Honam Mathematical J. **41** (2019), No. 3, pp. 651–659
https://doi.org/10.5831/HMJ.2019.41.3.651

IRREDUCIBILITY OF POLYNOMIALS WITH A LARGE COEFFICIENT

DoYong Kwon

Abstract. A certain class of polynomials with integer coefficients are considered. If one of the coefficients is large enough in modulus with additional assumptions, then the irreducibility over the field of rationals is proved.

1. Introduction and preliminaries

All too often, irreducible polynomials play what prime numbers do in the set of integers. And no general irreducibility criterion for polynomials is known just as primality test is hard. We consider single-variate polynomials with integer coefficients, and devise irreducibility criteria over the field of rationals $\mathbb{Q}$. No serious algebra will be involved in the proof. Instead, Rouché's theorem in complex analysis brings forth the main results. This situation is neither surprising nor new. In [1, 4], the author obtained, in an analogous vein, irreducibility criteria for some variants of reciprocal polynomials. The present paper demonstrates another case where analysis proves algebra as in the proof for the fundamental theorem of algebra.

Let $f, g : \mathbb{C} \to \mathbb{C}$ be holomorphic on and inside a simple closed contour C. Rouché's theorem states that the number of zeros of $f(x)$ inside C is robust under the perturbation $g(x)$ as long as $|g(x)| < |f(x)|$ on C. That is, $f(x)$ and $f(x) + g(x)$ have the same number of zeros inside C.

Let $f(x) \in \mathbb{Z}[x]$ be a polynomial with integer coefficients. If one of the coefficients is large enough compared to the others, then we prove its irreducibility over $\mathbb{Z}$, or equivalently over $\mathbb{Q}$.

Received March 25, 2019. Accepted April 28, 2019.
2010 Mathematics Subject Classification. 11R09, 30C10.
Key words and phrases. irreducible polynomial, irreducibility criterion

Given a polynomial $f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{C}[x]$ with degree d, let us write

$$\tilde{f}(x) := x^d f(x^{-1}) = a_0 x^d + a_1 x^{d-1} + \cdots + a_d.$$

We mean by a cyclotomic polynomial, a monic one with integer coefficients all of whose zeros α satisfy $\alpha^n = 1$ for some positive integer n. Suppose that a polynomial $f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{Z}[x]$ with degree d has zeros $\alpha_1, \alpha_2, \ldots, \alpha_d \in \mathbb{C}$. Then the *Mahler measure* of f is defined by

$$M(f) := |a_n| \prod_{i=1}^{d} \max\{1, |\alpha_i|\} \geq 1.$$

One readily notes that any cyclotomic polynomial has its Mahler measure 1. Furthermore, the converse is almost true by the next proposition that is due to Kronecker [3]. Because the original paper is old and written in German, the proof is included for readers' convenience.

Proposition 1.1. *Let $f(x) = \prod_{i=1}^{d}(x - \alpha_i) \in \mathbb{Z}[x]$ be monic with $f(0) \neq 0$. If $M(f) = 1$, then f is cyclotomic.*

Proof. From $M(f) = 1$ and $f(0) \neq 0$, it follows that $0 < |\alpha_i| \leq 1$ for all $i = 1, 2, \ldots, d$. For an integer $k \geq 1$, set

$$f_k(x) = \prod_{i=1}^{d}(x - \alpha_i^k).$$

The coefficient of x^i in f_k is the elementary symmetric polynomial in α_1^k, α_2^k, ..., α_d^k up to $\pm$ sign, whence its absolute value assumes at most $\binom{d}{i}$. Moreover, each coefficient in f_k is also a symmetric polynomial in $\alpha_1, \alpha_2, \ldots, \alpha_d$. Hence, the fundamental theorem of symmetric polynomials proves that it is a polynomial in the elementary symmetric polynomials of $\alpha_1, \alpha_2, \ldots, \alpha_d$, which are the coefficients of f. Gathering these, one deduces that all the coefficients in f_k are integers and bounded independently of k, and then that the set $\{f_k : k \geq 1\}$ is finite. Pick $k_1 < k_2 < \cdots < k_{d!+1}$ so that $f_{k_1} = f_{k_2} = \cdots = f_{k_{d!+1}}$. There exist $1 \leq m < n \leq d! + 1$ such that $\alpha_i^{k_m} = \alpha_i^{k_n}$ or $\alpha_i^{k_n - k_m} = 1$ holds for every $i = 1, 2, \ldots, d$. □

In 1933, Lehmer [5] showed that an irreducible polynomial

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$$

has Mahler measure $\tau_0 \approx 1.17628$, the unique real zero greater than 1, i.e., the other zeros lie on the closed unit disk. Since then, no polynomial whose Mahler measure is in between 1 and τ_0 is known, while it is strongly believed none.

Let $f(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{C}[x]$ be a monic polynomial with complex coefficients. Recall that each zero of $f(x)$ is a continuous function of the coefficients $a_0, a_1, \ldots, a_{d-1}$ [2]. We state this fact more precisely, which is excerpted from a modern book [7, Section 1.3]. For $c \in \mathbb{C}$ and $r > 0$, we write $D(c, r)$ for an open disc centered at c with radius r.

Proposition 1.2. *Let*

$$f(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 = \prod_{i=1}^{k}(x - \alpha_i)^{m_i} \in \mathbb{C}[x],$$

with $m_1 + m_2 + \cdots + m_k = d$, be a monic polynomial with distinct zeros $\alpha_1, \alpha_2, \ldots, \alpha_k$ of multiplicities $m_1, m_2, \ldots, m_k$, respectively. Given a positive $\varepsilon < \min_{1 \le i < j \le k} |\alpha_i - \alpha_j|/2$, there exists a $\delta > 0$ such that any monic polynomial $g(x) = x^d + b_{d-1}x^{d-1} + \cdots + b_0$ with $|b_i - a_i| < \delta$ for all $i = 0, 1, \ldots, d-1$ has exactly m_j zeros in each disc $D(\alpha_j, \varepsilon)$ for $j = 1, 2, \ldots, k$.

2. Irreducible polynomials

Given a polynomial $f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{Z}[x]$, we assume, throughout the paper, that the coefficients are coprime: $\gcd(a_d, a_{d-1}, \ldots, a_0) = 1$.

We first consider polynomials with a large prime leading coefficient.

Theorem 2.1. *Let $f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{Z}[x]$ with $a_d \neq 0$ and $a_0 \neq 0$. Suppose that one of the following conditions holds:*

(i) *a_d is a prime and $a_d > \sum_{i=0}^{d-1} |a_i|$,*

(ii) *a_0 is a prime and $a_0 > \sum_{i=1}^{d} |a_i|$.*

Then $f(x^n)$ is irreducible over $\mathbb{Q}$ for every integer $n \ge 1$.

Proof. Since each $f(x^n)$ also satisfies the corresponding condition on its coefficients, it suffices to prove irreducibility of $f(x)$.

(i) On the unit circle in $\mathbb{C}$, one notes that

$$|a_d x^d| = a_d > \sum_{i=0}^{d-1} |a_i| \ge |a_{d-1}x^{d-1} + \cdots + a_1 x + a_0|. \tag{1}$$

Now Rouché's theorem guarantees that the number of zeros of $f(x)$ inside the unit circle is equal to that of $a_d x^d$ in the same place. In other words, all the zeros of $f(x)$ lie inside the unit circle.

For some $g(x), h(x) \in \mathbb{Z}[x]$, suppose $f(x) = g(x)h(x)$ with $\deg g = m$, $\deg h = n$, and $0 < m, n < d$. Since a_d is a prime, either $g(x)$ or $h(x)$ should be monic. One may assume that $g(x)$ is monic. All the zeros of $g(x)$ lie inside the unit circle, and the absolute value of their product is $|g(0)|$ because $g(x)$ is monic. Therefore, $|g(0)| < 1$. But $g(0)$ is a nonzero integer, which is a contradiction.

(ii) Suppose $f(x) = g(x)h(x)$ as above. Then

$$\tilde{f}(x) = x^d f(x^{-1}) = x^m g(x^{-1})\ x^n h(x^{-1}) = \tilde{g}(x)\tilde{h}(x),$$

which contradicts (i). $\square$

In the previous theorem, the primality of the leading coefficient is indispensable.

Example 2.2. *The coefficients of a polynomial* $16x^2+8x+1$ *satisfies* $16 > 8+1$, *but* $16x^2+8x+1 = (4x+1)^2$ *is reducible.*

Next, we turn to polynomials whose second leading coefficients are large in moduli.

Theorem 2.3. *Let* $f(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ *with* $a_0 \neq 0$. *Suppose* $|a_{d-1}| > 1 + \sum_{i=0}^{d-2} |a_i|$. *Then* $f(x)$ *is irreducible over* $\mathbb{Q}$. *In addition, if* $a_{d-1} > 0$, *then* $f(x^2)$ *is also irreducible over* $\mathbb{Q}$.

Remark 2.4. After having obtained the whole results in this paper, the author recognized that the irreducibility of $f(x)$ had been proved by Perron [6]. All the other results in Theorem 2.3 and throughout the paper seem to be new. Since published in 1907, Perron's paper has been cited 7 times according to MathSciNet (*https://mathscinet.ams.org*). We include the proof for completeness.

Remark 2.5. As in Theorem 2.1, one also obtains irreducibility for $\tilde{f}(x)$. In the case, $|a_1|$ should be large and $|a_0|$ be equal to 1.

Proof of Theorem 2.3. On the unit circle in $\mathbb{C}$, the hypothesis implies that

$$|a_{d-1}x^{d-1}| = |a_{d-1}| > 1 + \sum_{i=0}^{d-2} |a_i| \geq |x^d + a_{d-2}x^{d-2} + \cdots + a_1 x + a_0|.$$

From Rouché's theorem it follows that the number of zeros of $f(x)$ inside the unit circle is equal to that of $a_{d-1}x^{d-1}$. That is, the $d-1$ zeros of

$f(x)$ lie inside the unit circle, and the only one zero is outside the unit circle.

For some $g(x), h(x) \in \mathbb{Z}[x]$, suppose $f(x) = g(x)h(x)$ with $\deg g = m$, $\deg h = n$, and $0 < m, n < d$. We may assume that $h(x)$ has a zero outside the unit circle. Then all the zeros of $g(x)$ lie inside the unit circle. Note that $g(x)$ is monic. As before, $g(0)$ is a nonzero integer, and at once $|g(0)| < 1$. This is a contradiction.

The only zero of $f(x)$ outside the unit circle should be real. Otherwise, its complex conjugate is another zero outside the unit circle. Suppose $a_{d-1} > 0$. If d is even, then $f(-1) < 0$ whereas $\lim_{x\to-\infty} f(x) = \infty$. For an odd d, we have $f(-1) > 0$ while $\lim_{x\to-\infty} f(x) = -\infty$. In either case, $f(x)$ has a negative real zero $\alpha_1 < -1$. Let $\alpha_2, \alpha_3, \ldots, \alpha_d$ be the other zeros. Then $f(x^2)$ has zeros $\pm\sqrt{\alpha_1}, \pm\sqrt{\alpha_2}, \ldots, \pm\sqrt{\alpha_d}$. Consequently, only two of them lie outside the unit circle, i.e., $|\pm\sqrt{\alpha_1}| > 1$ and $|\pm\sqrt{\alpha_i}| < 1$ for $i = 2, 3, \ldots, d$. Suppose $f(x^2) = g_2(x)h_2(x)$ for some nonconstant polynomials $g_2(x), h_2(x) \in \mathbb{Z}[x]$. If $h_2(x)$ has one of zeros $\pm\sqrt{\alpha_1}$, then it also has the other as a zero, because $\pm\sqrt{\alpha_1}$ are complex conjugates to each other. Accordingly, all the zeros of $g_2(x)$ lie inside the unit circle. The claimed contradiction follows. $\square$

For the irreducibility of $f(x^2)$ in the previous theorem, the positivity of a_{d-1} is essential as the next example says.

Example 2.6. *If $a_{d-1} < 0$ in Theorem 2.3, then the polynomial $f(x^2)$ is not necessarily irreducible over $\mathbb{Q}$. For instance, $f(x) = x^2 - 3x + 1$ is irreducible according to Theorem 2.3, but one finds that*

$$f(x^2) = x^4 - 3x^2 + 1 = (x^2 - x - 1)(x^2 + x - 1).$$

Even though $a_{d-1} > 0$ in Theorem 2.3, it appears that the polynomial $f(x^n)$ is not necessarily irreducible over $\mathbb{Q}$ unless $n = 1, 2$.

Example 2.7. *By Theorem 2.3, a polynomial $f(x) = x^3 + 11x^2 + 3x + 1$ is irreducible, but one derives that*

$$f(x^3) = x^9 + 11x^6 + 3x^3 + 1 = (x^3 + 2x^2 + 1)(x^6 - 2x^5 + 4x^4 + 2x^3 - 2x^2 + 1).$$

Similarly, both $g_1(x) = x^2 + 14x + 1$ and $g_2(x) = x^2 + 34x + 1$ are irreducible, but $g_1(x^4)$ and $g_2(x^4)$ are reducible, and have irreducible factors as follows:

$$g_1(x^4) = x^8 + 14x^4 + 1 = (x^4 - 2x^3 + 2x^2 + 2x + 1)(x^4 + 2x^3 + 2x^2 - 2x + 1),$$

$$g_2(x^4) = x^8 + 34x^4 + 1 = (x^4 - 4x^3 + 8x^2 - 4x + 1)(x^4 + 4x^3 + 8x^2 + 4x + 1).$$

3. Reducible polynomials

This section specifies, via some (counter-)examples, the limitations of the theorems proved in Section 2. First, the inequalities in Theorem 2.1 cannot be improved as the next theorem and examples say.

Theorem 3.1. *Let $f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{Z}[x]$ with $a_d \neq 0$ and $a_0 \neq 0$. Suppose that one of the following conditions holds:*

(i) *a_d is a prime and $a_d = \sum_{i=0}^{d-1} |a_i|$,*

(ii) *a_0 is a prime and $a_0 = \sum_{i=1}^{d} |a_i|$.*

If $f(x^n)$ has no cyclotomic factor, then $f(x^n)$ is irreducible over $\mathbb{Q}$.

Proof. Since each $f(x^n)$ also satisfies the corresponding condition on its coefficients, it suffices to prove it for $n = 1$. We also prove only the case of (i).

Let $r > 1$. For $x = re^{i\theta}$, one has

$$\begin{aligned}|a_{d-1}x^{d-1} + a_{d-2}x^{d-2} + \cdots + a_0| &\leq |a_{d-1}|r^{d-1} + |a_{d-2}|r^{d-2} + \cdots + |a_0| \\ &\leq r^{d-1}\sum_{i=0}^{d-1}|a_i| = r^{d-1}a_d < |a_d x^d|.\end{aligned}$$

By Rouché's theorem, all the zeros of $f(x)$ lie in $D(0, r)$. Since $r > 1$ is arbitrary, all the zeros of $f(x)$ lie on the closed unit disk $\overline{D(0, 1)}$. Here, we have used the completeness of $\mathbb{C}$.

Suppose $f(x) = g(x)h(x)$ for some nonconstant $g(x), h(x) \in \mathbb{Z}[x]$. We may assume that $g(x)$ is monic and that $h(x)$ is irreducible. Then the Mahler measure $M(g)$ of $g(x)$ is equal to 1. Finally, Proposition 1.1 implies that $g(x)$ is a cyclotomic polynomial. □

Under the condition of the above theorem, reducible polynomials indeed exist.

Example 3.2. *Suppose that $f(x)$ fulfills the hypothesis of Theorem 3.1. If $f(x) = a_d x^d - a_{d-1}x^{d-1} - \cdots - a_1 x - a_0$, where $a_i \geq 0$ for $i = 0, 1, \ldots, d$, then $f(x)$ has a cyclotomic factor $x - 1$. Or if*

$$f(x) = \begin{cases} a_d x^d + a_{d-1}x^{d-1} - a_{d-2}x^{d-2} + \cdots + a_1 x - a_0, & \text{when } d \text{ is even,} \\ a_d x^d + a_{d-1}x^{d-1} - a_{d-2}x^{d-2} + \cdots - a_1 x + a_0, & \text{otherwise,} \end{cases}$$

where $a_i \geq 0$ for $i = 0, 1, \ldots, d$, then $f(x)$ has a cyclotomic factor $x + 1$.

Corollary 3.3. *Let $f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{Z}[x]$ be given as in Theorem 3.1, and suppose that $f(x)$ satisfies one of* (i) *and* (ii)*.*

Let $\{0 \le i \le d : a_i \ne 0\}$ be the set of the indices of nonzero coefficients, and assume $f(1) \ne 0$ and $f(-1) \ne 0$. If $f(x)$ is reducible over $\mathbb{Q}$, then every element in $\{0 \le i \le d : a_i \ne 0\}$ is a multiple of some fixed divisor $1 < k < d$ of d. In particular, if $\gcd\{0 \le i \le d : a_i \ne 0\} = 1$, then $f(x)$ is irreducible over $\mathbb{Q}$.

Proof. If $f(x)$ is reducible, then it necessarily has a zero $e^{i\theta} \ne \pm 1$ on the unit circle. Consequently, for $x = e^{i\theta}$,

$$a_d e^{id\theta} + a_{d-1} e^{i(d-1)\theta} + \cdots + a_0 = 0,$$

and hence, the inequalities in

$$\begin{aligned}|a_{d-1}e^{i(d-1)\theta} + a_{d-2}e^{i(d-2)\theta} + \cdots + a_0| &\le |a_{d-1}| + |a_{d-2}| + \cdots + |a_0| \\ &\le \sum_{i=0}^{d-1} |a_i| = a_d \le |a_d e^{id\theta}|\end{aligned}$$

are all indeed equalities. Since $a_0 \ne 0$, these equalities are possible only if $e^{il\theta}$ is real, i.e., $l\theta$ is a multiple of π for every $l \in \{0 \le i \le d : a_i \ne 0\}$. In particular, $\theta = m\pi/d$ for some $m \in \{0, 1, \ldots, 2d-1\} \setminus \{0, d\}$, because $a_d \ne 0$ and $e^{i\theta} \ne \pm 1$. Note that there is $0 < l' < d$ in $\{0 \le i \le d : a_i \ne 0\}$. Otherwise, if $\{0 \le i \le d : a_i \ne 0\} = \{0, d\}$, then the coefficients of $f(x)$ are no more coprime. Suppose $\theta = m\pi/d = n\pi/k$ for some positive $n, k \in \mathbb{Z}$ with $\gcd(n, k) = 1$. Then $1 < k < d$. If $\gcd(m, d) = 1$, then the fact that $e^{il'\theta}$ is real implies that d divides ml', and so divides l', which is a contradiction.

Now, for every $l \in \{0 \le i \le d : a_i \ne 0\}$, we find that $e^{iln\pi/k}$ is real, whence k divides l. □

The next special case immediately follows.

Corollary 3.4. *Suppose that $f(x)$ satisfies all the hypotheses in Corollary 3.3. If, in addition, the degree d of $f(x)$ is an odd prime, then $f(x)$ is irreducible over $\mathbb{Q}$.*

Example 3.5. *Let $f(x) = 17x^{13} + a_{12}x^{12} + \cdots + a_0 \in \mathbb{Z}[x]$ with $f(\pm 1) \ne 0$. Then $f(x)$ is irreducible over $\mathbb{Q}$ as long as $\sum_{i=0}^{12} |a_i| \le 17$.*

The example below verifies both Theorem 3.1 and Corollary 3.3.

Example 3.6. *Let $f(x) = 5x^3 + 4x^2 + 1$. Then $f(x)$ satisfies the condition of Theorem 3.1. Noting $f(x) = (x+1)(5x^2 - x + 1)$, we find that $f(x^{2n})$ is factored into the following polynomials:*

$$f(x^{2n}) = 5x^{6n} + 4x^{4n} + 1 = (x^{2n} + 1)(5x^{4n} - x^{2n} + 1).$$

Here, $5x^{4n} - x^{2n} + 1$ is irreducible by Theorem 2.1, while $x^{2n} + 1$ is a cyclotomic polynomial that is not necessarily irreducible.

Next, we turn to Theorem 2.3. We discuss the case where the equality holds in Theorem 2.3.

Example 3.7. *Let a_i be nonnegative integers for $i = 0, 1, \ldots, d-1$, and suppose $a_{d-1} = 1 + \sum_{i=0}^{d-2} a_i$. Then $f(x) = x^d - a_{d-1}x^{d-1} + a_{d-2}x^{d-2} + a_{d-3}x^{d-3} + \cdots + a_0$ has a factor $x - 1$. On the other hand,*

$$f(x) = \begin{cases} x^d + a_{d-1}x^{d-1} & \\ \quad + a_{d-2}x^{d-2} - a_{d-3}x^{d-3} + \cdots - a_1 x + a_0, & \text{when } d \text{ is even,} \\ x^d + a_{d-1}x^{d-1} & \\ \quad + a_{d-2}x^{d-2} - a_{d-3}x^{d-3} + \cdots + a_1 x - a_0, & \text{otherwise} \end{cases}$$

has a factor $x + 1$.

The next theorem tells us that the above example is the only exception to the irreducibility when the coefficients satisfy $|a_{d-1}| = 1 + \sum_{i=0}^{d-2} |a_i|$.

Theorem 3.8. *Let $f(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ with $a_0 \neq 0$. Suppose $|a_{d-1}| = 1 + \sum_{i=0}^{d-2} |a_i|$. If $f(1) \neq 0$ and $f(-1) \neq 0$, then $f(x)$ is irreducible over $\mathbb{Q}$.*

Proof. Pick a sequence of complex vectors

$$\mathbf{b}_k = (b_{d-1,k}, b_{d-2,k}, \ldots, b_{0,k}) \in \mathbb{C}^d$$

so that $|b_{d-1,k}| > 1 + \sum_{i=0}^{d-2} |b_{i,k}|$ for any $k \geq 1$ and

$$\lim_{k\to\infty} \mathbf{b}_k = (a_{d-1}, a_{d-2}, \ldots, a_0).$$

For each $k \geq 1$, let us define $f_k(x) = x^d + b_{d-1,k}x^{d-1} + \cdots + b_{1,k}x + b_{0,k}$. A similar argument as in the proof of Theorem 2.3 shows that, for each $k \geq 1$, $d-1$ zeros of $f_k(x)$ lie inside the unit circle whereas only one zero is outside the unit circle. Since each zero of a polynomial is a continuous function of the coefficients (Proposition 1.2), at least $d-1$ zeros of $f(x)$ belong to the closed unit disk $\overline{D(0,1)}$.

Suppose $f(x) = g(x)h(x)$ for some nonconstant $g(x), h(x) \in \mathbb{Z}[x]$. We may assume that all zeros of $g(x)$ lie on $\overline{D(0,1)}$, and thus its Mahler measure is equal to 1. From Proposition 1.1, it follows that $g(x)$ is a cyclotomic polynomial, i.e., $e^{i\theta} \neq \pm 1$ is a zero of $f(x)$ for some $\theta \in \mathbb{R}$:

$$e^{id\theta} + a_{d-1}e^{i(d-1)\theta} + \cdots + a_1 e^{i\theta} + a_0 = 0.$$

But the equality $|a_{d-1}| = 1 + \sum_{i=0}^{d-2} |a_i|$ leads us to find that $e^{il\theta}$ is a real numbers whenever the coefficient of x^l in $f(x)$ is nonzero. In particular, both $e^{id\theta}$ and $e^{i(d-1)\theta}$ should be real. Therefore, one has $\theta = m\pi/d = n\pi/(d-1)$ for some $m, n \in \mathbb{Z}$, which is followed by $dn = (d-1)m$. So, d divides m, or equivalently, $e^{i\theta} = \pm 1$, which is a contradiction. □

Acknowledgments.

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2016R1D1A1B02010219).

References

[1] M. Bang and D.Y. Kwon. On the irreducibility of sum of two reciprocal polynomials. *Bull. Korean Math. Soc.* **53** (2016), no.3, 833–842.

[2] J.L. Coolidge. The continuity of the roots of an algebraic equation. *Ann. of Math. (2)* **9** (1908) 116–118.

[3] L. Kronecker. Zwei Sätze über Gleichungen mit ganzzahligen Coefficienten. *J. Reine Angew. Math.* **53** (1857), 173–175.

[4] D.Y. Kwon. Irreducible polynomials with all but one zero close to the unit disk. *Colloq. Math.* **143** (2016), no.2, 265–270.

[5] D.H. Lehmer. Factorization of certain cyclotomic functions. *Ann. of Math. (2)* **34** (1933), no.3, 461–479.

[6] O. Perron. Neue Kriterien für die Irreduzibilität algebraischer Gleichungen. *J. Reine Angew. Math.* **132** (1907), 288–307.

[7] Q.I. Rahman and G. Schmeisser. *Analytic theory of polynomials.* The Clarendon Press, Oxford University Press, Oxford, 2002.

DoYong Kwon
Department of Mathematics,
Chonnam National University,
Gwangju 61186, Republic of Korea
E-mail: doyong@jnu.ac.kr