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EXISTENCE OF THE SOLUTION OF COUNTABLY INFINITE SYSTEM OF DIFFERENTIAL EQUATIONS IN SEQUENCE SPACES $m^p(\phi)$ AND $n^p(\phi)$ WITH THE HELP OF MEASURE OF NON-COMPACTNESS

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ABSTRACT. The Banach spaces $m^p(\phi)$ and $n^p(\phi)$ are very important sequence spaces related to l_p, which were defined to fill the gaps between l_p $(1 \leq p \leq \infty)$. In this paper, we investigated the solubility of the infinite system of differential equations in $m^p(\phi)$ and $n^p(\phi)$ by proving related theorems. Moreover, one example has been included for the justification of the claim of this paper.

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1. Introduction

Differential equations have been used in various fields to model real-world problems. Therefore, the examination of its solubility in different situations become essential. In the past recent years, the researcher examined the solubility of diverse kinds of differential equations in numerous situations. Bougoffa and Khanfer [5] examined the solubility of the solution of the second-order nonlinear differential equation with the nonlocal boundary conditions. A system of countably infinite differential equations or simply infinite systems of differential equations (ISDEs) have lots of applications in various fields such as; branching processes [3], neural nets [6], dissociation of polymers [8], parabolic differential equations investigation [24, 25]. Moreover, several problems appeared in the mechanics laid down by the help of ISDEs [20, 19, 26]. Therefore, the examination of its solubility in different situations become essential. Since, almost all times, the provided experimental data-sets are discreet nature; thus, the role of

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the sequence spaces comes into the picture. This implies, every problem of ISDEs embellished to some sequence space. In the past, different sequence spaces and the solubility of the ISDEs has been checked viz. l_p, c_0, bv_p by Mursaleen [11, 16, 18], Banas [1], Mursaleen and Mohiuddin [17], Demiriz [7], Benchohra et al. [4], Khan [12], Khan and Mursaleen [13] etc. But, the solubility of ISDEs in one space does not imply its solubility in other space. Therefore, in this paper, we examine the solubility of ISDEs in the Banach spaces $m^p(\phi)$ and $n^p(\phi)$ which were defined to fill the gaps between the l_p $(1 \leq p \leq \infty)$ as well as to preserve its some mathematical properties [21]. Due to close relation of $m^p(\phi)$ and $n^p(\phi)$ with the traditional l_p space, researchers studied these spaces in different aspects. Tripathy and Sen [23], generalized these spaces and studied its mathematical properties. Again, in [22], Tripathy et al. generalized $m^p(\phi)$ and $n^p(\phi)$ spaces in fuzzy environment by using Orlicz function. Later, in [9], Karakas et al. proved $m^p(\phi)$ and $n^p(\phi)$ are BK spaces in its general form and studied its geometrical properties such as Banach-Saks property. Recently, in [10], Khan et al., extended these spaces into double sequences and proved both are BK spaces; moreover, they applied the extended double sequences to cluster existing real-world data-sets.

Throughout the paper, by ω, we represent the set of all sequences. Furthermore, for a linear metric space X, we called sequence $(b_n)_{n=1}^{\infty}$ a Schauder basis of X, if for any $x \in X$ there exist a unique scalars sequence $(\lambda_n)_{n=1}^{\infty}$ such that $x = \sum_{n=0}^{\infty} \lambda_n b_n$.

Let C be the collection of sets of the positive integers. For any given element σ of C, by $c(\sigma)$ we denote the sequence $\{c_n(\sigma)\}$ which is defined as $c_n(\sigma) = 1$ if $n \in \sigma$ else $c_n(\sigma) = 0$. Further, let

$$C_s = \left\{ \sigma \in C : \sum_{n=1}^{\infty} c_n(\sigma) \leq s \right\},$$

be the set of those σ whose support has cardinality at most s, and

$$\Phi = \left\{ \phi \in \omega : \phi_1 > 0, \Delta\phi_n \geq 0 \text{ and } \Delta\left(\frac{\phi_n}{n}\right) \leq 0,\ n = 1, 2, \cdots \right\},$$

where $\Delta\phi_n = \phi_n - \phi_{n-1}$ and $\phi_0 = 0$.

Then, for $\phi \in \Phi$, the sequence spaces $m^p(\phi)$ and $n^p(\phi)$ [21, 16, 18] are defined as:

$$m^p(\phi) = \left\{ x \in \omega : \sup_{s \geq 1} \sup_{\sigma \in C_s} \left(\frac{1}{\phi_s} \sum_{n \in \sigma} |x_n|^p \right) < \infty \right\} \tag{1}$$

$$n^p(\phi) = \left\{ x \in \omega : \sup_{u \in S(x)} \left(\sum_{n=1}^{\infty} |u_n|^p \Delta\phi_n \right) < \infty \right\} \tag{2}$$

The sequence spaces $m^p(\phi)$ and $n^p(\phi)$ are closely related to the l_p sequence space and are dual of each other in the sense of Kothe and Toipletz [21]. Some interesting inclusion relations among $m^p(\phi)$ and $n^p(\phi)$ and l_p can be found in [21]. Let us discuss some important inclusion relations in the form of following Lemma (1.1);

Lemma 1.1 (Sargent [21,]). *For the Sequence spaces $m^p(\phi)$, $n^p(\phi)$ and l_p we have;*

(i) *$m^p(\phi)$ and $n^p(\phi)$ both are the BK spaces with their associated norms.*
(ii) *If $\phi_n = p = 1$ $(n = 1, 2, 3, ...)$ then $m^p(\phi) = l_1$ [$n^p(\phi) = l_\infty$] and if $\phi_n = n$ $(n = 1, 2, 3, ...), p = 1$ then $m^p(\phi) = l_\infty$ [$n^p(\phi) = l_1$].*
(iii) *$l_1 \subseteq m^p(\phi) \subseteq l_\infty$ [$l_1 \subseteq n^p(\phi) \subseteq l_\infty$] for all $\phi \in \Phi$.*
(iv) *For any $\phi \in \Phi$, $m^p(\phi) \neq l_p$ [$n^p(\phi) \neq l_q$], $1 < p < \infty$.*

2. Measures of non-compactness

In this section, we recall some basics of the measures of non-compactness in a general metric space (X, d). By $M_c(X)$ we denote the class of bounded subsets of (X, d). Further, let $M(X)$ and $N(X)$ are the subsets of (X, d). Then, $N(X)$ is called a ϵ-net of $M(X)$, if for each $m \in M(X)$ there exist $n \in N(X)$ such that $d(m, n) < \epsilon$ i.e. every element of $M(X)$ can be approximated by an element of $N(X)$ within ϵ-net. The ϵ-net $N(X)$ of $M(X)$ is said to be finite if $N(X)$ is finite and then $M(X)$ is said to be totally bounded. For metric space (X, d) and an open boll $B(x_0, r) = \{x \in X : d(x_0, x) < r\}$, the Hausdorff measure of non-compactness χ of subset $E \in M_c(X)$ is defined as [17]:

$$\chi(E) = \{\epsilon > 0 : E \subset \cup_{i=1}^{n} B(x_i, r_i), x_i \in X, r_i < \epsilon(i = 1, 2, \cdots), n \in \mathbb{N}_0\}$$

or in other words

$$\chi(E) = \inf\{\epsilon > 0 : E \text{ has } \epsilon - \text{net in } X\}.$$

Moreover, for further study of this paper, let us state the following Lemma (2.1);

Lemma 2.1 (Mursaleen and Mohiuddine [17,]). *If X is normed space then for bounded subsets E_1, E_2 and E_3 of X, we have;*

(i) *$\chi(E) = 0 \Leftrightarrow E$ is totally bounded.*
(ii) *$E_1 \subset E_2 \Rightarrow \chi(E_1) \leq \chi(E_2)$.*
(iii) *$\chi(E_1 + E_2) \leq \chi(E_1) + \chi(E_2)$.*
(iv) *$\chi(\alpha E) = |\alpha|\chi(E)\ \forall\ \alpha \in \mathbb{C}$.*

A linear operator L between Banach spaces X and Y is said to be (χ_1, χ_2)-bounded (where χ_1, χ_2 are the Hausdorff measure of non-compactness of X and Y) if for all $E \in M_c(X)$, $L(E) \in M_c(Y)$ there exist constant $c \geq 0$ such that $\chi_2(L(E)) \leq c\chi_1(E)$. Moreover, if L is (χ_1, χ_2)-bounded. Then, $\|L\|_{(\chi_1,\chi_2)}$ is called (χ_1, χ_2)-measure of non-compactness of L and is defined as:

$$\|L\|_{(\chi_1,\chi_2)} = \inf\{c \geq 0 : \chi_2(L(E)) \leq c\chi_1(E),\ \forall\ E \in\ M_c(X)\}.$$

For the case of $\chi_1 = \chi_2 = \chi$ we have;

$$\|L\|_{(\chi_1,\chi_2)} = \|L\|_{\chi}. \tag{3}$$

Moreover, L is said to be compact if and only if

$$\|L\|_{\chi} = 0. \tag{4}$$

Now, for each $x \in X$ and $n \in \mathbb{N}$, the operator $P_n : X \to X$ such that $P_n(x) = \sum_{k=0}^{n} \alpha_k(x) b_k$, is called projector onto the linear span $\{b_0, b_1, \cdots, b_n\}$ (see [17]).

Moreover, let us state the following two very important theorem useful for this study;

Theorem 2.2 (Malkowsky and Rakocević [14, 15,]). *Let X be a Banach space and (b_k) is the Schauder basis of X and E be the subset of $M_c(X)$. Again let $P_n : X \to X$ is the projector onto the linear span with $\{b_0, b_1, \cdots, b_n\}$. Then, we have;*

$$\frac{1}{a} \lim_{n\to\infty} \sup \left(\sup_{x\in Q} \|(I - P_n)(x)\| \right) \leq \chi(E) \leq \lim_{n\to\infty} \sup \left(\sup_{x\in Q} \|(I - P_n)(x)\| \right).$$

Theorem 2.3 (Malkowsky and Rakocević [14, 15,]). *Let E be the bounded subset of the Banach space l_p ($1 \leq p \leq \infty$). If $P_n : l_p \to l_p$ be the projector defined by $P_n(x) = x^{[n]} = (x_0, x_1, \cdots, x_n, 0, 0, \cdots)$ for all $x \in l_p$. Then, we have;*

$$\chi(E) = \lim_{n\to\infty} \left(\sup_{x\in E} \|(I - P_n)(x)\|_{l_p} \right).$$

If $E \in M_c(l_p)$, then

$$\chi(E) = \lim_{n\to\infty} \left(\sup_{x\in E} \sum_{k\geq n} |x_k|^p \right).$$

In this paper, we used the theory of measures of non-compactness into examining the existence of solutions of infinite system of differential equations in the Banach spaces $m^p(\phi)$ and $n^p(\phi)$. In the view of Lemma (1.1), the spaces $m^p(\phi)$ and $n^p(\phi)$ are closely related to l_p. But, for any $\phi \in \Phi$, $m^p(\phi) \neq l_p$ and $n^p(\phi) \neq l_q$ for each $1 < p, q < \infty$ $(1/p + 1/q = 1)$. This implies, our results is totally different than the results obtained for l_p by Mursaleen and Mohiuddin [17] and Banas and Lecko [1]. Moreover, we also find out the sufficient condition of the solubility of an infinite system of differential equations in $m^p(\phi)$ and $n^p(\phi)$.

As, in [14, 15], Theorem (2.2) has been proved for the general Banach spaces. But, it is necessary to prove Theorem 2.3 for the Banach spaces $m^p(\phi)$ and $n^p(\phi)$.

Theorem 2.4. *Let E be the bounded subset of the Banach space $m^p(\phi)[n^p(\phi)]$. If $P_n : m^p(\phi)[n^p(\phi)] \to m^p(\phi)[n^p(\phi)]$ be the projector defined by $P_n(x) = x^{[n]} = (x_0, x_1, \cdots, x_n, 0, 0, \cdots)$ for all $x \in m^p(\phi)[n^p(\phi)]$. Then, we have;*

$$\chi(E) = \lim_{n\to\infty} \left(\sup_{x\in E} \|(I - P_n)(x)\|_{m^p(\phi)[n^p(\phi)]} \right).$$

If $E \in M_c(m^p(\phi))$, then,

$$\chi(E) = \lim_{n\to\infty} \left(\sup_{x\in E} \left(\sup_{s\geq 1} \sup_{\sigma\in C_s} \left(\frac{1}{\phi_s} \sum_{n\in\sigma,\ k\geq n} |x_k|^p \right) \right) \right).$$

and If $E \in M_c(n^p(\phi))$, then,

$$\chi(E) = \lim_{n\to\infty} \left(\sup_{x\in E} \left(\sup_{u\in S(x)} \sum_{n=1}^{\infty} |u_n|^p \Delta\phi_n \right) \right).$$

Proof. Let E be a bounded subset of $m^p(\phi)$ and P_n be a projection mapping then, we have;

$$E \subset P_n E + (1 - P_n)E. \tag{5}$$

Now, by using Lemma (2.11), and Theorem (2.12), (2.36) of [15] in Eq. (5), we have;

$$\chi(E) \leq \chi(P_n E) + \chi((I - P_n)E) = \chi((I - P_n)E) \leq \sup_{x\in E} \|(I - P_n)(x)\|_{l_p}$$

$$\Rightarrow\ \chi(E) \leq \sup_{x\in E} \|(I - P_n)(x)\|_{l_p}. \tag{6}$$

The inequality $\chi(E) \leq \sup_{x\in E} \|(I - P_n)(x)\|_{l_p}$ holds for each $1 \leq p < \infty$. Thus, from Lemma (1.1), for each $\phi \in \Phi$, there exist $p \in (1,\infty)$ such that $m^p(\phi) \subset l_p$. Therefore, Eq. (6) can be written as:

$$\chi(E) \leq \sup_{x\in E} \|(I - P_n)(x)\|_{m^p(\phi)}. \tag{7}$$

Also, since the limit of the right hand side of Eq. (7) exist, we can write it as:

$$\chi(E) \leq \lim_{n\to\infty} \sup_{x\in E} \|(I - P_n)(x)\|_{m^p(\phi)}. \tag{8}$$

Moreover, let $Z = \{z_1, z_2, \cdots, z_k\}$ be a $[\chi(E) + \epsilon]$-net of E. Then, for all $x \in E$ there exist $z \in \{z_1, z_2, \cdots, z_k\}$ and $t \in B_X$ such that $x = z + [\chi(E) + \epsilon]t$. This implies,

$$\sup \|(I - P_n)(x)\|_{m^p(\phi)}$$

$$\leq \sup \|(I - P_n)(z_i)\|_{m^p(\phi)} + [\chi(E) + \epsilon]\ \forall\ x \in\ E,\ 1 \leq i \leq k \tag{9}$$

$$\Rightarrow \lim_{n\to\infty} \sup_{x\in E} \|(I - P_n)(x)\|_{m^p(\phi)} \leq \chi(E) + \epsilon \tag{10}$$

By combining Eq. (7) and (10), we have;

$$\chi(E) = \lim_{n\to\infty}\left(\sup_{x\in E}\|(I-P_n)(x)\|_{m^p(\phi)}\right) \tag{11}$$

or

$$\chi(E) = \lim_{n\to\infty}\left(\sup_{x\in E}\left(\sup_{s\geq 1}\sup_{\sigma\in C_s}\left(\frac{1}{\phi_s}\sum_{n\in\sigma,\ k\geq n}|x_k|^p\right)\right)\right). \tag{12}$$

As, $m^p(\phi)$ and $n^p(\phi)$ are Kothe Toeplitz dual of each other i.e. $n^p(\phi) = \{x_n \in \omega : \sum_n |x_n y_n| < \infty,\ \forall\ x_n \in m^p(\phi)\}$ thus, for $E \in M_c(n^p(\phi))$, we have;

$\chi(E) = \lim_{n\to\infty}\left(\sup_{x\in E}\left(\sup_{u\in S(x)}\sum_{n=1}^{\infty}|u_n|^p\Delta\phi_n\right)\right).$ □

3. Existence of the solution of ISDEs

Before going to our main results to discuss the solubility of the infinite systems of differential equations in the Banach sequence space $m^p(\phi)[n^p(\phi)]$. Let us take an ordinary differential equation given as:

$$x' = f(t,x) \text{ with initial condition } x(0) = x_0. \tag{13}$$

The existence results of Eq. (12) have been discussed in [1, 2]. Moreover, Theorem (3) we have borrowed from [1, 2] which is stated as follows:

Theorem 3.1 (Banaś and Lecko, Banaś and Sadarangani [1, 2,])**.** *Let X be a Banach space with norm $\|.\|$ and $B[x_0,s]$ is the closed ball in X with center x_0 and radius s. Again let I be the interval such that $I=[0,T],\ T>0$. Let us assume a function $f : I\times X \to X$ such that:*

$$\|f(t,x)\| \leq C_1 + C_2\|x\| \tag{14}$$

where $x \in X$ and C_1, C_2 are the non negative constants. Let f is uniformly continuous in $I_1 \times B[x_0,s]$, here $s = (C_1T_1 + C_2T_1\|x_0\|)/(1-C_2T_1)$ and $I_1 = [0,T_1] \subset I,\ C_2T_1 < 1$. Also, assume that for every nonempty subset $Y \subset B[x_0,s]$ and for almost all $t \in I$. Then, we have;

$$\mu(f(t,Y)) \leq q(t)\mu(Y) \tag{15}$$

where $q(t)$ is an integrable function on I and μ is the sublinear measure of non-compactness such that $\{x_0\} \in Ker\ \mu$. Then, Eq. (12) has a solution $x \in X$ with $\{x(t)\} \in Ker\ \mu,\ t \in I_1$; where, $Ker\ \mu = \{E \in M_c(X) : \mu(E) = 0\}$ is the kernel of measure μ.

Remark 3.1. For a special case, when $\mu = \chi$ then the requirement of f of uniform continuity can be replaced by simple continuity.

Now, for the study of the solvability of the infinite systems of differential equations in Banach space $m^p(\phi)$, we means the existence of the solution of $x(t) = (x_i(t))$ in the interval $I = [0,T]$ such that $x(t) \in m^p(\phi)$ and $t \in I$.

$$x_i' = f_i(t, x_0, x_1, x_2, x_3, \cdots),\ x_i(0) = (x_i)_0,\ i = 1, 2, \cdots \tag{16}$$

The existence theorem for Eq. (15) in the Banach space $m(\phi)$ can be establish by making the following assumptions:

Assumption 1. $x_o = (x_i)_0 \in m^p(\phi)$.

Assumption 2. The mapping $f_i : I \times \mathbb{R}^\infty \to \mathbb{R}$ continuously maps the set $I \times m^p(\phi)$ into $m^p(\phi)$.

Assumption 3. There exist two nonnegative functions $q_i(t)$ and $r_i(t)$ on I such that

$$|f_i(t,x)| = |f_i(t, x_0, x_1, x_2, x_3, \cdots)|^p \leq q_i(t) + r_i(t)|x_i|^p$$

where, $t \in I$, $(x_i) \in m^p(\phi)$.

Assumption 4. $q_i(t)$ are continues in I and $\frac{1}{\phi_s}\sum\limits_{i\in\sigma} q_i(t)$ is uniformly convergent in I.

Assumption 5. $r_i(t)$ is equibounded in I and $r(t) = \lim\limits_{i\to\infty} \sup\limits_{i\in\sigma}(r_i(t))$ is integrable in I.

Theorem 3.2. *Under the assumptions (i)-(v), the system of infinite order differential equations* $x_i^{'} = f_i(t, x_0, x_1, x_2, x_3, \cdots)$ *with initial condition* $x_i(0) = (x_i)_0$ *defined in* $I = [0,T]$ *has a solution* $x(t) = x_i(t)$ *in the Banach space* $m^p(\phi)$*, whenever,* $C_2T < 1$*, here* $C_2 = \sup\limits_{i\in\sigma}(r_i(t))$*.*

Proof. Let $x(t) \in m^p(\phi)$ for each t in I. Now, we have;

$$\|f(t,x)\|_{m^p(\phi)} = \sup_{s\geq 1}\sup_{\sigma\in C_s}\left(\frac{1}{\phi_s}\sum_{i\in\sigma}|f(t,x)|^p\right)$$

$$\leq \sup_{s\geq 1}\sup_{\sigma\in C_s}\frac{1}{\phi_s}\sum_{i\in\sigma}[q_i(t) + r_i(t)|x_i|^p] \text{ (using assumption-iii)}$$

$$\leq \sup_{s\geq 1}\sup_{\sigma\in C_s}\frac{1}{\phi_s}\sum_{i\in\sigma}q_i(t) + \sup_{i\in\sigma}(r_i(t))\left(\sup_{s\geq 1}\sup_{\sigma\in C_s}\frac{1}{\phi_s}\sum_{i\in\sigma}|x_i|^p\right)$$

$$\leq C_1 + C_2\|x\|_{m^p(\phi)};\ C_1 = \sup_{t\in I}\left(\sup_{s\geq 1}\sup_{\sigma\in C_s}\frac{1}{\phi_s}\sum_{i\in\sigma}q_i(t)\right).$$

Again in the view of Theorem (3.1), let $s = (C_1T + C_2T\|x_0\|)/(1 - C_2T)$. Consider the operator $f = (f_i)$ defined in the set $I \times B[x_0, s]$ and let Y be a set in $M(m^p(\phi))$. Then, by using Theorem (2.4), we have;

$$\chi(f(t,Y)) = \lim_{n\to\infty}\left(\sup_{x\in Y}\left(\sup_{s\geq 1}\sup_{\sigma\in C_s}\left(\frac{1}{\phi_s}\sum_{i\in\sigma, i\geq n}|f_i(t,x_0,x_1,x_2,x_3,\cdots)|^p\right)\right)\right)$$

$$\leq \lim_{n\to\infty}\left(\sup_{s\geq 1}\sup_{\sigma\in C_s}\frac{1}{\phi_s}\sum_{i\in\sigma, i\geq n}q_i(t) + \sup_{i\in\sigma, i\geq n}(r_i(t))\left(\sup_{s\geq 1}\sup_{\sigma\in C_s}\frac{1}{\phi_s}\sum_{i\in\sigma, i\geq n}|x_i|^p\right)\right).$$

Thus, by assumption (iv) and (v), we have;

$$\chi(f(t,Y)) \leq r(t)\chi(Y)$$

This implies, the operator f satisfies the conditions of Theorem (3.1). Therefore, in the view of Theorem (3.1) and Remark (3.1), there exist a solution $x = x(t)$ of Eq. (16) in the Banach space $m^p(\phi)$. □

4. Illustrative Example

Consider a countably infinite system of differential equations give as follows;

$$\frac{d}{dt}(x_n(t)) = \frac{\sqrt[n]{t}}{n^2} + \frac{t\sin(t)x_m(t)}{m^2}\ x_n(0) = (x_n)_0,\ m \geq n = 1,2,\cdots, t\in I = [0,T]. \tag{17}$$

Here, m and n are two independent indices of the sequence such that for each n, m can be any indices greater than or equal to n. Now, from assumption-1, $x_n(0) = (x_n)_0 \in m^p(\phi)$. Also, it is easy to prove that the two functions $\frac{\sqrt[n]{t}}{n^2}$ and $\frac{t\sin(t)x_m(t)}{m^2}$ are continuous in I. Now, for any sequence $\phi\in\Phi$, we have;

$$\sup_{s\geq 1}\sup_{\sigma\in C_s}\left(\frac{1}{\phi_s}\sum_{i\in\sigma}|f(t,x)|^p\right) = \sup_{s\geq 1}\sup_{\sigma\in C_s}\left(\frac{1}{\phi_s}\sum_{m\in\sigma}\left|\frac{\sqrt[n]{t}}{n^2} + \frac{t\sin(t)x_m(t)}{m^2}\right|^p\right)$$

$$\leq \sup_{s\geq 1}\sup_{\sigma\in C_s}\left(\frac{1}{\phi_s}\sum_{m\in\sigma}\frac{\sqrt[n]{t}}{n^2}\right) + \sup_{s\geq 1}\sup_{\sigma\in C_s}\left(\frac{1}{\phi_s}\sum_{m\in\sigma}\left|\frac{t\sin(t)x_m(t)}{m^2}\right|^p\right).$$

It is easy to prove that $\frac{1}{\phi_s}\sum_{i\in\sigma}\frac{\sqrt[n]{t}}{n^2}$ converges uniformly to $\frac{T\pi^2}{6}$ in I. Thus we have

$$\sup_{s\geq 1}\sup_{\sigma\in C_s}\left(\frac{1}{\phi_s}\sum_{i\in\sigma}|f(t,x)|^p\right) \leq \frac{T\pi^2}{6} + \frac{t}{m^2}\sup_{s\geq 1}\sup_{\sigma\in C_s}\left(\frac{1}{\phi_s}\sum_{m\in\sigma}|x_m(t)|^p\right)$$

$$\leq \frac{T\pi^2}{6} + T\|x(t)\|_{m^p(\phi)}.$$

Thus, for each $t\in I$, $x_i(t)\in m^p(\phi)$ implies $f_i(t,x(t))\in m^p(\phi)$ (it is to be noted that $x_i(t)\in m^p(\phi) \nRightarrow x_i(t)\in l_p$). Again, for $\epsilon > 0$ and $x_i(t), y_i(t)\in m^p(\phi)$ such that $\|x_i(t)-y_i(t)\|_{m^p(\phi)} < \delta$, we have

$\|f_i(t,x(t)) - f_i(t,x(t))\|_{m^p(\phi)} \leq \frac{t\|(x_i(t)-y_i(t)\|_{m^p(\phi)}}{m^2} \leq \frac{t\delta}{m^2} = \epsilon.$

Thus, $f_i : I \times \mathbb{R}^\infty \to \mathbb{R}$ continuously maps the set $I \times m^p(\phi)$ into $m^p(\phi)$. Moreover, we have

$|f_n(t,x_i(t))|^p = |\frac{\sqrt[n]{t}}{n^2} + \frac{t\sin(t)x_i(t)}{m^2}|^p \leq \frac{\sqrt{t}}{n^2} + \frac{t}{m^2}|x_i(t)|^p = q_i(t) + r_i(t)|x_m(t)|^p$

where, $q_i(t) = \frac{\sqrt{t}}{n^2}$, $r_i(t) = \frac{t}{m^2}$.

Obviously, the functions $q_i(t)$ and $\frac{1}{\phi_s}\sum\limits_{i\in\sigma} q_i(t)$ are continuous and uniformly convergent in I respectively. Also, $r_i(t)$ is equibounded in I, and the limit of $r(t) = \lim\limits_{i\to\infty} \sup\limits_{i\in\sigma}(r_i(t))$ is integrable in I. Thus, by using Theorem (3.2), Eq. (16) has a solution in $m^p(\phi)$. By using $n^p(\phi) = \{x_n \in \omega : \sum_n |x_n y_n| < \infty,\ \forall\ x_n \in m^p(\phi)\}$. We can similarly prove the existence of the solution in $n^p(\phi)$.

5. Conclusion

In this paper, we examined the solubility of the infinite system of differential equations in the Banach spaces $m^p(\phi)$ and $n^p(\phi)$ by proving the necessary theorems. Moreover, as $m^p(\phi)$ and $n^p(\phi)$ both are defined to fill the gaps between l_p. Thus, our result is different than other discussed works. In concluding remark, the established mathematical results of this paper can also be used to address the solubility of the perturbed diagonal infinite system of differential equations;

$$x_i^{'} = h_i(t)x_i + f_i(t,x_0,x_1,x_2,x_3,\cdots),\ x_i(0) = (x_i)_0,\ (i = 1,2,\cdots).$$

by making the following assumptions:

Assumption 1. $x_o = (x_i)_0 \in m^p(\phi)[n^p(\phi)]$.

Assumption 2. The mapping $f_i : I \times \mathbb{R}^\infty \to \mathbb{R}$ continuously maps the set $I \times m^p(\phi)[n^p(\phi)]$ into $m^p(\phi)[n^p(\phi)]$.

Assumption 3. there exist two nonnegative functions $q_i(t)$ such that

$$|f_i(t,x)|^p = |f_i(t,x_0,x_1,x_2,x_3,\cdots)|^p \leq q_i(t)$$

where, $t \in I$, $(x_i) \in m^p(\phi)[n^p(\phi)]$.

Assumption 4. $q_i(t)$ are continues in I, and $\frac{1}{\phi_s}\sum\limits_{i\in\sigma} q_i(t)$ is uniformly convergent in I.

Assumption 5. $|h_i(t)|$ is equibounded in I, and $r(t) = \lim\limits_{i\to\infty} \sup\limits_{i\in\sigma} |h_i(t)|$ is integrable in I.

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