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ANOTHER PROOF OF CLASSICAL DIXON'S SUMMATION THEOREM FOR THE SERIES $_3F_2$

Insuk Kim and Myunghyun Cho*

Abstract. In this short research note, we aim to provide a new proof of classical Dixon's summation theorem for the series $_3F_2$ with unit argument. The theorem is obtained by evaluating an infinite integral and making use of classical Gauss's and Kummer's summation theorem for the series $_2F_1$.

1. Introduction

It is well known that in the theory of generalized hypergeometric series, classical Dixon's summation theorem for the series $_3F_2$ [2, 4, 5] viz.

$$(1) \qquad {}_3F_2\left[\begin{matrix} a, & b, & c \\ 1+a-b, & 1+a-c \end{matrix}; 1\right] = \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)},$$

provided $\mathrm{Re}(a-2b-2c) > -2$, play a key role.

In a very well known, useful, interesting and popular research paper, Bailey [1] obtained a large number of very interesting results involving products of generalized hypergeometric functions by employing classical Dixon's summation theorem (1).

As pointed out by Berndt [3] that the following very interesting summations due to Ramanujan viz.

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*Corresponding author

$$(2)\qquad 1+\frac{1}{5^2}\left(\frac{1}{2}\right)+\frac{1}{9^2}\left(\frac{1\cdot 3}{2\cdot 4}\right)+\cdots=\frac{\pi^{5/2}}{8\sqrt{2}\,\Gamma^2(3/4)},$$

$$(3)\qquad 1+\left(\frac{1}{2}\right)^3+\left(\frac{1\cdot 3}{2\cdot 4}\right)^3+\cdots=\frac{\pi}{\Gamma^4(3/4)}$$

and

$$(4)\qquad 1+\frac{1}{5}\left(\frac{1}{2}\right)^2+\frac{1}{9}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2+\cdots=\frac{\pi^2}{4\,\Gamma^4(3/4)},$$

can be obtained very quickly by employing classical Dixon's summation theorem (1) by taking (i) $a=\frac{1}{2}, b=c=\frac{1}{4}$, (ii) $a=b=c=\frac{1}{2}$ and (iii) $a=b=\frac{1}{2}, c=\frac{1}{4}$, respectively.

In the standard text of Bailey [1], classical Dixon's summation theorem have been established with the help of the following classical Gauss's summation theorem [5] viz.

$$(5)\qquad {}_2F_1\left[\begin{matrix}a,\ b\\ c\end{matrix};\ 1\right]=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

provided $\mathrm{Re}(c-a-b)>0$, and the following Kummer's summation theorem [5] viz.

$$(6)\qquad {}_2F_1\left[\begin{matrix}a,\quad b\\ 1+a-b\end{matrix};-1\right]=\frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)}.$$

In our present investigation, we aim to provide a new proof of classical Dixon's summation theorem (1) by evaluating an infinite integral. For this, we need the following result, which is a special case of Gauss's summation theorem (5) viz.

$$(7)\qquad {}_2F_1\left[\begin{matrix}-k,\quad a+k\\ 1+a-c\end{matrix};\ 1\right]=\frac{(-1)^k(c)_k}{(1+a-c)_k}.$$

2. New proof of Dixon's summation theorem (1)

In order to derive the result (1), we proceed as follows. Consider the following integral

$$I = \int_0^\infty e^{-t}\ t^{d-1}\ {}_3F_3\left[\begin{matrix} a, & b, & c \\ d, & 1+a-b, & 1+a-c \end{matrix}; t\right] dt.$$

for $\mathrm{Re}(d) > 0$.

Expressing the generalized hypergeometric function ${}_3F_3$ in series, we have

$$I = \int_0^\infty e^{-t}\ t^{d-1} \sum_{n=0}^{\infty} \frac{(a)_n\ (b)_n\ (c)_n\ t^n}{(d)_n\ (1+a-b)_n\ (1+a-c)_n\ n!}\ dt.$$

Changing the order of integration and summation, which is permitted due to the uniform convergence of the series, we have

$$I = \sum_{n=0}^{\infty} \frac{(a)_n\ (b)_n\ (c)_n}{(d)_n\ (1+a-b)_n\ (1+a-c)_n\ n!} \int_0^\infty e^{-t}\ t^{d+n-1} dt.$$

Evaluating the well known gamma integral and making use of the relation of following Pochhammer symbol with gamma function

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)},$$

we have, after some simplification

$$I = \Gamma(d) \sum_{n=0}^{\infty} \frac{(a)_n\ (b)_n\ (c)_n}{(1+a-b)_n\ (1+a-c)_n\ n!}. \tag{8}$$

Finally, summing up the series, we have

$$I = \Gamma(d)\ {}_3F_2\left[\begin{matrix} a, & b, & c \\ 1+a-b, & 1+a-c \end{matrix}; 1\right]. \tag{9}$$

On the other hand, writing (8) in the form

$$I = \Gamma(d) \sum_{n=0}^{\infty} \frac{(-1)^n\ (a)_n\ (b)_n}{(1+a-b)_n\ n!} \left\{ \frac{(-1)^n\ (c)_n}{(1+a-c)_n} \right\}.$$

Now, using (7), we have

$$I=\Gamma(d)\sum_{n=0}^{\infty}\frac{(-1)^n\ (a)_n\ (b)_n}{(1+a-b)_n\ n!}\ {}_2F_1\left[\begin{matrix}-n, & a+n\\ 1+a-c\end{matrix};\ 1\right].$$

Writing ${}_2F_1$ as a series, we have after some simplification

$$I=\Gamma(d)\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^n\ (a)_n\ (b)_n\ (-n)_m\ (a+n)_m}{(1+a-b)_n\ (1+a-c)_m\ n!\ m!}.$$

Using the identities

$$(a)_n(a+n)_m=(a)_{n+m}\quad \text{and}\quad (-n)_m=\frac{(-1)^m\ n!}{(n-m)!},$$

we have, after some calculation

$$I=\Gamma(d)\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^{n+m}\ (a)_{n+m}\ (b)_n}{(1+a-b)_n\ (1+a-c)_m\ m!\ (n-m)!}.$$

Now, using a known result [5, p.57, Equ.(2)]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}B(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}B(k,n+k),$$

we have

$$I=\Gamma(d)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n\ (a)_{n+2m}\ (b)_{n+m}}{(1+a-b)_{n+m}\ (1+a-c)_m\ m!\ n!}.$$

Using the identities

$$(a)_{n+2m}=(a)_{2m}(a+2m)_n\quad \text{and}\quad (b)_{n+m}=(b)_m(b+m)_n,$$

and after some simplification, we have

$$I=\Gamma(d)\sum_{m=0}^{\infty}\frac{(a)_{2m}\ (b)_m}{(1+a-b)_m\ (1+a-c)_m\ m!}\times\sum_{n=0}^{\infty}\frac{(-1)^n(a+2m)_n\ (b+m)_n}{(1+a-b+m)_n\ n!}.$$

Summing up the inner series, we have

(10)

$$I=\Gamma(d)\sum_{m=0}^{\infty}\frac{(a)_{2m}\ (b)_m}{(1+a-b)_m\ (1+a-c)_m m!}\times{}_2F_1\left[\begin{matrix}a+2m, & b+m\\ 1+a-b+m\end{matrix};-1\right].$$

Now using Kummer's summation theorem (6) to the right-hand side of (10) and then applying the identity

$$(a)_{2m} = 2^{2m}\left(\frac{1}{2}a\right)_m\left(\frac{1}{2}a+\frac{1}{2}\right)_m,$$

we get after some simplification

$$I = \frac{\Gamma(d)\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)}\sum_{m=0}^{\infty}\frac{(\frac{1}{2}a)_m\,(b)_m}{(1+a-c)_m\,m!}.$$

Summing up the series, we get

$$I = \frac{\Gamma(d)\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}a, & b\\ 1+a-c\end{matrix};\ 1\right]. \tag{11}$$

With the help of Gauss's summation theorem (5) to the right-hand side of (11), we have

$$I = \frac{\Gamma(d)\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)}. \tag{12}$$

Finally, equating (9) and (12), we get the desired Dixon's summation theorem (1).

This completes our new proof of Dixon's summation theorem for the series ${}_3F_2(1)$.

Conclusion Remark

In this note, we established Dixon's summation theorem via evaluating an infinite integral. We conclude this research by remarking that a few new applications of Dixon's theorem are under investigations and will be published soon.

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Insuk Kim
Department of Mathematics Education,
Wonkwang University,
Iksan, 570-749, Korea
E-mail: iki@wku.ac.kr

Myunghyun Cho
Department of Mathematics Education,
Wonkwang University,
Iksan, 570-749, Korea
E-mail: mhcho@wku.ac.kr