ w_{1,i}^+) | \\
& \quad + | h_i(a_{12,i}^- w_{2,i}^- - a_{12,i-1}^+ w_{2,i-1}^+) - h_{i+1}(a_{12,i+1}^- w_{2,i+1}^- - a_{12,i}^+ w_{2,i}^+) | \\
&\leq \quad C\bar h_i[\|w_1\|_{L^\infty[x_{i-1},x_{i+1}]} + \|w_2\|_{L^\infty[x_{i-1},x_{i+1}]}] \\
&\leq \quad C\bar h_i[\max_{x\in\Omega_S} |e_{1,\mu}(x)| + \max_{x\in\Omega_S} |e_{2,\mu}(x)|], \quad \text{from Theorem 1.1 and (19)},
\end{aligned}$$

$$| m_i | \leq C\bar h_i N^{-\tau_0}. \tag{29}$$

Case 2: When $\frac{N}{4}+1 \leq i \leq \frac{3N}{8}-1$ or $\frac{3N}{8}+1 \leq i \leq \frac{N}{2}-1$ or $\frac{3N}{4}+1 \leq i \leq \frac{7N}{8}-1$. or $\frac{7N}{8}+1 \leq i \leq N-1$. That is, the subinterval $[x_{i-1}, x_{i+1}] \subset \Omega_0$. The layer part will be calculated by estimating m_i. We have

$$\begin{aligned}
m_i \quad = \quad & h_i(a_{11,i}^- w_{1,i}^- - a_{11,i-1}^+ w_{1,i-1}^+) - h_{i+1}(a_{11,i+1}^- w_{1,i+1}^- - a_{11,i}^+ w_{1,i}^+) \\
& +h_i(a_{12,i}^- w_{2,i}^- - a_{12,i-1}^+ w_{2,i-1}^+) - h_{i+1}(a_{12,i+1}^- w_{2,i+1}^- - a_{12,i}^+ w_{2,i}^+)
\end{aligned}$$

$$
\begin{aligned}
&= h_i(-a_{11,i+1}w_{1,i+1} + 2a_{11,i}w_{1,i} - a_{11,i-1}w_{1,i-1}) + (h_i - h_{i+1})(a_{11,i+1}w_{1,i+1} \\
&\quad -a_{11,i}w_{1,i}) + h_i(-a_{12,i+1}w_{2,i+1} + 2a_{12,i}w_{2,i} - a_{12,i-1}w_{2,i-1}) + (h_i - h_{i+1}) \\
&\quad (a_{12,i+1}w_{2,i+1} - a_{12,i}w_{2,i}) \\
&= a_{11,i}(h_i(-w_{1,i+1} + 2w_{11,i} - w_{1,i-1}) + (h_i - h_{i+1})(w_{1,i+1} - w_{1,i})) \\
&\quad +h_i(a_{11,i} - a_{11,i-1})(w_{1,i-1} - w_{1,i}) + h_{i+1}(a_{11,i+1} - a_{11,i})(w_{1,i} - w_{1,i+1}) \\
&\quad +w_{1,i}(-h_{i+1}a_{11,i+1} + (h_i + h_{i+1})a_{11,i} - h_i a_{11,i-1})a_{12,i} \\
&\quad \times(h_i(-w_{2,i+1} + 2w_{21,i} - w_{2,i-1}) + (h_i - h_{i+1})(w_{2,i+1} - w_{2,i})) \\
&\quad +h_i(a_{12,i} - a_{12,i-1})(w_{2,i-1} - w_{2,i}) + h_{i+1}(a_{12,i+1} - a_{12,i})(w_{2,i} - w_{2,i+1}) \\
&\quad +w_{2,i}(-h_{i+1}a_{12,i+1} + (h_i + h_{i+1})a_{12,i} - h_i a_{12,i-1}).
\end{aligned}
$$

Using the Taylor's expansion for each of the terms in the previous expression yields

$$
\begin{aligned}
h_i a_{11,i}(-w_{1,i+1} + 2w_{1,i} - w_{1,i-1}) &= h_i(h_i - h_{i+1})a_{11,i}w_{1,i}^{'} - \frac{h_i^3}{2}a_{11,i}w_1^{''}(\theta_i) \\
&\quad -\frac{h_i h_{i+1}^2}{2}a_{11,i}w_1^{''}(\theta_{i+1}), \\
h_i a_{12,i}(-w_{2,i+1} + 2w_{2,i} - w_{2,i-1}) &= h_i(h_i - h_{i+1})a_{12,i}w_{2,i}^{'} - \frac{h_i^3}{2}a_{12,i}w_2^{''}(\theta_i) \\
&\quad -\frac{h_i h_{i+1}^2}{2}a_{12,i}w_2^{''}(\theta_{i+1}), \\
(h_i - h_{i+1})a_{11,i}(w_{1,i+1} - w_{1,i}) &= h_{i+1}(h_i - h_{i+1})a_{11,i}w_1^{'}(\xi_{i+1}), \\
(h_i - h_{i+1})a_{12,i}(w_{2,i+1} - w_{2,i}) &= h_{i+1}(h_i - h_{i+1})a_{12,i}w_2^{'}(\xi_{i+1}), \\
h_i(a_{11,i} - a_{11,i-1})(w_{1,i-1} - w_{1,i}) &= -h_i^3 a_{11}^{'}(\rho_i)w_1^{'}(\xi_i), \\
h_i(a_{12,i} - a_{12,i-1})(w_{2,i-1} - w_{2,i}) &= -h_i^3 a_{12}^{'}(\rho_i)w_2^{'}(\xi_i), \\
h_{i+1}(a_{11,i+1} - a_{11,i})(w_{1,i} - w_{1,i+1}) &= -h_{i+1}^3 a_{11}^{'}(\rho_{i+1})w_1^{'}(\xi_{i+1}), \\
h_{i+1}(a_{12,i+1} - a_{12,i})(w_{2,i} - w_{2,i+1}) &= (h_i^2 - h_{i+1}^2)a_{12,i}^{'}w_{2,i} - \frac{1}{2}(h_i^3 a_{12}^{''}(\eta_k) \\
&\quad +h_{i+1}^3 a_{12}^{''}(\eta_{i+1}))w_{2,i},
\end{aligned}
$$

where $\theta_i, \xi_i, \rho_i, \eta_i \in [x_{i-1}, x_i]$.

Lemma 5.2. *For the points $x_{i-1}, x_i, x_{i+1} \in \Omega_0$, $x_i \neq d = x_{\frac{N}{2}}$ of the mesh with $\tau_0 \geq 2$ the following holds*

$$
\begin{aligned}
|\,(h_i - h_{i+1})(w_{1,i+1} - w_{1,i})\,| &\leq Ch_{i+1}N^{-2}, \\
|\,(h_i - h_{i+1})(w_{2,i+1} - w_{2,i})\,| &\leq Ch_{i+1}N^{-2}, \\
|\,(h_i - h_{i+1})w_{1,i}^{'}\,| &\leq CN^{-2} \\
\text{and} \quad |\,(h_i - h_{i+1})w_{2,i}^{'}\,| &\leq CN^{-2}.
\end{aligned}
$$

Proof. Let $x_{i-1},\, x_i,\, x_{i+1} \in \bar{\Omega}_2$ and $x_i \neq d = x_{\frac{N}{2}}$

$$\begin{aligned} |h_i - h_{i+1}| &= \frac{\tau_0}{\beta}(\mu-\varepsilon)N^{-1}|\varphi_1^{'}(\rho_i) - \varphi^{'}(\rho_{i+1})| \\ &\leq C(\mu-\varepsilon)N^{-2}|\varphi''(\xi_i)| \end{aligned}$$

for $\rho_i,\, \rho_{i+1}, \xi_i \in (t_{i-1}, t_{i+1})$. Also $|w_{1,i+1} - w_{1,i}| = h_{i+1}|w_1'(\alpha_{i+1})|, \;\; \alpha_{i+1} \in (x_i, x_{i+1})$

$$\begin{aligned} |(h_i - h_{i+1})(w_{1,i+1} - w_{1,i})| &\leq C(\mu-\varepsilon)h_{i+1}N^{-2}|\varphi_1''(\xi_i) \mid\mid w_1^{'}(\alpha_{i+1}) \mid \\ &\leq C(\mu-\varepsilon)h_{i+1}N^{-2}(\frac{\psi_1'(x_i)}{\psi_1(x_i)})^2 \\ &\quad \times[\varepsilon^{-1}e_{1,\varepsilon}(\alpha_{i+1}) + \mu^{-1}e_{1,\mu}(\alpha_{i+1})] \\ &\leq Ch_{i+1}N^{-2}(\frac{\max\psi_1^{'}}{\psi_1(x_i)})^2[|e_{1,\varepsilon}| + |e_{1,\mu}|] \\ &\leq Ch_{i+1}N^{-2}(\psi_1(t_{i+1}))^{-2}. \end{aligned}$$

Using the fact that $\max \mid \psi_1' \mid = C\ln N$ and $e_{1,\varepsilon}(\alpha_{i+1}) \leq \psi_1(t_i)^2 + N^{-\tau_0}$, $e_{1,\mu}(\alpha_{i+1}) \leq \psi_1(t_i)^2 + N^{-\tau_0}$, we have

$$\begin{aligned} \mid (h_i - h_{i+1})(w_{1,i+1} - w_{1,i}) \mid &\leq Ch_{i+1}N^{-2}(\psi_1(t_i)^2 + N^{-\tau_0})(\psi_1(t_{i+1}))^{-2} \\ \mid (h_i - h_{i+1})(w_{1,i+1} - w_{1,i}) \mid &\leq Ch_{i+1}N^{-2}, \end{aligned}$$

since $\tau_0 \geq 2$. When $[x_{i-1}, x_{i+1}] \subset \bar{\Omega}_3$ and $[x_{i-1}, x_{i+1}] \subset \Omega_0 \cap \Omega^+$, the above estimate is also true for these intervals. From the previous analysis, we get

$$\begin{aligned} h_i a_{11,i}(-w_{1,i+1} + 2w_{1,i} - w_{1,i-1}) &\leq Ch_iN^{-2} + Ch_iN^{-2}\max|\psi_1'|, \\ h_i a_{12,i}(-w_{2,i+1} + 2w_{2,i} - w_{2,i-1}) &\leq Ch_iN^{-2} + Ch_iN^{-2}\max|\psi_1'|, \end{aligned}$$

and

$$\begin{aligned} (h_i - h_{i+1})a_{11,i}(w_{1,i+1} - w_{1,i}) &\leq Ch_{i+1}N^{-2}, \\ (h_i - h_{i+1})a_{12,i}(w_{2,i+1} - w_{2,i}) &\leq Ch_{i+1}N^{-2}. \end{aligned}$$

□

Applying the above Lemma 5.2 to each of the terms in m_i of Case 2, we have

$$|m_i| \leq C\bar{h}_iN^{-2}\max|\psi'|^2. \tag{30}$$

Now it remains to prove the estimates at the transition points.

Case 3: When $x_i, i \in \{\frac{N}{4}, \frac{3N}{8}, \frac{3N}{4}, \frac{7N}{8}\}$ and $i \neq \frac{N}{2}$. At these points $w_{1,i}, w_{1,i\pm1}$ and $w_{2,i}, w_{2,i\pm1}$ are bounded by $CN^{-\tau_0}$. Then, using the expression for $|m_i|$ given in Case 2,

$$\mid m_i \mid \leq C\bar{h}_iN^{-\tau_0}. \tag{31}$$

Case 4: When $i = \frac{N}{2}$. That is, $x_i = d$

$$\begin{aligned} m_i &= h_i(a_{11,i}^- w_{1,i}^- - a_{11,i-1}^+ w_{1,i-1}^+) - h_{i+1}(a_{11,i+1}^- w_{1,i+1}^- - a_{11,i}^+ w_{1,i}^+) \\ &\quad + h_i(a_{12,i}^- w_{2,i}^- - a_{12,i-1}^+ w_{2,i-1}^+) - h_{i+1}(a_{12,i+1}^- w_{2,i+1}^- - a_{12,i}^+ w_{2,i}^+) \end{aligned}$$

$$\begin{aligned}
&= h_i(-a_{11,i+1}w_{1,i+1} + a^+_{11,i}w^+_{1,i} + a^-_{11,i}w^-_{1,i} - a_{11,i-1}w_{1,i-1})\\
&\quad h_i(-a_{12,i+1}w_{2,i+1} + a^+_{12,i}w^+_{2,i} + a^-_{12,i}w^-_{2,i} - a_{12,i-1}w_{2,i-1})\\
|m_i| &\leq h_i|(a^+_{11,i} - a_{11,i+1})w^+_{1,i} + (a^-_{11,i} - a_{11,i-1})w^-_{1,i}|\\
&\quad +h_i|a_{11+1,i}(w^+_{1,i} - w_{1,i+1}) + a_{11,i-1}(w^-_{1,i} - w_{1,i-1})|\\
&\quad +h_i|(a^+_{12,i} - a_{12,i+1})w^+_{2,i} + (a^-_{12,i} - a_{12,i-1})w^-_{2,i}|\\
&\quad +h_i|a_{12+1,i}(w^+_{2,i} - w_{2,i+1}) + a_{12,i-1}(w^-_{2,i} - w_{2,i-1})|\\
&\leq Ch_ih_{i+1}|w^+_{1,i}| + Ch_i^2|w^-_{1,i}| + Ch_i(h_i(a_{11,i-1} - a^-_{11,i})\bar{w}'_{1,i}\\
&\quad -\frac{1}{2}h^2_{i+1}a^-_{11,i}\bar{w}''_1(\vartheta_i) + \frac{1}{2}h_i^2a_{11,i-1}\bar{w}''_1(\vartheta_i) + R_1) + Ch_ih_{i+1}|w^+_{2,i}|\\
&\quad +Ch_i^2|w^-_{2,i}| + Ch_i(h_i(a_{12,i-1} - a^-_{12,i})\bar{w}'_{2,i} - \frac{1}{2}h^2_{i+1}a^-_{12,i}\bar{w}''_2(\vartheta_i)\\
&\quad +\frac{1}{2}h_i^2a_{12,i-1}\bar{w}''_2(\vartheta_i) + R_2), \quad \vartheta_i \in [x_{i-1}, x_i].
\end{aligned}$$

We use the asymptotic expansion of the layer components $w_1 = \bar{w}_1 + R_1$ and $w_2 = \bar{w}_2 + R_2$, that can be derived using the technique from [15]. It can be concluded that the leading part $\bar{w}'_1$ of w'_1 and $\bar{w}'_2$ of w'_2 are continuous at $x = d$, enabling us to use Taylor's expansions for estimating $w^+_{1,i} - w_{1,i+1}, w^-_{1,i} - w_{1,i-1}$ and $w^+_{2,i} - w_{2,i+1}, w^-_{2,i} - w_{2,i-1}$. Since R_1, R_2 contain lower order terms, we have

$$| m_i | \leq C\bar{h}_i\varepsilon N^{-1} + C\bar{h}_i\varepsilon N^{-2}\max|\psi'|^2 + C\bar{h}_iN^{-2}\max|\psi'|^2, \tag{32}$$

and we use the estimate of $\max|\psi'|$ in the above result to obtain

$$|m_i| \leq C\bar{h}_i(\varepsilon + N^{-1})N^{-1}\ln^2 N, \quad \text{for Shishkin mesh.}$$

Collecting estimates (29)–(32) from the previously analyzed cases and using $\varepsilon \leq CN^{-1}$ and $\mu \leq CN^{-1}$, we have

$$\begin{aligned}
|< (a_{11}w_1 + a_{12}w_2)^I, \lambda^k_1 >_h| &\leq \frac{1}{6}\sum_{i=1}^{N-1}|m_i|\lambda^k_{1,i}\\
&\leq C(N^{-\tau_0} + N^{-2}\max|\psi'|)\sum_{i=1}^{N-1}\bar{h}_i\lambda^k_{1,i}\\
&\leq CN^{-2}\max|\psi'|\|\lambda^k_1\|_{L^1(\Omega)}\\
&\leq CN^{-2}\max|\psi'|,
\end{aligned}$$

since $\tau_0 \geq 2$ and $\|\lambda^k_1\|_{L^1(\Omega)} \leq C$. From (26), (28) and the above estimate, we have

$$K^*_1(x_k) \leq C\varepsilon N^{-1} + CN^{-2}\max|\psi'|. \tag{33}$$

A similar estimate is also hold for $K^*_2(x_k)$, from (24). Therefore from equations (21)–(22), (33) and $\max|\psi'| = C\ln N$ (in case of Shishkin mesh), for $p = 1, 2$

we have

$$K_p(x_k) \quad \leq \quad C(\varepsilon+\mu)N^{-1} + CN^{-2}\ln^2 N.$$

Since $|\bar{K}(x_i)| = \max(|K_1(x_i)|, |K_2(x_i)|)$, we have

$$|\bar{P}\bar{u}(x_i) - \bar{u}_h(x_i)| = |\bar{K}(x_i)| \leq C(\varepsilon+\mu)N^{-1} + CN^{-2}\ln^2 N.$$

Lemma 5.3. *Let $\bar{u}$ and $\bar{u}_h$ be solutions of the BVP (1)–(2) and (13)–(14) respectively. Then for Shishkin mesh, the pointwise maximum norm of the error satisfies*

$$\mid \bar{u}(x_i) - \bar{u}_h(x_i) \mid \leq CN^{-2}\ln^2 N + C(\varepsilon+\mu)N^{-1} + CN^{1-\tau_0}. \quad \square$$

Now, since V_h uses linear Lagrange elements, we can easily derive a bound for the error $u_j - u_{jh}$ on each element $[x_{i-1}, x_i], i = 1(1)N, j = 1, 2$. For arbitrary $i \in \{1, 2, \cdots, N\}$ and $x \in [x_{i-1}, x_i]$, the triangle inequality implies

$$\mid u_j(x) - u_{jh}(x) \mid \leq \mid u_j(x) - u_j^I(x) \mid + \mid u_j^I(x) - u_{jh}(x) \mid, j = 1, 2.$$

The difference between the piecewise linear function u_j^I and u_{jh} at the point x is estimated by, for $i = 2(1)N-1$

$$\begin{aligned} \left|u_j^I(x) - u_{jh}(x)\right| &= \mid u_j(x_{i-1})\phi_{i-1}(x) + u_j(x_i)\phi_i(x) - u_{jh}(x_{i-1})\phi_{i-1}(x) \\ &\quad -u_{jh}(x_i)\phi_i(x) \mid \\ &\leq \mid u_j(x_{i-1}) - u_{jh}(x_{i-1}) \mid \phi_{i-1}(x) + \mid u_j(x_i) - u_{jh}(x_i) \mid \phi_i(x) \\ &\leq CN^{-2}\ln^2 N + C(\varepsilon+\mu)N^{-1} + CN^{1-\tau_0}\text{ , by Lemma 5.3} \end{aligned}$$

where ϕ_i are functions defined in Section 2.1. The same bound holds for $i = 1$ and $i = N$. Therefore for each interval $[x_{i-1}, x_i]$ we finally obtain the error estimate

$$\begin{aligned} \mid u_j(x) - u_{jh}(x) \mid \quad &\leq \quad \|\bar{u} - \bar{u}^I\|_{L^\infty[x_{i-1}, x_i]} \\ &\quad + CN^{-2}\ln^2 N + C(\varepsilon+\mu)N^{-1} + CN^{1-\tau_0}, j = 1, 2. \end{aligned} \tag{34}$$

6. Error Estimate

The following theorem gives us the result on the maximum norm of the error $\bar{u} - \bar{u}_h$ not just on each interval, but on the whole domain $[0, 1]$.

Theorem 6.1. *Let $\bar{u}$ and $\bar{u}_h$ be solutions of BVP (1)–(2) and (13)–(14) respectively, $\varepsilon \leq CN^{-1}, \mu \leq CN^{-1}$ and $\tau_0 > 3$. Then we have*

$$\|\bar{u} - \bar{u}_h\| \leq CN^{-2}\ln^2 N, \quad \text{for Shishkin mesh.}$$

Proof. For Shishkin mesh, the theorem follows from the inequality (17) and the results on interpolation error 4.1. $\square$

Remark 6.1. All the results in this article also hold good in case when the functions f_1 and f_2 have more than one point of discontinuity. We may also use Bakhavlov-Shishkin meshes [4] instead of Shishkin meshes. For this case, we will arrive $O(N^{-2})$ convergence and numerical experiments has been carried out for both Shishkin and Bakhavlov- Shishkin meshes in the next section.

7. Numerical Experiments

In this section we experimentally verify our theoretical results proved in the previous section.

Example 7.1. Consider the BVP

$$-\varepsilon u_1^{''} + u_1' + 2u_1 - u_2 = f_1(x), \tag{35}$$

$$-\mu u_2^{''} + u_2' - u_1 + 2u_2 = f_2(x), \quad x \in \Omega^- \cup \Omega^+, \tag{36}$$

where

$$f_1(x) = \begin{cases} 1.0, & 0 \le x \le 0.5, \\ -0.8, & 0.5 \le x \le 1 \end{cases}$$

and

$$f_2(x) = \begin{cases} -2.0, & 0 \le x \le 0.5, \\ 1.8, & 0.5 \le x \le 1. \end{cases}$$

For our tests, we take $\varepsilon = 2^{-18}$, which is sufficiently small to bring out the singularly perturbed nature of the problem. We measure the accuracy in various norms and the rates of convergence r^N are computed using the following formula:

$$r^N = log_2(\frac{E^N}{E^{2N}}),$$

where

$$E^N = \max_{j=1,2}\{\max_{x_i \in \bar{\Omega}_\varepsilon^N} \mid (u_{jh})^N(x_i) - (u_{jh}^I)^{2048}(x_i) \mid\}$$

and u_{jh}^I denotes the piecewise linear interpolant of u_{jh}.

Example 7.2. Consider the BVP

$$-\varepsilon u_1^{''} + 2u_1^{'} + 2(x+1)^2 u_1 - (1+x^3)u_2 = f_1(x), \tag{37}$$

$$-\mu u_2^{''} + 1.5u_2^{'} - 2cos(\pi x/4)u_1 + (1+\sqrt{2})u_2 = f_2(x), \quad x \in \Omega^- \cup \Omega^+, \tag{38}$$

where

$$f_1(x) = \begin{cases} -(1+x), & 0 \le x \le 0.5, \\ x^2, & 0.5 \le x \le 1 \end{cases}$$

and

$$f_2(x) = \begin{cases} -2x, & 0 \le x \le 0.5, \\ 1-x^2, & 0.5 \le x \le 1. \end{cases}$$

In Tables 1 and 2, we present values of E^N, r^N for the solutions of the BVPs (35)-(36) and (37)-(38) for Shishkin and Bakhavlov-Shishkin meshes respectively. The Figures and depict the numerical solution of the BVP (35)-(36) for Shishkin mesh with $N = 512$. From the tables it is obvious that the method presented in this paper works better than the standard upwind difference scheme on Shishkin mesh. Some extent the numerical results support the theoretical results.

TABLE 1. *Values of E^N and r^N for the solution of the BVP (35 - 36).*

N	Shishkin mesh		Bakhavlov-Shishkin mesh	
	E^N	r^N	E^N	r^N
32	6.1623e-002	0.8613	6.1611e-02	0.8610
64	3.3921e-002	0.9729	3.3921e-02	0.9728
128	1.7283e-002	1.0642	1.7283e-02	1.0640
256	8.2660e-003	1.2053	8.2666e-03	1.2052
512	3.5851e-003	1.5764	3.5854e-03	1.5766
1024	1.2020e-003	-	1.2021e-03	-

TABLE 2. *Values of E^N and r^N for the solution of the BVP (37 - 38).*

N	Shishkin mesh		Bakhavlov-Shishkin mesh	
	E^N	r^N	E^N	r^N
32	2.9345e-002	1.0687	2.9379e-002	1.0698
64	1.3990e-002	1.0703	1.3996e-002	1.0707
128	6.6622e-003	1.1111	6.6634e-003	1.1112
256	3.0842e-003	1.2282	3.0845e-003	1.2283
512	1.3165e-003	1.5880	1.3165e-003	1.5880
1024	4.3790e-004	-	4.3792e-004	-

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