Honam Mathematical J. **41** (2019), No. 3, pp. 595–607
https://doi.org/10.5831/HMJ.2019.41.3.595

STRONG COMPATIBILITY IN CERTAIN QUASIGROUP NONUNIFORM HOMOGENEOUS SPACES OF DEGREE 4

Bokhee Im* and Ji-Young Ryu

Abstract. We consider quasigroups $Q(\Gamma)$ obtained as certain double covers of the symmetric group S_3 of degree 3, for directed graphs Γ on the vertex set S_3. We completely characterize the strong compatibility of elements of $Q(\Gamma)$ for any quasigroup nonuniform homogeneous space of degree 4. For such homogeneous spaces, we classify all the strong and weak compatibility graphs of $Q(\Gamma)$.

1. INTRODUCTION AND PRELIMINARIES

If G is a group of permutations on a finite set Ω, two permutations g and h are said to be *compatible* precisely when xg is not equal to xh for any x in Ω. In other words, the quotient $g/h = gh^{-1}$ has no fixed points. One then defines the compatibility graph of G on Ω as the undirected graph on the vertex set G, in which an edge joins two permutations if and only if they are compatible.

Permutation representations have been extended from groups to loops [7], quasigroups [8], implementing various versions of approximate symmetry. The concept of compatiblity extends in parallel, actually splitting into distinct concepts of strong and weak compatibility (Definition 1.2). This paper is part of a program to study the compatibility graphs of quasigroup permutation representations and homogeneous spaces.

In [4], we considered certain quasigroups $Q(\Gamma)$ obtained as double covers of the symmetric group S_3 of degree 3, for directed graphs Γ on the vertex set S_3, and established a non-regular approximate symmetry.

Received January 18, 2019. Revised April 10, 2019. Accepted April 23, 2019.
2010 Mathematics Subject Classification. 20N05, 05C25, 05E15
Key words and phrases. quasigroup, quasigroup action, homogeneous space, intercalate, strongly compatible, compatibility graph, compatibility.
This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2011-0023226).
*Corresponding author

We also completely characterized the weak compatibility of $Q = Q(\Gamma)$ for any quasigroup homogeneous space $P\backslash Q$ of degree 4, even though the corresponding weak compatibility graph is missing in [4].

This paper is a continuation of [4]. We complete the strong compatibility characterization in terms of the corresponding adjacency matrix restriction and list all the strong and weak compatibility graphs of Q for such nonuniform homogeneous spaces. In Section 2, we show the unique weak compatibility graph in Figure 2 of Theorem 2.2. Applying the weak compatibility, we obtain the strong compatibility characterization in Theorem 2.3 and Theorem 2.4. And we characterize and list all the strong compatibility graphs of Q for $P\backslash Q$ in Theorem 3.3.

Definition 1.1 ([10]). *Let P be a subquasigroup of a finite, nonempty quasigroup Q.*

(1) The relative left multiplication group $\mathrm{LMlt}_Q P$ *of P in Q is the subgroup of the symmetric group $SymQ$ on the set Q which is generated by all the left multiplications*

$$L(p) : Q \to Q; x \mapsto px$$

for elements p of P.

(2) The homogeneous space $P\backslash Q$ *is defined as the set of all orbits of* $\mathrm{LMlt}_Q P$ *on Q. The cardinality $|P\backslash Q|$ is called the* degree d *of the homogeneous space $P\backslash Q$. For each element q of Q, the right multiplication by q is the permutation*

$$R(q) : Q \to Q; x \mapsto xq$$

of Q. The action matrix $R_{P\backslash Q}(q)$ *of q on $P\backslash Q$ is the $d\times d$ row-stochastic matrix with entry*

$$[R_{P\backslash Q}(q)]_{XY} = \frac{|XR(q) \cap Y|}{|X|}$$

in the row labeled by the $\mathrm{LMlt}_Q P$*-orbit X and column labeled by the* $\mathrm{LMlt}_Q P$*-orbit Y. The homogeneous space $P\backslash Q$ is understood as the set of all orbits of $LMlt_Q P$ on Q together with the action map $q \mapsto R_{P\backslash Q}(q)$ and called* nonuniform *in the case of having different orbit sizes.*

The above quasigroup action may be interpreted as a proper approximate symmetry if at least one action matrix is not a permutation matrix. A proper approximate symmetry has been defined as an exact symmetry holding at least one level of a hierarchial system in [9, 10].

We restrict ourselves to the following special case in this paper;

(1) a hierarchy with just two levels, *macroscopic* and *microscopic*,

(2) exact three-fold symmetry only at the macroscopic level as follows:

Macrostates:	A	B	C
Microstates:	$a,\ a'$	b	c

,

where $P\backslash Q = \{a = P, a', b, c\},\ \ A = \{a, a'\},\ B = \{b\},\ C = \{c\}$. Acknowledging the distinction between a and a', however, we see that the symmetry is approximate at the microscopic level. Note that our microstates are of different cardinality, hence our homogeneous space $P\backslash Q$ becomes nonuniform. We will abuse notation by considering a macrostate as the union of its orbits. That is to say, the first macrostate A is understood also as $a \cup a'$, $B = \{b\} = b$, and $C = \{c\} = c$(See Proposition 2.1). This abuse of notation leads us to classify elements of Q properly and also to use the block multiplication table in Figure 3 efficiently to calculate the inverse $R(q)^{-1}(Y)$ of a right multiplication $R(q)$ of $q \in Q$ for $Y \in P\backslash Q$.

We have the following definition.

Definition 1.2 (**Strong and weak compatibility**([3])). *Suppose that P is a subquasigroup of a finite, nonempty quasigroup Q.*

(1) Two distinct elements q_1 and q_2 of Q are said to be strongly compatible *(*in the action on $P\backslash Q$*) if*

$$R(q_1)^{-1}(Y) \cap R(q_2)^{-1}(Y) = \varnothing$$

for all points Y of $P\backslash Q$.

(2) Two distinct elements q_1 and q_2 of Q are said to be weakly compatible *(*in the action on $P\backslash Q$*) if*

$$[R_{P\backslash Q}(q_1)]_{XY} + [R_{P\backslash Q}(q_2)]_{XY} \leq 1$$

for all points X and Y of $P\backslash Q$.

(3) The strong *and* weak compatibility graphs *of Q for the space $P\backslash Q$ are the undirected graphs (without loops), on the vertex set Q, in which two vertices are joined by an edge if and only if they are respectively strongly or weakly compatible in the action of Q on $P\backslash Q$.*

We follow notations and terminologies as described in [4].

2. COMPATIBILITY CHARACTERIZATION

We denote the six elements of S_3 as three rotations $\rho_0 = (0)$, $\rho_1 = (021)$, $\rho_2 = (012)$, and three reflections $\sigma_0 = (12)$, $\sigma_1 = (02)$, $\sigma_2 = (01)$.

Take $\mathbb{Z}_2 = \{0,\ 1\}$ as in [3]. For convenience, let us use the notation π^ϵ to denote an element (π, ϵ) of $S_3 \times \mathbb{Z}_2$. Figure 1 displays the compatibility graph of the symmetric group S_3 in its natural action on the set $\{0, 1, 2\}$([10]).

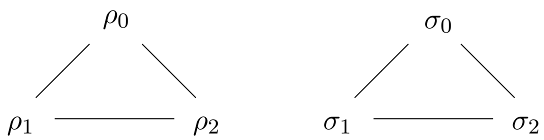

FIGURE 1. The compatibility graph of S_3.

We arrange the direct product $S_3 \times \mathbb{Z}_2$ as an ordered set

$$\{\rho_0^0, \rho_0^1, \sigma_0^0, \sigma_0^1, \rho_1^0, \rho_1^1, \sigma_1^0, \sigma_1^1, \rho_2^0, \rho_2^1, \sigma_2^0, \sigma_2^1\}$$

according to the ordered set S_3=$\{\rho_0, \sigma_0, \rho_1, \sigma_1, \rho_2, \sigma_2\}$, so that we have exact symmetry at the macroscopic level of hierarchial system throughout this paper.

Consider a relation $\lambda \cdot \mu = \nu$ in S_3. The corresponding fragment $\begin{bmatrix} \nu^0 & \nu^1 \\ \nu^1 & \nu^0 \end{bmatrix}$ of the body of the multiplication table of $S_3 \times \mathbb{Z}_2$, indexed by the respective rows labeled λ^0, λ^1 and columns labeled μ^0, μ^1, is known as the *intercalate* corresponding to the *source* λ and *sink* μ.

The body of the multiplication table of the group $S_3 \times \mathbb{Z}_2$ is a Latin square. It remains a Latin square if the intercalate $\begin{bmatrix} \nu^0 & \nu^1 \\ \nu^1 & \nu^0 \end{bmatrix}$ is changed to $\begin{bmatrix} \nu^1 & \nu^0 \\ \nu^0 & \nu^1 \end{bmatrix}$. Each such quasigroup is specified uniquely by a directed graph Γ (in which loops are allowed) on the vertex set S_3: The intercalate corresponding to the source λ and sink μ is changed precisely when there is a directed edge in Γ from the source λ to the sink μ. Write $Q(\Gamma)$ for the quasigroup specified in this way by a directed graph Γ. Let $M(\Gamma) = [\gamma_{\lambda\mu}]$ be the 6×6 adjacency matrix of Γ, interpreted as a matrix over the field $\mathbb{Z}_2$. Then for λ, μ in S_3 and l, m in $\mathbb{Z}_2$, the equation

$$\lambda^l \cdot \mu^m = \nu^{l+m+\gamma_{\lambda\mu}}$$

specifies the product of λ^l and μ^m in $Q(\Gamma)$, given the product relation $\lambda \cdot \mu = \nu$ in S_3. Note that the projection

$$\theta : Q(\Gamma) \to S_3; \lambda^l \mapsto \lambda$$

is a quasigroup homomorphism.

Let us summarize Lemma 2.1 and Lemma 2.2 of [4] properly as the following proposition which describes the nonuniform homogeneous space $P\backslash Q$ considered in this paper.

Proposition 2.1 ([4]). *Let $Q := Q(\Gamma)$ be the quasigroup specified by a directed graph Γ as explained above, and let $P\backslash Q$ be a nonuniform homogeneous space of degree 4 for any subquasigroup P of Q. Then without loss of generality we have*

(1) P is one of $\{\rho_0^0, \sigma_0^0\}$, $\{\rho_0^0, \sigma_0^1\}$, $\{\rho_0^1, \sigma_0^0\}$, and $\{\rho_0^1, \sigma_0^1\}$.

(2) $P\backslash Q$ consists of four microstates ordered as P, $A-P$, B, and C and also of three macrostates $A := \{\rho_0^0, \rho_0^1, \sigma_0^0, \sigma_0^1\}$, $B := \{\rho_1^0, \rho_1^1, \sigma_1^0, \sigma_1^1\}$ and $C := \{\rho_2^0, \rho_2^1, \sigma_2^0, \sigma_2^1\}$.

(3) The first two rows of the adjacency matrix $M(\Gamma) = [\gamma_{\lambda\mu}]$ should have the following restrictions: Let $\rho_0^\varepsilon \in P$. Then $\begin{bmatrix} \gamma_{\rho_0,\rho_0} & \gamma_{\rho_0,\sigma_0} \\ \gamma_{\sigma_0,\rho_0} & \gamma_{\sigma_0,\sigma_0} \end{bmatrix} = \begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix}$ and $\begin{bmatrix} \gamma_{\rho_0,\rho_i} & \gamma_{\rho_0,\sigma_i} \\ \gamma_{\sigma_0,\rho_i} & \gamma_{\sigma_0,\sigma_i} \end{bmatrix}$ is a matrix except $\begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix}$ and $\begin{bmatrix} \varepsilon & \varepsilon \\ 1-\varepsilon & 1-\varepsilon \end{bmatrix}$ for $i = 1, 2$.

The graph of Figure 2 describes precisely the weak compatibility characterization of Q for the homogeneous space $\{\rho_0^\varepsilon, \sigma_0^\tau\}\backslash Q(\Gamma)$ of degree 4.

Theorem 2.2 (**Weak compatibility graph**). *Let $Q = Q(\Gamma)$ and $P\backslash Q$ be as explained in Proposition 2.1. Then the weak compatibility graph of Q for $P\backslash Q$ is precisely as in Figure 2 for any directed graph Γ.*

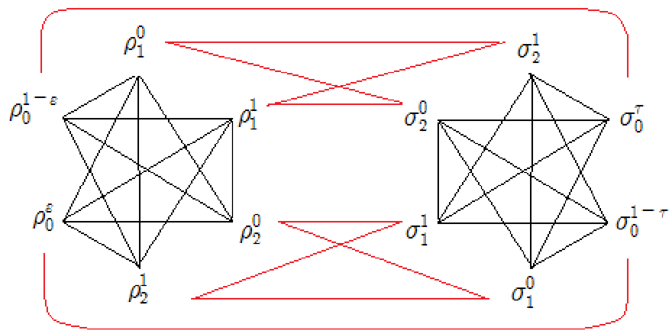

FIGURE 2. The unique weak compatibility graph for the homogeneous space $\{\rho_0^\varepsilon, \sigma_0^\tau\}\backslash Q$.

Proof. Theorem 3.3 of [4] can be expressed precisely as the above weak compatibility graph. All the action matrices of $Q(\Gamma)$ are independent of the choice of P and Γ, by Theorem 2.3 of [4]. Hence we obtain the result. □

We now discuss the strong compatibility of q_1 and q_2 in Q. Strong compatibility implies weak compatibility ([3]). That is, if two elements q_1 and q_2 of Q are not weakly compatible, then q_1 and q_2 are not strongly compatible. So we need to check the strong compatibility only for those vertices that are connected in the graph in Figure 2.

Figure 3 gives the block multiplication table of $Q = Q(\Gamma)$, where $\pi = \{\pi^0, \pi^1\}$ for each $\pi \in S_3$. Clearly, this table is independent of choice of Γ.

Q	ρ_0	σ_0	ρ_1	σ_1	ρ_2	σ_2
A	A	A	B	B	C	C
B	B	C	C	A	A	B
C	C	B	A	C	B	A

FIGURE 3. The block multiplication table of $Q = S_3 \times \mathbb{Z}_2$

Note that the body of the above block multiplication table of Q has 3 rows and 6 columns. Examining the second and third rows of the above table, it is not possible to reduce as 3×3 block multiplication table. In fact, Figure 3 is essential in determining strong compatibility of two distinct elements q_1 and q_2 of Q. In Theorem 2.3, we show that any two vertices in Figure 2 are strongly compatible, except for the vertices

$$\rho_i^{\varepsilon} \text{ and } \sigma_j^{\tau}, \text{ for } 1 \le i \ne j \le 2 \text{ and } 0 \le \varepsilon, \tau \le 1. \tag{1}$$

Theorem 2.3 (**Strong compatibility characterization I**). *Let Q and $P\backslash Q$ be as explained in Proposition 2.1. Then any strong compatibility graph must contain all the edges in the graph of Figure 4. Moreover, this strong compatibility graph is minimal.*

Proof. If both $\theta(q_1)$ and $\theta(q_2)$ are rotations or reflections, such that q_1 and q_2 belong to the different macrostates, then q_1 and q_2 are strongly compatible by the group structure.

Since $\rho_0^{\varepsilon} \in P$, the matrix $\begin{bmatrix} \gamma_{\rho_0,\rho_0} & \gamma_{\rho_0,\sigma_0} \\ \gamma_{\sigma_0,\rho_0} & \gamma_{\sigma_0,\sigma_0} \end{bmatrix}$ of the adjacency matrix of Γ is $\begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix}$ by (3) of Proposition 2.1. That is, if $q_1 = \rho_0^{\varepsilon}$ and $q_2 = \sigma_0^{\tau}$ for $0 \le$

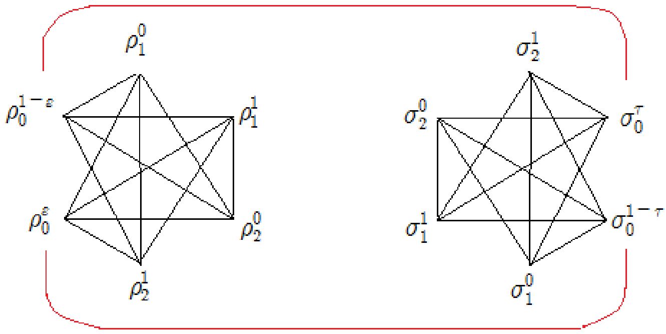

FIGURE 4. (Minimal) strong compatibility graph for $\{\rho_0^{\varepsilon}, \sigma_0^{\tau}\}\backslash Q$.

$\varepsilon, \tau \leq 1$, then $PR(q_1) = P$, $PR(q_2) = (A - P)$, $(A-P)R(q_1) = (A-P)$ and $(A-P)R(q_2) = P$. Independent of the choices of a directed graph Γ, we have $BR(q_1) = B$, $BR(q_2) = C$, $CR(q_1) = C$ and $CR(q_2) = B$. Therefore, for all orbits X of $P\backslash Q$, $R(q_1)^{-1}(X) \cap R(q_2)^{-1}(X) = \varnothing$. Hence q_1 and q_2 are strongly compatible.

The minimality of this strong compatibility graph will be shown in Theorem 3.1. □

The case (1) is completely characterized in Theorem 2.4.

Theorem 2.4 (**Strong compatibility characterization II**). *Let $Q = Q(\Gamma)$ be the quasigroup specified by a directed graph Γ, and let $P\backslash Q$ be the homogeneous space of degree 4 for any subquasigroup P of Q as described in Proposition 2.1. Let q_1 and $q_2 \in Q$ so that $\theta(q_1)$ is a rotation and $\theta(q_2)$ is a reflection satisfying that q_1 and q_2 are contained in the different orbits B, C. i.e. $q_1 = \rho_i^{\varepsilon}$ and $q_2 = \sigma_j^{\tau}$, for $1 \leq i \neq j \leq 2$ and $0 \leq \varepsilon, \tau \leq 1$.*

(1) If $P = \{\rho_0^0, \sigma_0^0\}$ and $\varepsilon = \tau$, then q_1 and q_2 are strongly compatible if and only if the entries of each row of the 2×2 matrix $\begin{bmatrix} \gamma_{\rho_j,\rho_i} & \gamma_{\rho_j,\sigma_j} \\ \gamma_{\sigma_j,\rho_i} & \gamma_{\sigma_j,\sigma_j} \end{bmatrix}$ are different, that is, the corresponding matrix is one of the following four matrices $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(2) If $P = \{\rho_0^0, \sigma_0^0\}$ and $\varepsilon \neq \tau$, then q_1 and q_2 are strongly compatible if and only if the entries of each row of the matrix $\begin{bmatrix} \gamma_{\rho_j,\rho_i} & \gamma_{\rho_j,\sigma_j} \\ \gamma_{\sigma_j,\rho_i} & \gamma_{\sigma_j,\sigma_j} \end{bmatrix}$ are

the same, that is, the corresponding matrix is one of the following four matrices $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$, $\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$, $\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$ *and* $\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}$.

(3) For the following 3 cases $P = \{\rho_0^0, \sigma_0^1\}$ *and* $\varepsilon \neq \tau$, $P = \{\rho_0^1, \sigma_0^0\}$ *and* $\varepsilon \neq \tau$, *and* $P = \{\rho_0^1, \sigma_0^1\}$ *and* $\varepsilon = \tau$, *the result of the above case (1) holds.*

(4) For the following 3 cases $P = \{\rho_0^0, \sigma_0^1\}$ *and* $\varepsilon = \tau$, $P = \{\rho_0^1, \sigma_0^0\}$ *and* $\varepsilon = \tau$, *and* $P = \{\rho_0^1, \sigma_0^1\}$ *and* $\varepsilon \neq \tau$, *the result of the above case (2) holds.*

Proof. Let us supply the proof for the case (1) only. Then we obtain the result of the case (2), because we have constructed our quasigroup $Q(\Gamma)$ only with intercalate changes while preserving the group structure of S_3. Analogously we obtain the case (3) and (4).

Suppose that $i = 1 < j = 2$, i.e., $q_1 = \rho_1^\varepsilon$ and $q_2 = \sigma_2^\varepsilon$ for $\varepsilon = 0, 1$. Then since $Aq_1 = B$ and $Bq_2 = B$, we have $R(q_1)^{-1}(B) \cap R(q_2)^{-1}(B) = A \cap B = \varnothing$. In the same way, since $Bq_1 = C$ and $Aq_2 = C$, we have $R(q_1)^{-1}(C) \cap R(q_2)^{-1}(C) = B \cap A = \varnothing$. So q_1 and q_2 are strongly compatible if and only if $R(q_1)^{-1}(P) \cap R(q_2)^{-1}(P) = \varnothing$. It is now enough to observe carefully the following subtable of the multiplication table of the quasigroup $Q(\Gamma)$ with no intercalate changes:

Q	ρ_1^0	ρ_1^1	σ_2^0	σ_2^1
ρ_2^0	ρ_0^0	ρ_0^1	σ_0^0	σ_0^1
ρ_2^1	ρ_0^1	ρ_0^0	σ_0^1	σ_0^0
σ_2^0	σ_0^0	σ_0^1	ρ_0^0	ρ_0^1
σ_2^1	σ_0^1	σ_0^0	ρ_0^1	ρ_0^0

Then we find out that q_1 and q_2 are strongly compatible if and only if any of $\rho_2^0 q_1 \cup \rho_2^0 q_2$ and $\sigma_2^0 q_1 \cup \sigma_2^0 q_2$ are neither P nor $A - P$, i.e., if and only if the entries of each row of the matrix $\begin{bmatrix}\gamma_{\rho_2,\rho_1} & \gamma_{\rho_2,\sigma_2}\\ \gamma_{\sigma_2,\rho_1} & \gamma_{\sigma_2,\sigma_2}\end{bmatrix}$ of $M(\Gamma)$ are different.

For the case $i = 2 > j = 1$, i.e., $q_1 = \rho_2^\varepsilon$ and $q_2 = \sigma_1^\varepsilon$, consider the matrix $\begin{bmatrix}\gamma_{\rho_1,\rho_2} & \gamma_{\rho_1,\sigma_1}\\ \gamma_{\sigma_1,\rho_2} & \gamma_{\sigma_1,\sigma_1}\end{bmatrix}$ whose columns are changed from the matrix $\begin{bmatrix}\gamma_{\rho_1,\sigma_1} & \gamma_{\rho_1,\rho_2}\\ \gamma_{\sigma_1,\sigma_1} & \gamma_{\sigma_1,\rho_2}\end{bmatrix}$ of $M(\Gamma)$.

Therefore, we obtain the result, for any $q_1 = \rho_i^\varepsilon$ and $q_2 = \sigma_j^\varepsilon$ with $1 \leq i \neq j \leq 2$, that q_1 and q_2 are strongly compatible if and only if the entries of each row of $\begin{bmatrix}\gamma_{\rho_j,\rho_i} & \gamma_{\rho_j,\sigma_j}\\ \gamma_{\sigma_j,\rho_i} & \gamma_{\sigma_j,\sigma_j}\end{bmatrix}$ are different. □

Remark 2.5. *As explained in the above proof of Theorem 2.4, one should keep in mind that the columns of* $\begin{bmatrix} \gamma_{\rho_1,\rho_2} & \gamma_{\rho_1,\sigma_1} \\ \gamma_{\sigma_1,\rho_2} & \gamma_{\sigma_1,\sigma_1} \end{bmatrix}$ *are in the inverse order from the ordinary submatrix* $\begin{bmatrix} \gamma_{\rho_1,\sigma_1} & \gamma_{\rho_1,\rho_2} \\ \gamma_{\sigma_1,\sigma_1} & \gamma_{\sigma_1,\rho_2} \end{bmatrix}$ *of* $M(\Gamma)$.

Corollary 2.6. *Let* $Q = Q(\Gamma)$ *be a quasigroup as in Proposition 2.1. Then* ρ_i^ε *is strongly compatible with at most one of* $\{\sigma_j^\varepsilon, \sigma_j^{1-\varepsilon}\}$ *for* $1 \le i \ne j \le 2$ *and* $0 \le \varepsilon \le 1$.

Corollary 2.7. *Let* $Q = Q(\Gamma)$ *be a quasigroup as in Proposition 2.1. If* ρ_i^ε *and* σ_j^τ *are strongly compatible, then* $\rho_i^{1-\varepsilon}$ *and* $\sigma_j^{1-\tau}$ *are also strongly compatible for* $1 \le i \ne j \le 2$ *and* $0 \le \varepsilon, \tau \le 1$.

3. STRONG COMPATIBILITY GRAPHS

Let $Q = Q(\Gamma)$ be the quasigroup specified by a directed graph Γ, and let $P\backslash Q$ be a nonuniform homogeneous space of degree 4 for any subquasigroup P of Q as described in Proposition 2.1. To draw the strong compatibility graphs, we take P as $\{\rho_0^0, \sigma_0^0\}$ without loss of generality. We characterize and list all the strong compatibility graphs of Q for $P\backslash Q$ in Theorem 3.3.

Theorem 3.1. *Let* $Q = Q(\Gamma)$ *and* $\{\rho_0^0, \sigma_0^0\}\backslash Q(\Gamma)$ *be as explained above. Then the graph of Figure 4 with* $\varepsilon = \tau = 0$ *is the unique minimal strong compatibility graph of* Q *for the homogeneous space* $\{\rho_0^0, \sigma_0^0\}\backslash Q(\Gamma)$ *of degree* 4.

Proof. Figure 2 is the unique weak compatibility graph for the homogeneous space $\{\rho_0^0, \sigma_0^0\}\backslash Q(\Gamma)$. But it is not the strong compatibility graph of Q, due to Corollary 2.6. So the strong compatibility graphs of Q for $P\backslash Q$ must have fewer edges than those of the weak compatibility graph in Figure 2. By Theorem 2.3, all edges between vertices ρ_i^ε's, as well as σ_i^ε's ($0 \le \varepsilon \le 1$, and $0 \le i \le 2$) must remain as shown in Figure 4. The outside two edges between ρ_0^ε and σ_0^τ ($0 \le \varepsilon \ne \tau \le 1$) must also remain, by Theorem 2.3. Then applying Theorem 2.4, we can take the corresponding entries of the adjacency matrix $M(\Gamma)$ of Γ properly to obtain no more (inside) edges between the left rotation island and right reflection island. □

Theorem 3.2. *Let $Q = Q(\Gamma)$ and $\{\rho_0^0, \sigma_0^0\}\backslash Q(\Gamma)$ be as explained above. Then the 4 graphs of Figure 5 are precisely the strong compatibility graph of Q for the nonuniform homogeneous space $\{\rho_0^0, \sigma_0^0\}\backslash Q(\Gamma)$ of degree 4 with the maximal number of edges.*

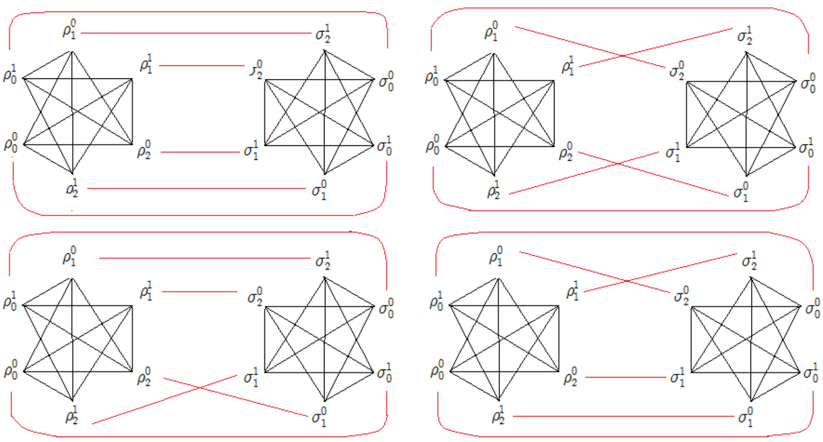

FIGURE 5. Maximal strong compatibility graphs for $\{\rho_0^0, \sigma_0^0\}\backslash Q(\Gamma)$.

Proof. If there exist one inside edge between two islands, say ρ_1^1 and σ_2^0 as in the first graph, then by Corollary 2.7 we must have the edge between ρ_1^0 and σ_2^1. But by Corollary 2.6 there are no more inside upper edges. Note that inside upper edges and lower edges appear independently by Theorem 2.4. □

By Theorem 3.1 and the proof of Theorem 3.2, we obtain the following Theorem 3.3 characterizing the strong compatibility graphs.

Theorem 3.3. *Let $Q = Q(\Gamma)$ be the quasigroup specified by a directed graph Γ, and let $P\backslash Q$ be a nonuniform homogeneous space of degree 4 for any subquasigroup P of Q, as described in Proposition 2.1. Take P as $\{\rho_0^0, \sigma_0^0\}$ without loss of generality. Then there exist precisely 9 different strong compatibility graphs of Q for $P\backslash Q$ as shown in Figure 4, Figure 5 and Figure 6.*

Finally, for the upper left strong compatibility graph, say G in Figure 6, we show how to determine all the adjacency matrices $M(\Gamma) = [\gamma_{\lambda\mu}]$ which give the strong compatibility graph G in the following example.

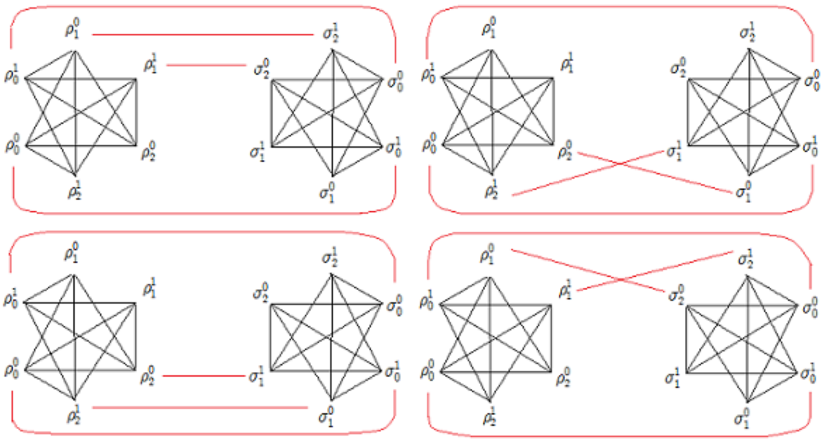

FIGURE 6. Remaining strong compatibility graphs for $\{\rho_0^0, \sigma_0^0\}\backslash Q(\Gamma)$.

Example 3.4. *Let G be the upper left strong compatibility graph for a quasigroup Q, for the nonuniform homogeneous space $\{\rho_0^0, \sigma_0^0\}\backslash Q$ of degree 4. Then by (3) of Proposition 2.1, we should take* $\begin{bmatrix} \gamma_{\rho_0,\rho_0} & \gamma_{\rho_0,\sigma_0} \\ \gamma_{\sigma_0,\rho_0} & \gamma_{\sigma_0,\sigma_0} \end{bmatrix}$ *as* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. *Since $|P\backslash Q| = 4$, we should restrict entries of the first two rows of the adjacency matrix $M(\Gamma) = [\gamma_{\lambda\mu}]$. Indeed, again by (3) of Proposition 2.1, the matrix* $\begin{bmatrix} \gamma_{\rho_0,\rho_i} & \gamma_{\rho_0,\sigma_i} \\ \gamma_{\sigma_0,\rho_i} & \gamma_{\sigma_0,\sigma_i} \end{bmatrix}$ *is one of 14 matrices, except for* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, *for $i = 1, 2$.*

Since there are no edges between ρ_2^ε and σ_1^τ $(\varepsilon, \tau = 0, 1)$ in the strong compatibility graph G, we need to restrict the third and fourth rows of $M(\Gamma)$: Due to both (1) and (2) of Theorem 2.4, the matrix $\begin{bmatrix} \gamma_{\rho_1,\rho_2} & \gamma_{\rho_1,\sigma_1} \\ \gamma_{\sigma_1,\rho_2} & \gamma_{\sigma_1,\sigma_1} \end{bmatrix}$ *must have only one entry different from the others, i.e. be one of* $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *and* $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Now for the last two rows of $M(\Gamma)$, notice in the graph G that ρ_1^ε and σ_2^τ are strongly compatible when $\varepsilon \neq \tau$, and that ρ_1^ε and σ_2^τ are not strongly compatible when $\varepsilon = \tau$. Then due to (2) of Theorem 2.4, *the matrix* $\begin{bmatrix}\gamma_{\rho_2,\rho_1} & \gamma_{\rho_2,\sigma_2}\\ \gamma_{\sigma_2,\rho_1} & \gamma_{\sigma_2,\sigma_2}\end{bmatrix}$ *must be one of* $\begin{bmatrix}0&0\\0&0\end{bmatrix}$, $\begin{bmatrix}1&1\\1&1\end{bmatrix}$, $\begin{bmatrix}1&1\\0&0\end{bmatrix}$ *and* $\begin{bmatrix}0&0\\1&1\end{bmatrix}$.

There are no further restrictions for the remainung 16 entries of the 6×6 *adjacency matrix* $M(\Gamma)$. *Hence we see that* G *appears as the compatibility graph of quasigroups* Q *which correspond to* $14 \times 14 \times 8 \times 4 \times 2^{16}$ *different number of adjacency matrices* $M(\Gamma)$. *The adjacency matrix* $M(\Gamma)$ *in Figure 7 is one such example.*

$$M(\Gamma) = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&1&1&0\\ 1&1&0&1&0&0\\ 1&0&0&0&0&0\\ 1&1&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}$$

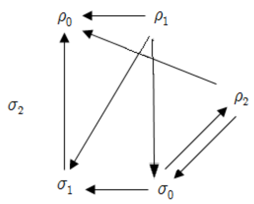

Figure 7. An adjacency matrix $M(\Gamma)$ and its corresponding directed graph Γ.

Remark 3.5. *As shown in Figure 2, all* ρ_i^{ε}*'s for* $0 \leq \varepsilon \leq 1$ *and* $0 \leq i \leq 2$ *do not form a clique* K_6 *of order 6. Moreover, no sharply transitive sets are involved in this paper, since our homogeneous space is nonuniform, i.e. our microstates are of different cardinality. In the case of uniform homogeneous space (of degree 6) as in* [5], *however, the clique* K_6 *appears in the weak and strong compatibility graph and there exist various sharply transitive sets.*

ACKNOWLEDGMENT

We are grateful to the referee for helpful and thoughtful comments on an earlier version of this paper.

References

[1] P.J. Cameron and C.Y. Ku, *Intersecting families of permutations*, Eur. J. Comb., **24** (2003), 881-890.

[2] K. Heinrich and W.D. Wallis, *The maximum number of intercalates in a latin square*, Combinatorial Mathematics VIII, K.L. McAvaney, Ed., Springer, Berlin, (1981), 221-233.

[3] B. Im, J.-Y. Ryu and J.D.H. Smith, *Sharply transitive sets in quasigroup actions*, J. Algebr. Comb. **33** (2011), 81-93.

[4] B. Im and J.-Y. Ryu, *Compatibility in certain quasigroup homogeneous space*, Bull.Korean Math. Soc **50** (2013), no. 2, 667-674.
[5] H.-Y. Lee, B. Im and J.D.H. Smith, *Stochastic tensors and approximate symmetry*, Discrete Math. **34D** (2017), 1335-1350.
[6] B.D. McKay and I.M. Wanless, *Most Latin squares have many subsquares*, J. Combin. Theory Ser. A **86** (1999), 322-347.
[7] J.D.H. Smith, *Permutation representations of loops*, J. Alg. **264** (2003), 342-357.
[8] J.D.H. Smith, *Quasigroup permutation representations*, Quasigroups and Related Systems **10** (2003), 115-134
[9] J.D.H. Smith, *Symmetry and entropy: a hierarchical perspective*, Symmetry **16** (2005), 37-45.
[10] J.D.H. Smith, *An Introduction to Quasigroups and Their Representations*, Chapman and Hall/CRC, Boca Raton, FL, 2007.

Bokhee Im
Department of Mathematics, Chonnam National University,
Gwangju 61186, Korea.
E-mail: bim@jnu.ac.kr

Ji-Young Ryu
Department of Mathematics, Chonnam National University,
Gwangju 61186, Korea.
E-mail: jryu@jnu.ac.kr