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RELATIONSHIPS BETWEEN CUSP POINTS IN THE EXTENDED MODULAR GROUP AND FIBONACCI NUMBERS

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Abstract. Cusp (parabolic) points in the extended modular group $\overline{\Gamma}$ are basically the images of infinity under the group elements. This implies that the cusp points of $\overline{\Gamma}$ are just rational numbers and the set of cusp points is $Q_\infty = Q \cup \{\infty\}$ .The Farey graph F is the graph whose set of vertices is Q_∞ and whose edges join each pair of Farey neighbours. Each rational number x has an integer continued fraction expansion (ICF) $x = [b_1, ..., b_n]$. We get a path from ∞ to x in F as $<\infty, C_1, ..., C_n>$ for each ICF. In this study, we investigate relationships between Fibonacci numbers, Farey graph, extended modular group and ICF. Also, we give a computer program that computes the geodesics, block forms and matrix represantations.

1. Modular and Extended Modular Group

The most important Hecke group $H(\lambda_3)$ is the modular group $\Gamma = PSL(2, \mathbb{Z})$, i.e.

$$PSL(2,\mathbb{Z}) = \{\frac{az+b}{cz+d} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1\}.$$

This group is equal to $SL(2, \mathbb{Z})/\{\pm I\}$.

Then, the modular group Γ is isomorphic to the free product of two finite cyclic groups of orders 2 and 3 and it has a presentation

$$\Gamma =< T, S \mid T^2 = S^3 = I >= C_2 * C_3.$$

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The extended modular group $\overline{\Gamma} = PGL(2,\mathbb{Z}) \simeq GL(2,\mathbb{Z})/\{\pm I\}$ is defined by adding the reflection $R(z) = 1/\overline{z}$ to the generators of the modular group Γ . Thus, the extended modular group $\overline{\Gamma}$ has the presentation

$$\overline{\Gamma} =< T,S,R \mid T^2 = S^3 = R^2 = (TR)^2 = (SR)^2 = I >= D_2 *_{Z_2} D_3.$$

The extended modular group $\overline{\Gamma}$ is also known to be an amalgamated free product of two dihedral groups of orders 4 and 6 with a cyclic group of orders 2. Also $\overline{\Gamma} = \Gamma \cup G'$ where $G' = \{\frac{a\overline{z}+b}{c\overline{z}+d} : a,b,c,d \in \mathbb{Z},\ ad - bc = -1\}$. Thus, extended modular group contains automorphisms and anti-automorphisms respectively. Modular and extended modular group have especially been of great interest in many fields of Mathematics, for example number theory, automorphic function theory, group theory and graph theory. (more information for modular and extended modular group see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11].

2. Continued Fractions and Farey Graph

The Farey sequence of order n is the sequence of completely reduced fractions between 0 and 1 which when in the lowest terms have denominators less than or equal to n, arranged in order of increasing size.

Each Farey sequence starts with the value 0, denoted by the fraction $\frac{0}{1}$, and ends with the value 1, denoted by the fraction $\frac{1}{1}$. Fractions $\frac{a}{b}, \frac{c}{d} \in Q$ are called neighbours if $|ad - bc| = 1$. The Farey sum is $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$ and it is called as mediant. Also $\frac{a}{b}$ and $\frac{c}{d}$ are the parents of $\frac{a+c}{b+d}$. The Farey sequences of orders 1 to 4 are

$$F_1 = \{\frac{0}{1}, \frac{1}{1}\}$$
$$F_2 = \{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\}$$
$$F_3 = \{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\}$$
$$F_4 = \{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\}$$

In particular, F_n contains all of the members of F_{n-1} and also contains an additional fraction for each number that is less than n and coprime to n. and $|F_n| = |F_{n-1}| + \varphi(n)$.

Now, we will introduce the Farey graph. Firstly $\frac{1}{0}$ is defined as the reduced form of ∞ and $\frac{a}{b}$ to be a Farey neighbour of ∞ if and only if $|a.0 - b.1| = 1$. The Farey graph F is the graph whose set of vertices is Q_∞ and whose edges join each pair of Farey neighbours. We denote the path as $< v_1, v_2, ..., v_n >$. The vertices of all the triangles are labeled with fractions $\frac{a}{b}$, including the fraction $\frac{1}{0}$ for ∞. In the upper half of the diagram first label the vertices of the big triangles $\frac{0}{1}$, $\frac{1}{1}$, and $\frac{1}{0}$ Then by induction, if the labels at the two ends of the long edge of a triangle are $\frac{a}{b}$ and $\frac{c}{d}$, the label on the third vertex of the triangle is $\frac{a+c}{b+d}$.

In recent years, mathematicians such as Alan Beardon, Caroline Series, Svetlana Katok, Ian Short and Mairi Walker have contributed to the theory of continued fractions by considering the action of particular groups of Mobius transformations [12],[13],[14],[15],[16],[17].

Definition 2.1. *In* [18], *Rosen defined $\lambda-$continued fractions related the real number λ as*

$$r_0\lambda - \cfrac{1}{r_1\lambda - \cfrac{1}{r_2\lambda - \cfrac{1}{r_3\lambda - \ldots - \cfrac{1}{r_n\lambda}}}} = [r_0\lambda; r_1\lambda, r_2\lambda, ..., r_n\lambda]$$

There are strong connections between Hecke groups and continued fractions.

In this paper, we study modular and extended modular group. We recall that if $\lambda = 1$, we get the finite integer continued fraction (ICF)

$$r_0 - \cfrac{1}{r_1 - \cfrac{1}{r_2 - \cfrac{1}{r_3 - \ldots - \cfrac{1}{r_n}}}} = [r_0; r_1, r_2, ..., r_n]$$

where all r_i are integers. The integers $r_1, r_2, ..., r_n$ are called the partial quotients of the continued fraction.

Corollary 2.2. *Let $V(z) = \frac{az+b}{cz+d} = U^{r_0}TU^{r_1}T...U^{r_n}T(z)$ be an automorphism in extended modular group $\overline{\Gamma}$, then*

$$V(\infty) = \frac{a}{c} = r_0 - \cfrac{1}{r_1 - \cfrac{1}{r_2 - \cfrac{1}{r_3 - \ldots - \cfrac{1}{r_n}}}}$$

Similarly, for an anti-automorphism $V'(z) = \frac{a\overline{z}+b}{c\overline{z}+d} =$ $U^{r_0}TU^{r_1}T...U^{r_n}R(z)$ in the extended modular group $\overline{\Gamma}$, we find

$$V^{'}(\infty)=\frac{a}{c}=r_0-\cfrac{1}{r_1-\cfrac{1}{r_2-\cfrac{1}{r_3-...-\frac{1}{r_n}}}}$$

Definition 2.3. *The n^{th}convergents of an integer continued fraction $[r_0;r_1,r_2,...]$ are defined as $C_n=[r_0;r_1,r_2,...,r_n]$.*

Theorem 2.4. [15] *Let $x=[r_0;r_1,r_2,...]$ be an integer continued fraction. Then we can get $C_n=[r_0;r_1,r_2,...,r_n]=\frac{p_n}{q_n}$ where $p_0=r_0$, $q_0=1$,*

$$p_1=r_0r_1+1, q_1=r_1, \quad \text{and } p_k=r_kp_{k-1}+p_{k-2}, q_k=r_kq_{k-1}+q_{k-2}$$

$(k=2,3,...,n)$

We need some connections between Farey graph and integer continued fractions. Let the real number x has an ICF expansion $x=[r_0;r_1,r_2,...,r_n]$ in which all r_i are integers. The convergents of an ICF-expansion of x, namely $[r_0;r_1,r_2,...,r_i]$ for $i=0,...,n$, form a finite sequence $C_0,...,C_n$ of vertices of F, where C_0 is an integer and $C_n=x$. We shall see that if we express C_i as an irreducible rational Ai/B_i then $|AiB_{i+1}-B_iA_{i+1}|=1$, so that C_i and C_{i+1} are Farey neighbors, and this implies that $<\infty,C_0,...,C_n>$ is a path from ∞ to x in F. The shortest ICF expansions of x as geodesic paths in F from ∞ to x; we shall call these shortest expansions the geodesic expansions of x.

Theorem 2.5. [15]*Suppose that x is rational and that $\rho(\infty,x)=n$. Then there are at most F_n geodesics from ∞ to x.(Where $\rho(\infty,x)$ is the legnth of the path and F_n is the n^{th} Fibonacci number.*

Cusp points are basically the images of infinity under the group elements in $\overline{\Gamma}$. All coefficients of the elements of the extended modular group $\overline{\Gamma}$ are rational integers. This implies that the parabolic points of $\overline{\Gamma}$ are just rational number and the set of parabolic points of $\overline{\Gamma}$ is equal to $Q\cup\{\infty\}$. In the literature, there has been several attempts to find these points. In [19], Schmidt and Sheingorn give the relationship between cusp points and fundamental domain of Hecke groups. In [20], Özgür and Cangül determine all parabolic points of $H(\lambda),\lambda\geq 2$. In [18], Rosen, shows $V(\infty)=\frac{a}{c}=[r_0\lambda,-1/r_1\lambda,...,-1/r_{n-1}\lambda]$ for $V(z)=\frac{az+b}{cz+d}=U^{r_0}TU^{r_1}T...U^{r_n}$ in Hecke groups. In this study, we know that each rational number $\frac{m}{n}\in Q_\infty$ is a cusp point of the extended modular group. Firstly we get the geodesics and ICFs of $\frac{m}{n}$ by F_n. Then, using these geodesics, ICFs and the presentation of extended modular group, we obtain the products of matrix representations whose entries are Fibonacci numbers; for each $\frac{m}{n} \in Q_\infty$ that is cusp point of the element of extended modular group. Therefore, we get important connections between Farey graph, continued fractions, extended modular group and Fibonacci numbers.

3. Main Results

The following transformations are needed.to get relationships between the integer continued fractions and the path in Farey graph

$$TS : z \longmapsto z+1, \qquad TS^2 : z \longmapsto \frac{z}{z+1}.$$

Let $W(T,S,R)$ is a reduced word in $\overline{\Gamma}$ such that the sum of exponents of R is even number, then this word in Γ is

$$S^i(TS)^{m_0}(TS^2)^{n_0}...(TS)^{m_k}(TS^2)^{n_k}T^j$$

and $W(T,S,R)$ is a reduced word in $\overline{\Gamma}$ such that the sum of exponents of R is odd number, then this word is

$$S^i(TS)^{m_0}(TS^2)^{n_0}...(TS)^{m_k}(TS^2)^{n_k}T^jR$$

for $i=0,1,2$ and $j=0,1$. The exponents of blocks are positive integers but m_0 and n_k may be zero. This representation is general and called a *block reduced form* abbreviated as BRF in [22].

We can write any reduced word in BRF by these blocks. For examples, the word $TSTSTSTS^2TS^2TS$ in BRF is $(TS)^3(TS^2)^2(TS)$ and the word RTS^2RTS^2R in BRF is $(TS)(TS^2)R$.

By using these BRF's, in [21], Fine has studied trace classes in the modular group Γ. Then, in [22], Koruoğlu et. al. investigated trace classes in the extended modular group $\overline{\Gamma}$.

Theorem 3.1. *Let $x=[r_0;r_1,r_2,...,r_n]$ be an integer continued fraction.*

(i) An automorphism element of the extended modular group whose parabolic point is x can be written

$$W=(TS)^{r_0-1}(TS^2)(TS)^{r_1-2}(TS^2)(TS)^{r_2-2}(TS^2)...(TS^2)(TS)^{r_n-1}T$$

(ii) An anti-automorphism element of the extended modular group whose parabolic point is x can be written

$$W=(TS)^{r_0-1}(TS^2)(TS)^{r_1-2}(TS^2)(TS)^{r_2-2}(TS^2)...(TS^2)(TS)^{r_n-1}R$$

Proof. (i) Let $W = U^{r_0}TU^{r_1}T...TU^{r_n}T$ be an element of the modular group Γ. It is easily seen that

$$r_0 - \cfrac{1}{r_1 - \cfrac{1}{r_2 - \cfrac{1}{r_3 - \ldots - \frac{1}{r_n}}}} = U^{r_0}TU^{r_1}T...TU^{r_n}T(\infty)$$

Therefore, the parabolic point of this word is $[r_0; r_1, r_2, ..., r_n]$ If we put $U = TS$ in this word, we get

$$\begin{aligned} &(TS)^{r_0}T(TS)^{r_1}T...T(TS)^{r_n}T \\ =\, &\underbrace{TS.TS\ldots TS}_{r_0 \text{ times}}\,T\,\underbrace{TS.TS\ldots TS}_{r_1 \text{ times}}\,T...\underbrace{TS.TS\ldots TS}_{r_n \text{ times}}\,T \end{aligned}$$

Hence, we obtain the word in $\overline{\Gamma}$

$$(TS)^{r_0-1}(TS^2)(TS)^{r_1-2}(TS^2)(TS)^{r_2-2}(TS^2)...(TS^2)(TS)^{r_n-1}T$$

(ii) From $R(z) = \frac{1}{\bar{z}}$ and definition of ICF it is easily proven. □

In [23], authors obtained the sequences which are the generalized version of the Fibonacci sequence given in [9] for the extended modular group $\overline{\Gamma}$, in the extended Hecke groups $\overline{H}(\lambda_q)$. Then, they applied their results to $\overline{\Gamma}$ to find all elements of the extended modular group $\overline{\Gamma}$.These sequences are

$$h^k = \begin{pmatrix} a_k & a_{k-1} \\ a_{k-1} & a_{k-2} \end{pmatrix} \text{ and } f^k = \begin{pmatrix} a_{k-1} & a_k \\ a_k & a_{k+1} \end{pmatrix}$$

The definition and boundary contitions of this sequence are

$$\begin{aligned} &a_k = \lambda_q a_{k-1} + a_{k-2}, \text{ for } k \geq 2, \\ &a_0 = 1, \ a_1 = \lambda_q. \end{aligned}$$

If we put $\lambda_q = 1$, we get the usual Fibonacci sequence. In [23], they defined the new block *reduced form* abbreviated as $NBRF$ in the extended modular group $\overline{\Gamma}$ as

$$f = RTS = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \qquad h = RTS^2 = TSR = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Here, the k^{th} power of f and h are

$$f^k = \begin{pmatrix} f_{k-1} & f_k \\ f_k & f_{k+1} \end{pmatrix} \text{ and } h^k = \begin{pmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{pmatrix}$$

where f_k is the Fibonacci sequence. Then, they showed that an element of the extended modular group can be obtained by powers of h and f.

Lemma 3.2. [23] *There are relations between BRF's and NBRF'S*

as

$$TS = Rf = hR, TS^2 = Rh = fR$$

Theorem 3.3. *Let $x = [r_0; r_1, r_2, ..., r_n]$ be a parabolic point of the*
$W = (TS)^{r_0-1}(TS^2)(TS)^{r_1-2}(TS^2)(TS)^{r_2-2}(TS^2)...(TS)^{r_{n-1}-2}$
$(TS^2)(TS)^{r_n-1}T \in \Gamma$.

Then we can obtain a $NBRF$ of W as follows

if r_0 is odd $(TS)^{r_0-1}(TS^2) = (hf)^{\frac{r_0-1}{2}} fR$ and r_0 is even
$(TS)^{r_0-1}(TS^2) = (hf)^{\frac{r_0-2}{2}} h^2$

if r_i is odd $(TS)^{r_i-2}(TS^2) = (hf)^{\frac{r_i-3}{2}} h^2$ and r_i is even
$(TS)^{r_i-1}(TS^2) = (hf)^{\frac{r_i-2}{2}} fR$ $(i = 1, 2, ..., n-1)$

if r_n is odd $(TS)^{r_n-1}T = (hf)^{\frac{r_n-1}{2}} T$ and r_n is even
$(TS)^{r_n-1}T = (hf)^{\frac{r_n-2}{2}} hRT$

Proof. Let us take
$W = (TS)^{r_0-1}(TS^2)(TS)^{r_1-2}(TS^2)(TS)^{r_2-2}(TS^2)...(TS^2)$
$(TS)^{r_n-1}T \in \Gamma$

and its parabolic point $[r_0; r_1, r_2, ..., r_n]$. If we use the above Lemma, we can write this word as
$W = (Rf)^{r_0-1}(fR)(Rf)^{r_1-2}(fR)(Rf)^{r_2-2}(fR)...(Rf)^{r_{n-1}-2}(fR)$
$(Rf)^{r_n-1}T$

We separete this word as
$(Rf)^{r_0-1}(fR)$,$(Rf)^{r_1-2}(fR)$,$(Rf)^{r_2-2}(fR)$,...$(Rf)^{r_{n-1}-2}(fR)$,
$(Rf)^{r_n-1}T$.

Firstly we consider the part $(Rf)^{r_0-1}(fR)$. There are two cases: If r_0 is odd then we can write
$(Rf)^{r_0-1}(fR) = (Rf)(Rf)...(Rf)(Rf)(fR) = (hR)(Rf)...(hR)$
$(Rf)(fR) = (hf)^{\frac{r_0-1}{2}} fR$

If r_0 is even we get
$(Rf)^{r_0-1}(fR) = (hR)(Rf)...(hR)(Rf)(hR)(fR)$
$= (hR)(Rf)...(hR)(Rf)/hR)(Rh) = (hf)^{\frac{r_0-2}{2}} h^2$

Now we consider parts $i.(i = 1, 2, ..., n-1)$If r_i is odd then, we obtain

$(TS)^{r_i-2}(TS^2) = (hR)(Rf)...(hR)(Rf)/hR)(fR)$
$= (hR)(Rf)...(hR)(Rf)/hR)(Rh) = (hf)^{\frac{r_i-3}{2}}h^2$

Similarly if r_i is even then we write

$(TS)^{r_i-2}(TS^2) = (Rf)(Rf)...(Rf)(Rf)(fR)$
$= (hR)(Rf)...(hR)(Rf)(fR) = (hf)^{\frac{r_i-2}{2}}fR$

For the last part, $(TS)^{r_n-1}T = (hf)^{\frac{r_n-1}{2}}T$ if r_n is odd and $(TS)^{r_n-1}T = (hf)^{\frac{r_n-2}{2}}hRT$ if r_n is even, the results can be easily proven. □

Theorem 3.4. *Let $x = [r_0; r_1, r_2, ..., r_n]$ be parabolic point of the*
$W = (TS)^{r_0-1}(TS^2)(TS)^{r_1-2}(TS^2)(TS)^{r_2-2}(TS^2)...(TS)^{r_{n-1}-2}$
$(TS^2)(TS)^{r_n-1}R \in \overline{\Gamma}$.

Then it can obtained a $NBRF$ of W as follows

if r_0 is odd $(TS)^{r_0-1}(TS^2) = (hf)^{\frac{r_0-1}{2}}fR$ and r_0 is even
$(TS)^{r_0-1}(TS^2) = (hf)^{\frac{r_0-2}{2}}h^2$

if r_i is odd $(TS)^{r_i-2}(TS^2) = (hf)^{\frac{r_i-3}{2}}h^2$ and r_i is even
$(TS)^{r_i-1}(TS^2) = (hf)^{\frac{r_i-2}{2}}fR$ $(i = 1, 2, ..., n-1)$

if r_n is odd $(TS)^{r_n-1}R = (hf)^{\frac{r_n-1}{2}}R$ and r_i is even
$(TS)^{r_n-1}R = (hf)^{\frac{r_n-2}{2}}h$

Proof. It can easily proven by using $TS = Rf = hR$,
$TS^2 = Rh = fR$. □

Example 3.5. *In table 1, we find the geodesic paths, integer continued fractions, BRFs, NBRFs of $\frac{2}{7}$ in the extended modular group.*

Remark 3.6. *Farey rational numbers are in $[0, 1]$. However, each rational numbers can be obtained by generator $U(z) = z + 1$ in the extended modular group. Hence, each rational number as cusp points can be written some matrices products whose all entries are Fibonacci numbers.*

4. Computer Program

We prepared a program that is written in the Python programming language, designed using the principles of procedural and structural programming and was implemented by importing the networkX, symPy, python standard math and tkinter libraries.

In the main code block, the variable definition and function calls are made to display the graphical user interface on the screen and interact

Automorphisms		Anti-automorphisms
$<\infty, 0, \frac{1}{3}, \frac{2}{7}>$ $<\infty, 0, \frac{1}{4}, \frac{2}{7}>$	Geodesics	$<\infty, 0, \frac{1}{3}, \frac{2}{7}>$ $<\infty, 0, \frac{1}{4}, \frac{2}{7}>$
$[0, -3, 2]$	ICF	$[0, -3, 2]$
$[0, -4, -2]$		$[0, -4, -2]$
$\left(TS^2\right)^3 . (TS)^2 .T$	BRF	$\left(TS^2\right)^3 . (TS)^2 .R$
$\left(TS^2\right)^3 . (TS) . \left(TS^2\right)$		$\left(TS^2\right)^3 . (TS) . \left(TS^2\right) . T.R$
fhf^2hRT	NBRF	fhf^2h
fhf^3R		fhf^3T
$\begin{pmatrix} f_0 & f_1 \\ f_1 & f_2 \end{pmatrix} \begin{pmatrix} h_2 & h_1 \\ h_1 & h_0 \end{pmatrix} \begin{pmatrix} f_1 & f_2 \\ f_2 & f_3 \end{pmatrix} \begin{pmatrix} h_2 & h_1 \\ h_1 & h_0 \end{pmatrix} RT$	Matrices	$\begin{pmatrix} f_0 & f_1 \\ f_1 & f_2 \end{pmatrix} \begin{pmatrix} h_2 & h_1 \\ h_1 & h_0 \end{pmatrix} \begin{pmatrix} f_1 & f_2 \\ f_2 & f_3 \end{pmatrix} \begin{pmatrix} h_2 & h_1 \\ h_1 & h_0 \end{pmatrix}$
$\begin{pmatrix} f_0 & f_1 \\ f_1 & f_2 \end{pmatrix} \begin{pmatrix} h_2 & h_1 \\ h_1 & h_0 \end{pmatrix} \begin{pmatrix} f_2 & f_3 \\ f_3 & f_4 \end{pmatrix} R$		$\begin{pmatrix} f_0 & f_1 \\ f_1 & f_2 \end{pmatrix} \begin{pmatrix} h_2 & h_1 \\ h_1 & h_0 \end{pmatrix} \begin{pmatrix} f_2 & f_3 \\ f_3 & f_4 \end{pmatrix} T$
$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	Fibonacci	$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$		$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

TABLE 1. Geodesic paths, ICFs, BRFs, NBRFs of 2/7 in $\overline{\Gamma}$

with the user. The algorithms used by these functions are based on the theorems we found in our previous studies.

Thus, in response to the rational number we entered from the user interface, related geodesic paths, integer continued fractions, block and new block forms for the automorphism and anti-automorphism elements of the extended modular group, matrices consisting of Fibonacci numbers are displayed on the screen. One can access this program by the following link:

https://github.com/kaymakf/Sule-Sarica/releases/download/0.1.1/sule.exe

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