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# $\eta$-RICCI SOLITONS ON $\epsilon - LP-$SASAKIAN MANIFOLDS WITH A QUARTER-SYMMETRIC METRIC CONNECTION

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**Abstract.** In this paper, we study $\eta$-Ricci solitons on $\epsilon$-$LP$-Sasakian manifolds with a quarter-symmetric metric connection satisfying certain curvature conditions. In particular, we have discussed that the Ricci soliton on $\epsilon - LP-$Sasakian manifolds with a quarter-symmetric metric connection satisfying certain curvature conditions is expanding or steady according to the vector field $\xi$ being timelike or spacelike. Moreover, we construct 3-dimensional examples of an $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection to verify some results of the paper.

## 1. Introduction

The study of manifolds with indefinite metrics is of high interest in physics and relativity theory. In [1], A. Bejancu and K. L. Duggal introduced the concept of $\epsilon$-Sasakian manifolds. Later, it was shown by X. Xufeng and C. Xiaoli [29] that these manifolds are real hypersurfaces of indefinite Kahlerian manifolds. In 2010, M. M. Tripathi et al. [16] have studied $\epsilon$-almost paracontact manifolds and in particular, $\epsilon$-para-Sasakian manifolds. On the other hand, the concept of $\epsilon$-Kenmotsu manifolds was introduced by U. C. De and A. Sarkar [27], they have studied some curvature properties on this manifold. $\epsilon$-Kenmotsu manifolds have also been studied A. Haseeb [2], A. Haseeb et. al [4], R. N. Singh et. al [21] and many others. In 2012, R. Prasad and V. Srivastava have studied $\epsilon$-Lorentzian para-Sasakian manifolds and shown its existence by an example [18]. Recently, A. Haseeb, A. Prakash and

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M. D. Siddiqi have studied $\epsilon$-Lorentzian para-Sasakian manifolds with a quarter-symmetric metric connection and obtained some interesting results [3].

In 1982, R. S. Hamilton [22] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. G. Perelman ([9], [10]) used Ricci flow and its surgery to prove Poincare conjecture. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}.$$

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling. A Ricci soliton $(g, V, \lambda)$ on a Riemannian manifold $(M, g)$ is a generalization of an Einstein metric such that [19, 23, 24, 26]

$$(\pounds_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \tag{1}$$

where $S$ is the Ricci tensor, $\pounds_V$ is the Lie derivative operator along the vector field $V$ on $M$ and $\lambda$ is a real number. The Ricci soliton is said to be shrinking, steady or expanding according to $\lambda$ being negative, zero or positive, respectively. Ricci solitons, in the context of general relativity, have been studied by M. Ali and Z. Ahsan [15].

As a generalization of Ricci soliton, the notion of $\eta$-Ricci soliton was introduced by J. T. Cho and M. Kimura [11]. They have studied Ricci soliton of real hypersurfaces in a non-flat complex space form and defined $\eta$-Ricci soliton, which satisfies the equation

$$(\pounds_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0, \tag{2}$$

where $\lambda$ and $\mu$ are real numbers. In particular, if $\mu = 0$, then the notion of $\eta$-Ricci soliton $(g, V, \lambda, \mu)$ reduces to the notion of Ricci soliton $(g, V, \lambda)$. Recenty, $\eta-$Ricci solitons have been studied by various authors such as A. Singh and S. Kishor [5], A. M. Blaga [6], D. G. Prakasha and B. S. Hadimani [8] and many others.

## 2. Preliminaries

A differentiable manifold $M$ of dimension $n$ is called an $\epsilon$-Lorentzian para-Sasakian (briefly, $\epsilon - LP-$Sasakian), if it admits a $(1, 1)$-tensor field $\phi$, a contravariant vector field $\xi$, a 1-form $\eta$ and a Lorentzian like metric g which satisfy

$$\phi^2 X = X + \eta(X)\xi,\ \eta(\xi) = -1, \tag{3}$$

$$g(\xi,\xi) = -\epsilon,\ \eta(X) = \epsilon g(X,\xi),\ \phi\xi = 0,\ \eta(\phi X) = 0, \tag{4}$$

$$g(\phi X, \phi Y) = g(X,Y) + \epsilon\eta(X)\eta(Y), \tag{5}$$

$$(\nabla_X \phi)(Y) = g(X,Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi, \tag{6}$$

$$\nabla_X \xi = \epsilon\phi X \tag{7}$$

for all vector fields $X,\ Y \in \chi(M)$, where $\epsilon$ is -1 or 1 according to the vector field $\xi$ being spacelike or timelike and $\nabla$ denotes the Levi-Civita connection with respect to $g$. If we put

$$\Phi(X,Y) = g(\phi X, Y) \tag{8}$$

for all vector fields $X$ and $Y$ on $M$, then $\Phi(X,Y)$ is a symmetric $(0,2)$ tensor field. Also since the 1-form $\eta$ is closed in an $\epsilon - LP-$Sasakian manifold, so we have [18]

$$(\nabla_X \eta)(Y) = \Phi(X,Y),\ \Phi(X,\xi) = 0 \tag{9}$$

for all vector fields $X,\ Y \in \chi(M)$.
Moreover, the curvature tensor $R$, the Ricci tensor $S$ and the Ricci operator $Q$ in an $\epsilon - LP-$Sasakian manifold with the Levi-Civita connection satisfy the following equations [18]:

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{10}$$

$$R(\xi,X)Y = \epsilon g(X,Y)\xi - \eta(Y)X, \tag{11}$$

$$R(\xi,X)\xi = -R(X,\xi)\xi = X + \eta(X)\xi, \tag{12}$$

$$\eta(R(X,Y)Z) = \epsilon[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{13}$$

$$S(X,\xi) = (n-1)\eta(X),\ Q\xi = \epsilon(n-1)\xi, \tag{14}$$

where $X,Y,Z \in \chi(M)$ and $g(QX,Y) = S(X,Y)$.
We note that if $\epsilon = 1$ and the structure vector field $\xi$ is timelike, then an $\epsilon - LP-$Sasakian manifold is usual $LP-$Sasakian manifold.

**Definition 2.1.** *An $\epsilon - LP-$Sasakian manifold is said to be a generalized $\eta$-Einstein manifold if its Ricci tensor $S$ of type $(0,2)$ satisfies* [13]

$$S(Y,Z) = l_1 g(Y,Z) + l_2 \eta(Y)\eta(Z) + l_3 g(\phi Y, Z),$$

*where $l_1, l_2$ and $l_3$ are the scalar functions on $M$. If $l_3 = 0$, then the manifold reduces to an $\eta-$Einstein manifold.*

## 3. Curvature tensor on an $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection

A linear connection $\bar{\nabla}$ in a Riemannian manifold $M$ is said to be a quarter-symmetric connection [25] if the torsion tensor $T$ of the connection $\bar{\nabla}$ is of the form

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] = \eta(Y)\phi X - \eta(X)\phi Y, \tag{15}$$

where $\eta$ is a 1-form and $\phi$ is a $(1,1)$ tensor field. If moreover, a quarter-symmetric connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y,Z) = 0 \tag{16}$$

for all $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields on $M$, then $\bar{\nabla}$ is said to be a quarter-symmetric metric connection, otherwise it is said to be a quarter-symmetric non-metric connection.

Let $M$ be an $n$-dimensional $\epsilon - LP-$Sasakian manifold and $\nabla$ be the Levi-Civita connection on $M$. A quarter-symmetric metric connection $\bar{\nabla}$ in an $\epsilon - LP-$Sasakian manifold is defined by [25]

$$\bar{\nabla}_X Y = \nabla_X Y + H(X,Y), \tag{17}$$

where $H(X,Y)$ is a tensor of type $(1,1)$ such that

$$H(X,Y) = \frac{1}{2}[T(X,Y) + T'(X,Y) + T'(Y,X)], \tag{18}$$

where $T'$ and $T$ are related by

$$g(T'(X,Y), Z) = g(T(Z,X), Y). \tag{19}$$

From (15) and (19), we get

$$T'(X,Y) = \eta(X)\phi(Y) - \epsilon g(\phi X, Y)\xi. \tag{20}$$

Now combining (15), (18) and (20), we obtain

$$H(X,Y) = \eta(Y)\phi(X) - \epsilon g(\phi X, Y)\xi. \tag{21}$$

Thus a quarter-symmetric metric connection $\bar{\nabla}$ in an $\epsilon - LP-$Sasakian manifold is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - \epsilon g(\phi X, Y)\xi. \tag{22}$$

A quarter symmetric metric connection have been studied by many authors in several ways to a different extent such as K. Mandal and U. C. De [12], M. Ahmad et al. [14], R. Prasad and A. Haseeb [17], R. N. Singh and S. K. Pandey [20], U. C. De and A. K. Mondal [28] and many others.

The curvature tensor $\bar{R}$ of quarter-symmetric metric connection $\bar{\nabla}$ in $M$ is defined by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z. \tag{23}$$

From (3)-(7), (9), (22) and (23), we obtain

$$\begin{aligned}\bar{R}(X,Y)Z = R(X,Y)Z &+ (2-\epsilon)[g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X] \\ &+\epsilon\eta(Z)[\eta(Y)X - \eta(X)Y] \\ &+[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)]\xi,\end{aligned} \tag{24}$$

where $X, Y, Z \in \chi(M)$ and $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ is the Riemannian curvature tensor of the Levi-Civita connection $\nabla$.

Let $\{e_1, e_2, e_3......., e_{n-1}, \xi\}$ be a frame of orthonormal basis of the tangent space at any point of the manifold. The Ricci tensor $\bar{S}$ and the scalar curvature $\bar{r}$ of the manifold with a quarter-symmetric metric connection are defined by [7]

$$\begin{aligned}\bar{S}(X,Y) &= \sum_{i=1}^{n}\epsilon_i g(\bar{R}(e_i, X)Y, e_i), \\ \bar{r} &= \sum_{i=1}^{n}\epsilon_i \bar{S}(e_i, e_i),\end{aligned}$$

respectively. Also we have

$$g(X,Y) = \sum_{i=1}^{n}\epsilon_i g(X, e_i)g(Y, e_i),$$

where $X, Y \in \chi(M)$ and $\epsilon_i = g(e_i, e_i) = +1$ or $-1$. Contracting $X$ in (24), we get

$$\begin{aligned}\bar{S}(Y,Z) = S(Y,Z) &+ (1-\epsilon)g(Y,Z) + (n\epsilon - 1)\eta(Y)\eta(Z) \\ &-(2-\epsilon)g(\phi Y, Z)\psi,\end{aligned} \tag{25}$$

where $\psi = trace\phi$ and is defined by $\psi = \sum_{i=1}^{n} \epsilon_i g(\phi e_i, e_i)$.
Contracting again $Y$ and $Z$ in (25), it follows that

$$\bar{r} = r - (n-1)\epsilon - (2-\epsilon)\psi^2, \tag{26}$$

where $r$ is the scalar curvature of the connection $\nabla$.
From the equations (22), (24) and (25), we can easily prove the following Lemma:

**Lemma 3.1.** *Let $M$ be an $n$-dimensional $\epsilon-LP-$Sasakian manifold with a quarter-symmetric metric connection. Then we have*

$$\bar{R}(X,Y)\xi = (1-\epsilon)[\eta(Y)X - \eta(X)Y], \tag{27}$$

$$\bar{R}(\xi,X)Y = -(1-\epsilon)[g(X,Y)\xi + \eta(Y)X], \tag{28}$$

$$\bar{R}(\xi,X)\xi = (1-\epsilon)[X + \eta(X)\xi],$$

$$\bar{S}(X,\xi) = (1-\epsilon)(n-1)\eta(X),\ \bar{S}(\xi,\xi) = -(1-\epsilon)(n-1), \tag{29}$$

$$\bar{\nabla}_X\xi = -(1-\epsilon)\phi X \tag{30}$$

*for any vector fields $X, Y \in \chi(M)$.*

Now, we express the Lie derivative along $\xi$ on $M$ with a quarter-symmetric metric connection as follows:

$$(\bar{\pounds}_\xi g)(Y,Z) = \bar{\pounds}_\xi g(Y,Z) - g(\bar{\pounds}_\xi Y, Z) - g(Y, \bar{\pounds}_\xi Z). \tag{31}$$

By virtue of (15), (31) takes the form

$$(\bar{\pounds}_\xi g)(Y,Z) = \bar{\nabla}_\xi g(Y,Z) - g(\bar{\nabla}_\xi Y - \bar{\nabla}_Y\xi - \phi Y, Z) \tag{32}$$

$$+g(Y, \bar{\nabla}_\xi Z - \bar{\nabla}_Z\xi - \phi Z)$$

which after computation gives

$$(\bar{\pounds}_\xi g)(Y,Z) = 2\epsilon g(\phi Y, Z) \tag{33}$$

for all $Y, Z \in \chi(M)$.

## 4. $\eta$-Ricci solitons on $\epsilon - LP-$Sasakian manifolds with a quarter-symmetric metric connection

Suppose that an $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection admits an $\eta$-Ricci soliton $(g, \xi, \lambda, \mu)$. Then (2) holds and we have

$$(\bar{\pounds}_\xi g)(Y,Z) + 2\bar{S}(Y,Z) + 2\lambda g(Y,Z) + 2\mu\eta(Y)\eta(Z) = 0. \tag{34}$$

By virtue of (33), (34) takes the form

$$\bar{S}(Y,Z) = -\epsilon g(\phi Y, Z) - \lambda g(Y,Z) - \mu\eta(Y)\eta(Z). \tag{35}$$

Putting $Y = Z = \xi$ in (35) and using (3), (4) and (29), we obtain

$$\lambda - \epsilon\mu = (1-\epsilon)(n-1). \tag{36}$$

Thus we have the following:

**Theorem 4.1.** *If $(g, \xi, \lambda, \mu)$ is an $\eta-$Ricci soliton on an $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection, then the scalars $\lambda$ and $\mu$ are related by $\lambda - \epsilon\mu = (1-\epsilon)(n-1)$.*

**Example 4.2.** *We consider the 3-dimensional manifold*

$$M = \left\{(x,y,z) \in R^3\right\},$$

*where $(x,y,z)$ are the standard coordinates in $R^3$. Let $e_1$, $e_2$ and $e_3$ be the vector fields on $M$ given by*

$$e_1 = e^z\frac{\partial}{\partial y},\ e_2 = e^z(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}),\ e_3 = \frac{\partial}{\partial z} = \xi,$$

*which are linearly independent at each point of $M$ and hence form a basis of $T_pM$. Let $g$ be the Lorentzian like (semi-Riemannian) metric on $M$ defined by*

$$g(e_1,e_1)=g(e_2,e_2)=1,\ g(e_3,e_3)=-\epsilon,\ g(e_1,e_2)=g(e_1,e_3)=g(e_2,e_3)=0.$$

*Let $\eta$ be the 1-form on $M$ defined by $\eta(X) = \epsilon g(X, e_3) = \epsilon g(X, \xi)$ for all $X \in \chi(M)$. Let $\phi$ be the $(1,1)$ tensor field on $M$ defined by*

$$\phi e_1 = -\epsilon e_1,\ \phi e_2 = -\epsilon e_2,\ \phi e_3 = 0.$$

*The linearity property of $\phi$ and $g$ yields*

$$\eta(e_3) = -1,\ \phi^2 X = X + \eta(X)\xi,\ g(\phi X, \phi Y) = g(X,Y) + \epsilon\eta(X)\eta(Y)$$

*for all $X, Y \in \chi(M)$.*
*Now, by direct computations, we obtain*

$$[e_1, e_2] = 0, \quad [e_2, e_3] = -e_2, \quad [e_3, e_1] = e_1.$$

*The Riemannian connection* $\nabla$ *of the metric tensor* $g$ *is given by*

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]),$$

*which is known as Koszul's formula. Using Koszul's formula, we can easily calculate*

$$(37)\quad \nabla_{e_1}e_1 = -\epsilon e_3,\ \nabla_{e_2}e_1 = 0,\ \nabla_{e_3}e_1 = 0,\ \nabla_{e_1}e_2 = 0,\ \nabla_{e_2}e_2 = -\epsilon e_3,\ \nabla_{e_3}e_2 = 0,\ \nabla_{e_1}e_3 = -e_1,\ \nabla_{e_2}e_3 = -e_2,\ \nabla_{e_3}e_3 = 0.$$

*Let* $X = \sum_{i=1}^{3} X^i e_i = X^1 e_1 + X^2 e_2 + X^3 e_3 \in \chi(M).$

*Also, one can easily verify that*

$$\nabla_X \xi = \epsilon\phi X \quad \text{and} \quad (\nabla_X \phi)Y = g(X,Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi.$$

*Thus the manifold* $M$ *is an* $\epsilon - LP-$*Sasakian manifold. From (22) and (37), we find*

$$(38)\quad \bar{\nabla}_{e_1}e_1 = (1-\epsilon)e_3, \bar{\nabla}_{e_2}e_1 = 0, \bar{\nabla}_{e_3}e_1 = 0, \bar{\nabla}_{e_1}e_2 = 0,\ \bar{\nabla}_{e_2}e_2 = (1-\epsilon)e_3,\ \bar{\nabla}_{e_3}e_2 = 0,\ \bar{\nabla}_{e_1}e_3 = -(1-\epsilon)e_1,\ \bar{\nabla}_{e_2}e_3 = -(1-\epsilon)e_2,\ \bar{\nabla}_{e_3}e_3 = 0.$$

*It is known that*

$$(39)\quad R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

*By using the above results, we can easily calculate the components of the curvature tensors as follows:*

$$(40)\quad R(e_1,e_2)e_1 = -\epsilon e_2,\ R(e_1,e_2)e_2 = \epsilon e_1,\ R(e_1,e_2)e_3 = 0,$$
$$R(e_2,e_3)e_1 = 0,\ R(e_2,e_3)e_2 = -\epsilon e_3,\ R(e_2,e_3)e_3 = -e_2,$$
$$R(e_1,e_3)e_1 = -\epsilon e_3,\ R(e_1,e_3)e_2 = 0,\ R(e_1,e_3)e_3 = -e_1,$$

*and*

$$(41)\quad \bar{R}(e_1,e_2)e_1 = 2(1-\epsilon)e_2,\ \bar{R}(e_1,e_2)e_2 = -2(1-\epsilon)e_1,\ \bar{R}(e_1,e_2)e_3 = 0,$$
$$\bar{R}(e_2,e_3)e_1 = 0,\ \bar{R}(e_2,e_3)e_2 = (1-\epsilon)e_3,\ \bar{R}(e_2,e_3)e_3 = -(1-\epsilon)e_2,$$
$$\bar{R}(e_1,e_3)e_1 = (1-\epsilon)e_3,\ \bar{R}(e_1,e_3)e_2 = 0,\ \bar{R}(e_1,e_3)e_3 = -(1-\epsilon)e_1.$$

*From these curvature tensors, we obtain*

$$(42)\quad S(e_1,e_1) = S(e_2,e_2) = 2\epsilon,\ S(e_3,e_3) = -2.$$

$$(43)\quad \bar{S}(e_1,e_1) = \bar{S}(e_2,e_2) = -3(1-\epsilon),\ \bar{S}(e_3,e_3) = -2(1-\epsilon).$$

*Therefore, from (42) and (43) we get* $r = 6\epsilon$ *and* $\bar{r} = -8(1-\epsilon)$*, respectively. Thus it can be seen that the equation (26) is satisfied, where*

$\psi = \sum_{i=1}^{3} \epsilon_i g(\phi e_i, e_i) = -2\epsilon$. *Now from the equations (8), (15) and (16) we find*

$$\Phi(e_1, e_1) = \Phi(e_2, e_2) = -\epsilon, \Phi(e_3, e_3) = 0, \tag{44}$$

$$T(e_1, e_1) = T(e_2, e_2) = T(e_3, e_3) = T(e_1, e_2) = 0, \tag{45}$$

$$T(e_1, e_3) = e_1, T(e_2, e_3) = e_2,$$

$$(\bar{\nabla}_X g)(Y, Z) = 0 \tag{46}$$

*for any vector fields* $X, Y, Z \in \chi(M)$, *respectively. Thus the connection* $\bar{\nabla}$ *defined on* $M$ *is a quarter-symmetric metric connection.*
*Now from (35), we find* $\bar{S}(e_1, e_1) = 1 - \lambda$ *and* $\bar{S}(e_3, e_3) = \epsilon\lambda - \mu$, *therefore from (43) it follows that* $\lambda = 4 - 3\epsilon$ *and* $\mu = 2\epsilon - 1$. *Thus* $(g, \xi, \lambda, \mu)$ *for* $\lambda = 4 - 3\epsilon$ *and* $\mu = 2\epsilon - 1$ *defines an* $\eta-$*Ricci soliton on an* $\epsilon - LP-$*Sasakian manifold with a quarter-symmetric metric connection. By using these values of the scalars* $\lambda$ *and* $\mu$ *in (35), we obtain*

$$\bar{S}(Y, Z) = -\epsilon g(\phi Y, Z) - (4 - 3\epsilon) g(Y, Z) - (2\epsilon - 1)\eta(Y)\eta(Z). \tag{47}$$

*Thus the manifold* $M$ *(defined in Example 4.2) admitting an* $\eta-$*Ricci soliton* $(g, \xi, \lambda, \mu)$ *with a quarter-symmetric metric connection is a generalized* $\eta-$*Einstein manifold of the form (47).*

**Definition 4.3.** *An* $\epsilon$*-LP-Sasakian manifold is said to be*
$(i)$ *conformally flat with a quarter-symmetric metric connection if*

$$\bar{C}(X, Y)Z = 0, \tag{48}$$

$(ii)$ *quasi conformally flat with a quarter-symmetric metric connection if*

$$g(\bar{C}(X, Y)Z, \phi W) = 0, \tag{49}$$

$(iii)$ $\xi$*-conformally flat with a quarter-symmetric metric connection if*

$$\bar{C}(X, Y)\xi = 0, \tag{50}$$

*where* $\bar{C}$ *is the conformal curvature tensor with a quarter-symmetric metric connection and is given by*

$$\bar{C}(X, Y)Z = \bar{R}(X, Y)Z - \frac{1}{(n-2)}[\bar{S}(Y, Z)X - \bar{S}(X, Z)Y \tag{51}$$

$$+ g(Y, Z)\bar{Q}X - g(X, Z)\bar{Q}Y] + \frac{\bar{r}}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y]$$

*for all* $X, Y, Z, W \in \chi(M)$, $\bar{Q}$ *is the Ricci operator with a quarter-symmetric metric connection such that* $g(\bar{Q}X, Y) = \bar{S}(X, Y)$.

In [3], A. Haseeb, A. Prakash and M. D. Siddiqi proved that conformally flat, quasi-conformally flat and $\xi$-conformally flat $\epsilon-LP-$Sasakian manifolds with a quarter-symmetric metric connection are an $\eta-$Einstein manifold. In the same way, we can easily prove:

**Proposition 4.4.** *An $n$-dimensional conformally flat, quasi-conformally flat and $\xi$-conformally flat $\epsilon-LP-$Sasakian manifold with a quarter-symmetric metric connection $\bar{\nabla}$ defined by (22) are an $\eta$-Einstein manifold of the form*

$$\bar{S}(X,W)=[\frac{\bar{r}}{n-1}+(1-\epsilon)]g(X,W)+\epsilon[\frac{\bar{r}}{n-1}+n(1-\epsilon)]\eta(X)\eta(W), \tag{52}$$

$$\bar{S}(X,W)=[\frac{\bar{r}}{(n-1)}+(1-\epsilon)(2n-3)]g(X,W) +[\frac{\epsilon\bar{r}}{(n-1)}-(1-\epsilon)(3n-4)]\eta(X)\eta(W), \tag{53}$$

$$\bar{S}(X,W)=[\frac{\bar{r}}{n-1}+(1-\epsilon)]g(X,W)+[\frac{\epsilon\bar{r}}{n-1}-n(1-\epsilon)]\eta(X)\eta(W), \tag{54}$$

*respectively.*

Now we prove the following:

**Theorem 4.5.** *If $(g,\xi,\lambda,\mu)$ is an $\eta-$Ricci soliton on conformally flat, quasi-conformally flat and $\xi$-conformally flat $\epsilon-LP-$Sasakian manifold with a quarter-symmetric metric connection, then the scalars $\lambda$ and $\mu$ are related by $\lambda-\epsilon\mu=(1-\epsilon)(n-1)$.*

*Proof.* Firstly, we assume that the manifold $M$ with a quarter-symmetric metric connection is conformally flat. Then from (34), (35) and (52) we have

$$\epsilon g(\phi Y,Z)+[\lambda+\frac{\bar{r}}{(n-1)}+(1-\epsilon)]g(Y,Z) +\epsilon[\frac{\bar{r}}{n-1}+n(1-\epsilon)+\epsilon\mu]\eta(Y)\eta(Z)=0.$$

Putting $Y=Z=\xi$ in the last equation then using (3) and (4), we find

$$\lambda-\epsilon\mu=(1-\epsilon)(n-1). \tag{55}$$

Secondly, we assume that the manifold $M$ with a quarter-symmetric metric connection is quasi-conformally flat. Then from (34), (35) and (53), we have

$$\epsilon g(\phi Y,Z)+[\lambda+\frac{\bar{r}}{(n-1)}+(1-\epsilon)(2n-3)]g(Y,Z)$$

$$+[\frac{\epsilon\bar{r}}{(n-1)} - (1-\epsilon)(3n-4) + \mu]\eta(Y)\eta(Z) = 0.$$

Putting $Y = Z = \xi$ in the last equation then using (3) and (4), we find

$$\lambda - \epsilon\mu = (1-\epsilon)(n-1). \tag{56}$$

Next, we assume that the manifold $M$ with a quarter-symmetric metric connection is $\xi$-conformally flat. Then from (34), (35) and (54) we have

$$\epsilon g(\phi Y, Z) + [\lambda + \frac{\bar{r}}{(n-1)} + (1-\epsilon)]g(Y,Z)$$
$$+ [\frac{\epsilon\bar{r}}{(n-1)} - n(1-\epsilon) + \mu]\eta(Y)\eta(Z) = 0.$$

Putting $Y = Z = \xi$ in the last equation then using (3) and (4), we find

$$\lambda - \epsilon\mu = (1-\epsilon)(n-1). \tag{57}$$

Thus from (55), (56) and (57), Theorem 4.4 is proved. □

In particular, if $\mu = 0$, then from (55)-(57) it follows that $\lambda = (1-\epsilon)(n-1)$. Thus we have

**Corollary 4.6.** *If $(g, \xi, \lambda)$ is a Ricci soliton on conformally flat, quasi-conformally flat and $\xi$-conformally flat $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection, then the Ricci soliton on $M$ is steady or expanding according to the vector field $\xi$ being timelike or spacelike.*

Let $(g, V, \lambda, \mu)$ be an $\eta-$Ricci soliton on an $\epsilon-LP-$Sasakian manifold with a quarter-symmetric metric connection. Then we have

$$(\bar{\mathcal{L}}_V g)(Y,Z) + 2\bar{S}(Y,Z) + 2\lambda g(Y,Z) + 2\mu\eta(Y)\eta(Z) = 0, \tag{58}$$

where $\bar{\mathcal{L}}_V$ is the Lie derivative along the vector field $V$ on $M$ with a quarter-symmetric metric connection.

Now, let $(g, V, \lambda, \mu)$ be an $\eta-$Ricci soliton on an $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection such that $V$ is pointwise collinear with $\xi$, i.e., $V = b\xi$, where $b$ is a function. Then (58) holds and we have

$$bg(\bar{\nabla}_Y\xi, Z) + (Yb)\eta(Z) + bg(Y, \bar{\nabla}_Z\xi) + (Zb)\eta(Y) + 2bg(\phi Y, Z) \tag{59}$$
$$+2\bar{S}(Y,Z) + 2\lambda g(Y,Z) + 2\mu\eta(Y)\eta(Z) = 0$$

which in view of (30) takes the form

$$2b\epsilon g(\phi Y, Z) + (Yb)\eta(Z) + (Zb)\eta(Y) \tag{60}$$
$$+2\bar{S}(Y,Z) + 2\lambda g(Y,Z) + 2\mu\eta(Y)\eta(Z) = 0.$$

Putting $Z = \xi$ in (60) and using (3), (4) and (29) it follows that

(61) $-(Yb) + (\xi b)\eta(Y) + 2(1-\epsilon)(n-1)\eta(Y) + 2\epsilon\lambda\eta(Y) - 2\mu\eta(Y) = 0.$

Again putting $Y = \xi$ in (61) and using (3), we get

$$(\xi b) + (1-\epsilon)(n-1) + \epsilon\lambda - \mu = 0. \tag{62}$$

Combining the equations (61) and (62), we have

$$db = [(1-\epsilon)(n-1) + \epsilon\lambda - \mu]\eta. \tag{63}$$

Now applying $d$ on (63), we find

$$[(1-\epsilon)(n-1) + \epsilon\lambda - \mu]d\eta = 0. \tag{64}$$

Since $d\eta \neq 0$, so it follows from (64) that

$$\lambda - \epsilon\mu = (1-\epsilon)(n-1). \tag{65}$$

By using (64) in (63), we find $db = 0$, i.e., $b$ is a constant. Therefore (60) reduces to

$$\bar{S}(Y,Z) = -\lambda g(Y,Z) - b\epsilon g(\phi Y,Z) - \mu\eta(Y)\eta(Z)$$

which in view of (25), takes the form

$$\begin{aligned} S(Y,Z) = &-(\lambda + 1 - \epsilon)g(Y,Z) - (n\epsilon - 1 + \mu)\eta(Y)\eta(Z) \\ &+[(2-\epsilon)\psi - b\epsilon]g(\phi Y,Z). \end{aligned} \tag{66}$$

Thus we have the following theorem:

**Theorem 4.7.** *If $(g,V,\lambda,\mu)$ is an $\eta-$Ricci soliton on an $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection such that $V$ is pointwise collinear with $\xi$, then $V$ is a constant multiple of $\xi$ and the manifold is a generalized $\eta$-Einstein manifold with the Levi-Civita connection of the form* (66) *and the scalars $\lambda$ and $\mu$ are related by $\lambda - \epsilon\mu = (1-\epsilon)(n-1)$.*

In particular, if $\mu = 0$, then from (65) it follows that $\lambda = (1-\epsilon)(n-1)$. Thus we have

**Corollary 4.8.** *If $(g,V,\lambda)$ is a Ricci soliton on an $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection such that $V$ is pointwise collinear with $\xi$, then the Ricci soliton on $M$ is steady or expanding according to the vector field $\xi$ being timelike or spacelike.*

## 5. $\eta-$Ricci solitons on $\epsilon - LP-$Sasakian manifolds with a quarter-symmetric metric connection satisfying $\bar{R}(X,Y) \cdot \bar{S} = 0$

Let $M$ be an $n-$dimensional $\epsilon-LP-$Sasakian manifold with a quarter-symmetric metric connection admits an $\eta-$Ricci soliton $(g, \xi, \lambda, \mu)$ satisfying $\bar{R}(X,Y) \cdot \bar{S} = 0$. Then we have

$$\bar{S}(\bar{R}(X,Y)U,V) + \bar{S}(U, \bar{R}(X,Y)V) = 0 \tag{67}$$

for any $X, Y, U, V \in \chi(M)$. Putting $X = \xi$ in (67), we have

$$\bar{S}(\bar{R}(\xi,Y)U,V) + \bar{S}(U, \bar{R}(\xi,Y)V) = 0. \tag{68}$$

In view of (28), (68) becomes

$$g(Y,U)\bar{S}(\xi,V) + \eta(U)\bar{S}(Y,V) + g(Y,V)\bar{S}(U,\xi) + \eta(V)\bar{S}(U,Y) = 0, \quad (1-\epsilon) \neq 0$$

which by putting $U = \xi$ and then using (3), (29) reduces to

$$\bar{S}(Y,V) = -(1-\epsilon)(n-1)g(Y,V). \tag{69}$$

In view of (25), (69) takes the form

$$S(Y,V) = -n(1-\epsilon)g(Y,V) - (n\epsilon-1)\eta(Y)\eta(V) + (2-\epsilon)g(\phi Y,V)\psi. \tag{70}$$

From (33), (34) and (69), we find

$$\epsilon g(\phi Y,V) - (1-\epsilon)(n-1)g(Y,V) + \lambda g(Y,V) + \mu\eta(Y)\eta(V) = 0. \tag{71}$$

Taking $V = Y = \xi$ in (71) then using (3) and (4), it follows that

$$\lambda - \epsilon\mu = (1-\epsilon)(n-1). \tag{72}$$

Thus we have the following theorem:

**Theorem 5.1.** *If $(g, \xi, \lambda, \mu)$ is an $\eta-$Ricci soliton on an $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection satisfying $\bar{R}(X,Y) \cdot \bar{S} = 0$, then the manifold is a generalized $\eta-$Einstein manifold with the Levi-Civita connection of the form* (70) *and the scalars $\lambda$ and $\mu$ are related by $\lambda - \epsilon\mu = (1-\epsilon)(n-1)$.*

In particular, if $\mu = 0$, then from (72) it follows that $\lambda = (1-\epsilon)(n-1)$. Thus we have

**Corollary 5.2.** *If $(g, \xi, \lambda)$ is a Ricci soliton on an $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection satisfying $\bar{R} \cdot \bar{S} = 0$, then the Ricci soliton on $M$ is always expanding according to the vector field $\xi$ being spacelike.*

## 6. $\eta-$Ricci solitons on $\epsilon - LP-$Sasakian manifolds with a quarter-symmetric metric connection satisfying $\bar{S} \cdot \bar{R} = 0$

Let $M$ be an $n-$dimensional $\epsilon-LP-$Sasakian manifold with a quarter-symmetric metric connection admits an $\eta-$Ricci soliton $(g, \xi, \lambda, \mu)$ satisfying $\bar{S} \cdot \bar{R} = 0$. Then we have

$$(X \wedge_{\bar{S}} Y)\bar{R}(U, V)Z + \bar{R}((X \wedge_{\bar{S}} Y)U, V)Z + \bar{R}(U, (X \wedge_{\bar{S}} Y)V)Z + \bar{R}(U, V)(X \wedge_{\bar{S}} Y)Z = 0 \tag{73}$$

for any $X, Y, Z, U, V \in \chi(M)$, where the endomorphism $X \wedge_{\bar{S}} Y$ is defined by

$$(X \wedge_{\bar{S}} Y)Z = \bar{S}(Y, Z)X - \bar{S}(X, Z)Y. \tag{74}$$

Taking $Y = \xi$ in (73), we have

$$(X \wedge_{\bar{S}} \xi)\bar{R}(U, V)Z + \bar{R}((X \wedge_{\bar{S}} \xi)U, V)Z + \bar{R}(U, (X \wedge_{\bar{S}} \xi)V)Z + \bar{R}(U, V)(X \wedge_{\bar{S}} \xi)Z = 0 \tag{75}$$

which in view of (74) becomes

$$\bar{S}[\xi, \bar{R}(U, V)Z]X - \bar{S}[X, \bar{R}(U, V)Z]\xi + \bar{S}(\xi, U)\bar{R}(X, V)Z - \bar{S}(X, U)\bar{R}(\xi, V)Z + \bar{S}(\xi, V)\bar{R}(U, X)Z - \bar{S}(X, V)\bar{R}(U, \xi)Z + \bar{S}(\xi, Z)\bar{R}(U, V)X - \bar{S}(X, Z)\bar{R}(U, V)\xi = 0. \tag{76}$$

By using (29) in (76), we find

$$(1 - \epsilon)(n - 1)[\eta(\bar{R}(U, V)Z)X + \eta(U)\bar{R}(X, V)Z + \eta(V)\bar{R}(U, X)Z + \eta(Z)\bar{R}(U, V)X] - \bar{S}[X, \bar{R}(U, V)Z]\xi - \bar{S}(X, U)\bar{R}(\xi, V)Z - \bar{S}(X, V)\bar{R}(U, \xi)Z - \bar{S}(X, Z)\bar{R}(U, V)\xi = 0. \tag{77}$$

Taking inner product of (77) with $\xi$, we get

$$-(1 - \epsilon)(n - 1)[\eta(\bar{R}(U, V)Z)\eta(X) + \eta(U)\eta(\bar{R}(X, V)Z) + \eta(V)\eta(\bar{R}(U, X)Z) + \eta(Z)\eta(\bar{R}(U, V)X)] + \epsilon\bar{S}[X, \bar{R}(U, V)Z] - \epsilon\bar{S}(X, U)\eta(\bar{R}(\xi, V)Z) - \epsilon\bar{S}(X, V)\eta(\bar{R}(U, \xi)Z) - \epsilon\bar{S}(X, Z)\eta(\bar{R}(U, V)\xi) = 0$$

which by putting $U = Z = \xi$ then using (27) and (28) reduces to

$$-(1-\epsilon)^2(n-1)[g(X, V) - \eta(X)\eta(V)] + \epsilon\bar{S}[X, (1-\epsilon)(V + \eta(V)\xi)] = 0 \tag{78}$$

from which we obtain

$$\bar{S}(X, V) = -(1 - \epsilon)(n - 1)g(X, V), \quad (1 - \epsilon) \neq 0. \tag{79}$$

In view of (25), (79) takes the form

$$S(X, V) = -n(1 - \epsilon)g(X, V) - (n\epsilon - 1)\eta(X)\eta(V) + (2 - \epsilon)g(\phi X, V)\psi. \tag{80}$$

From (33), (34) and (79), we find

$$\epsilon g(\phi Y, V) - (1-\epsilon)(n-1)g(Y,V) + \lambda g(Y,V) + \mu\eta(Y)\eta(V) = 0. \tag{81}$$

Taking $V = Y = \xi$ in (81) then using (3) and (4), it follows that

$$\lambda - \epsilon\mu = (1-\epsilon)(n-1). \tag{82}$$

Thus we have the following theorem:

**Theorem 6.1.** *If $(g, \xi, \lambda, \mu)$ is an $\eta-$Ricci soliton on an $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection satisfying $\bar{S} \cdot \bar{R} = 0$, then the manifold is a generalized $\eta-$Einstein manifold with the Levi-Civita connection of the form* (80) *and the scalars $\lambda$ and $\mu$ are related by $\lambda - \epsilon\mu = (1-\epsilon)(n-1)$.*

In particular, if $\mu = 0$, then from (82) it follows that $\lambda = (1-\epsilon)(n-1)$. Thus we have

**Corollary 6.2.** *If $(g, \xi, \lambda)$ is a Ricci soliton on an $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection satisfying $\bar{S} \cdot \bar{R} =$ 0, then the Ricci soliton on $M$ is always expanding according to the vector field $\xi$ being spacelike.*

## 7. $\epsilon - LP-$Sasakian manifolds with a quarter-symmetric metric connection with second order parallel tensor

Let us suppose that $\alpha$ be a symmetric parallel tensor of type $(0, 2)$ with respect to the connection $\bar{\nabla}$, that is, $\bar{\nabla}\alpha = 0$. Then we have

$$\alpha[\bar{R}(X,Y)Z, W] + \alpha[Z, \bar{R}(X,Y)W] = 0 \tag{83}$$

for any $X, Y, Z, W \in \chi(M)$.
Replacing $X = Z = W = \xi$ in (83), we have

$$\alpha[\bar{R}(\xi,Y)\xi, \xi] + \alpha[\xi, \bar{R}(\xi,Y)\xi] = 0 \tag{84}$$

which in view of (28) yields

$$\alpha(Y,\xi) = -\eta(Y)\alpha(\xi,\xi) = -\epsilon g(Y,\xi)\alpha(\xi,\xi), \; (1-\epsilon) \neq 0. \tag{85}$$

Differentiating (85) with respect to $\bar{\nabla}$ along any arbitrary vector field $X$, we find

(86)
$$\begin{aligned} \alpha(\bar{\nabla}_X Y, \xi) + \alpha(Y, \bar{\nabla}_X \xi) = &- \epsilon[(\bar{\nabla}_X g)(Y,\xi) + g(\bar{\nabla}_X Y, \xi) \\ &+ g(Y, \bar{\nabla}_X \xi)]\alpha(\xi,\xi) - 2\epsilon g(Y,\xi)\alpha(\bar{\nabla}_X \xi, \xi). \end{aligned}$$

By virtue of (16) and (85), (86) reduces to

$$\alpha(Y, \bar{\nabla}_X\xi) = -\epsilon g(Y, \bar{\nabla}_X\xi)\alpha(\xi, \xi) - 2\epsilon g(Y, \xi)\alpha(\bar{\nabla}_X\xi, \xi)$$

which in view of (30) takes the form

$$-\alpha(Y, \phi X) = \epsilon g(Y, \phi X)\alpha(\xi, \xi) + 2\epsilon g(Y, \xi)\alpha(\phi X, \xi). \tag{87}$$

Replacing $Y$ by $\phi X$ in (85), we get

$$\alpha(\phi X, \xi) = 0. \tag{88}$$

Combining (87) and (88), we obtain

$$-\alpha(Y, \phi X) = \epsilon g(Y, \phi X)\alpha(\xi, \xi). \tag{89}$$

By replacing $X$ by $\phi X$ in (89) and using (3), it follows that

$$\alpha(Y, X) = -\epsilon g(Y, X)\alpha(\xi, \xi).$$

The fact that $\alpha(\xi, \xi)$ is a constant can be checked by differentiating it along any vector field on $M$. Thus we have the following theorem:

**Theorem 7.1.** *On an $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection any symmetric parallel second order covariant tensor is a constant multiple of the metric tensor.*

**Example 7.2.** *We consider the $(2n+1)$-dimensional manifold $M = \{(x_1, x_2, x_3 \cdots, x_{2n}, x_{2n+1}) \in R^{2n+1}\}$, where $(x_1, x_2, x_3 \cdots, x_{2n}, x_{2n+1})$ are the standard coordinates in $R^{2n+1}$. Let $e_1, e_2, e_3 \cdots, e_{2n}$ and $e_{2n+1}$ be the vector fields on $M^{2n+1}$ given by*

$$e_k = coshx_{2n+1}\frac{\partial}{\partial x_k} + sinhx_{2n+1}\frac{\partial}{\partial x_{k+1}}, \ \text{if } k = 1, 3, ....(2n-1),$$

$$e_k = sinhx_{2n+1}\frac{\partial}{\partial x_{k-1}} + coshx_{2n+1}\frac{\partial}{\partial x_k}, \ \text{if } k = 2, 4..., 2n, \ \text{and } e_{2n+1} = \xi$$

*which are linearly independent at each point of $M$. Let $g$ be the Lorentzian like (semi-Riemannian) metric on $M$ defined for $1 \leq i, j \leq (2n+1)$ by*

$$g(e_i, e_j) = 0, \ \text{if } \ i \neq j; \ g(e_i, e_j) = 1, \ \text{if } \ i = j < (2n+1);$$

$$g(e_i, e_j) = -\epsilon, \ \text{if } \ i = j = (2n+1).$$

*Let $\eta$ be the 1-form on $M$ defined by $\eta(X) = \epsilon g(X, e_{2n+1}) = \epsilon g(X, \xi)$ for all $X \in \chi(M)$. Let $\phi$ be the $(1,1)$ tensor field on $M$ defined by*

$$\phi e_k = -\epsilon e_{k+1}, \text{if } k = 1, 3, ....(2n-1); \phi e_k = -\epsilon e_{k-1}, \text{if } k = 2, 4..., 2n,$$

$$\text{and} \quad \phi e_{2n+1} = 0.$$

*The linearity property of* $\phi$ *and* $g$ *yields*

$$\eta(e_{2n+1}) = -1,\ \phi^2 X = X + \eta(X)\xi,\ g(\phi X, \phi Y) = g(X,Y) + \epsilon\eta(X)\eta(Y)$$

*for all* $X, Y \in \chi(M)$.
*Now, by direct computations, we obtain*

$$[e_k, \xi] = -e_{k+1}, \text{if } k = 1, 3, ...., (2n-1); [e_k, \xi] = -e_{k-1}, \text{if } k = 2, 4, .., 2n;$$
$$[e_i, e_j] = 0,\ \text{if } 1 \leq i, j \leq 2n.$$

*Using Koszul's formula for the metric g, we can easily calculate*

(90)
$$\nabla_{e_i} e_{2n+1} = -e_{i+1}, \text{if } i = 1, 3, ...(2n-1);$$
$$\nabla_{e_i} e_{2n+1} = -e_{i-1}, \text{if } i = 2, 4, ...2n;$$
$$\nabla_{e_i} e_{i+1} = \nabla_{e_{i+1}} e_i = -\epsilon e_{2n+1}, \text{if } i = 1, 3, 5, ....(2n-1);$$
$$\nabla_{e_i} e_{i-1} = \nabla_{e_{i-1}} e_i = -\epsilon e_{2n+1}, \text{if } i = 2, 4, 6....2n;$$

$$\nabla_{e_i} e_{i+j} = \nabla_{e_{i+j}} e_i = 0, \text{if } 1 \leq i \leq (2n-1), 2 \leq j \leq (2n-1) \text{ but } i+j < (2n+1),$$

*where n is a natural number. Also, one can easily verify that*

$$\nabla_X \xi = \epsilon\phi X \quad \text{and} \quad (\nabla_X \phi)(Y) = g(X,Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi.$$

*Thus the manifold* $M$ *is an* $\epsilon - LP-$*Sasakian manifold.*

*In particular, for the* $3-$*dimensional* $\epsilon - LP-$*Sasakian manifold we consider* $M = \{(x_1, x_2, x_3) \in R^3\}$, *where* $(x_1, x_2, x_3)$ *are the standard coordinates in* $R^3$. *Let* $e_1$, $e_2$ *and* $e_3$ *be the vector fields on* $M$ *given by*

$$e_1 = coshx_3 \frac{\partial}{\partial x_1} + sinhx_3 \frac{\partial}{\partial x_2},\ e_2 = sinhx_3 \frac{\partial}{\partial x_1} + coshx_3 \frac{\partial}{\partial x_2},\ e_3 = \frac{\partial}{\partial x_3} = \xi,$$

*which are linearly independent at each point of* $M^3$ *and hence form a basis of* $T_pM$. *Let* $g$ *be the Lorentzian like (semi-Riemannian) metric on* $M$ *defined by*

$$g(e_1, e_1) = g(e_2, e_2) = 1,\ g(e_3, e_3) = -\epsilon,\ g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0.$$

*Let* $\eta$ *be the 1-form on* $M$ *defined by* $\eta(X) = \epsilon g(X, e_3) = \epsilon g(X, \xi)$ *for all* $X \in \chi(M)$. *Let* $\phi$ *be the* $(1,1)$ *tensor field on* $M$ *defined by*

$$\phi e_1 = -\epsilon e_2,\ \phi e_2 = -\epsilon e_1,\ \phi e_3 = 0.$$

*The linearity property of* $\phi$ *and* $g$ *yields*

$$\eta(e_3) = -1,\ \phi^2 X = X + \eta(X)\xi,\ g(\phi X, \phi Y) = g(X,Y) + \epsilon\eta(X)\eta(Y)$$

*for all* $X, Y \in \chi(M)$.
*Now, by direct computations, we obtain*

$$[e_1, e_2] = 0, \quad [e_2, e_3] = -e_1, \quad [e_3, e_1] = e_2.$$

*Using Koszul's formula, we can easily calculate*

$$(91)\quad \nabla_{e_1}e_1 = 0,\ \nabla_{e_2}e_1 = -\epsilon e_3,\ \nabla_{e_3}e_1 = 0,\ \nabla_{e_1}e_2 = -\epsilon e_3,\ \nabla_{e_2}e_2 = 0,$$
$$\nabla_{e_3}e_2 = 0,\ \nabla_{e_1}e_3 = -e_2,\ \nabla_{e_2}e_3 = -e_1,\ \nabla_{e_3}e_3 = 0.$$

*Also, one can easily verify that*

$$\nabla_X\xi = \epsilon\phi X \quad \text{and} \quad (\nabla_X\phi)Y = g(X,Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi.$$

*Thus the manifold $M$ is an $\epsilon - LP-$Sasakian manifold. From (22) and (91), we find*

$$(92)\quad \bar{\nabla}_{e_1}e_1 = 0,\ \bar{\nabla}_{e_2}e_1 = (1-\epsilon)e_3,\ \bar{\nabla}_{e_3}e_1 = 0,\ \bar{\nabla}_{e_1}e_2 = (1-\epsilon)e_3,$$
$$\bar{\nabla}_{e_2}e_2 = 0,\ \bar{\nabla}_{e_3}e_2 = 0,\ \bar{\nabla}_{e_1}e_3 = 0,\ \bar{\nabla}_{e_2}e_3 = 0,\ \bar{\nabla}_{e_3}e_3 = 0.$$

*It is known that*

$$(93)\quad R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

*By using the above results, we can easily obtain the components of the curvature tensors as follows:*

$$(94)\quad R(e_1,e_2)e_1 = \epsilon e_2,\ R(e_1,e_2)e_2 = -\epsilon e_1,\ R(e_1,e_2)e_3 = 0,$$
$$R(e_2,e_3)e_1 = 0,\ R(e_2,e_3)e_2 = -\epsilon e_3,\ R(e_2,e_3)e_3 = -e_2,$$
$$R(e_1,e_3)e_1 = -\epsilon e_3,\ R(e_1,e_3)e_2 = 0,\ R(e_1,e_3)e_3 = -e_1,$$

*and*

$$(95)\quad \bar{R}(e_1,e_2)e_1 = \epsilon e_2,\ \bar{R}(e_1,e_2)e_2 = 2(1-\epsilon)e_1,\ \bar{R}(e_1,e_2)e_3 = 0,$$
$$\bar{R}(e_2,e_3)e_1 = 0,\ \bar{R}(e_2,e_3)e_2 = (1-\epsilon)e_3,\ \bar{R}(e_2,e_3)e_3 = -(1-\epsilon)e_2,$$
$$\bar{R}(e_1,e_3)e_1 = (1-\epsilon)e_3,\ \bar{R}(e_1,e_3)e_2 = 0,\ \bar{R}(e_1,e_3)e_3 = -(1-\epsilon)e_1.$$

*From these curvature tensors, we calculate*

$$(96)\quad S(e_1,e_1) = S(e_2,e_2) = 0,\ S(e_3,e_3) = -2.$$

$$(97)\quad \bar{S}(e_1,e_1) = \bar{S}(e_2,e_2) = (1-\epsilon),\ \bar{S}(e_3,e_3) = -2(1-\epsilon).$$

*Therefore, from (96) and (97) we obtain $r = 2\epsilon$ and $\bar{r} = 0$, respectively. Thus it can be seen that the equation (26) is satisfied, where $\psi = \sum_{i=1}^{3}\epsilon_i g(\phi e_i, e_i) = 0$.*
*If we take $\mu = 0$ in (35), then putting $Y = Z = e_i$ and sum up with respect to $i(1 \le i \le 3)$, we have*

$$\sum_{i=1}^{3}\epsilon_i\bar{S}(e_i,e_i) = -\epsilon\sum_{i=1}^{3}\epsilon_i g(\phi e_i,e_i) - \lambda\sum_{i=1}^{3}\epsilon_i g(e_i,e_i)$$

*which after computation gives $\lambda = 0$. Thus a Ricci soliton $(g,\xi,\lambda)$ on an $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection* *is steady for the timelike $\epsilon - LP-$Sasakian manifold with a quarter-symmetric metric connection, which verifies corollaries 4.6 and 4.8.*

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