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FUZZY PROPER UP-FILTERS OF UP-ALGEBRAS

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Abstract. The concept of fuzzy sets in UP-algebras was first introduced by Somjanta et al. [Fuzzy sets in UP-algebras; 2016]. In this paper, we introduce and study fuzzy proper UP-filters of UP-algebras and prove their generalizations and characteristic fuzzy sets of proper UP-filters. Moreover, we discuss the relations between fuzzy proper UP-filters and their level subsets.

1. Introduction

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [9], BCI-algebras [10], B-algebras [20], UP-algebras [7] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [10] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [9, 10] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

A fuzzy set f in a nonempty set X is a function from X to a closed interval $[0, 1]$ and the fuzzy set $\overline{f}$ defined by $\overline{f}(x) = 1 - f(x)$ for all $x \in X$ is said to be the *complement* of f in X. The concept of a fuzzy subset of a set was first considered by Zadeh [36] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere. After the introduction of the concept of fuzzy sets by Zadeh [36], several researches were conducted on the generalizations of the notion of fuzzy set and application to many

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logical algebras such as: In 2000, Jun et al. [12] studied fuzzy $\mathcal{I}$-ideals in IS-algebras. In 2002, Jun et al. [13] studied fuzzy B-algebras in B-algebras. Yonglin and Xiaohong [35] studied fuzzy a-ideals in BCI-algebras. In 2004, Jun [11] studied (α, β)-fuzzy ideals of BCK/BCI-algebras. Akram and Dar [1] studied fuzzy subalgebras and d-ideals in d-algebras. In 2007, Akram and Dar [2] studied fuzzy ideals in K-algebras. In 2009, Hadipour [6] studied (α, β)-fuzzy BF-algebras. In 2010, Song et al. [30] studied fuzzy ideals in BE-algebras. In 2011, Mostafa et al. [18] studied fuzzy KU-ideals in KU-algebras. In 2012, Mostafa et al. [19] studied fuzzy Q-ideals in Q-algebras. In 2013, Rao [22] studied fuzzy filters in BE-algebras. In 2014, Yamini and Kailasavalli [34] studied fuzzy B-ideals in B-algebras. In 2019, Koam [17] introduced the concept of pseudometric on KU-algebras.

The branch of the logical algebra, UP-algebras was introduced by Iampan [7]. Later Somjanta et al. [29] studied fuzzy UP-subalgebras, fuzzy UP-ideals and fuzzy UP-filters of UP-algebras. Guntasow et al. [5] studied fuzzy translations of a fuzzy set in UP-algebras. Kesorn et al. [15] studied intuitionistic fuzzy sets in UP-algebras. Kaijae et al. [14] studied anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras. Tanamoon et al. [33] studied Q-fuzzy sets in UP-algebras. Sripaeng et al. [32] studied anti Q-fuzzy UP-ideals and anti Q-fuzzy UP-subalgebras of UP-algebras. Dokkhamdang et al. [4] studied Generalized fuzzy sets in UP-algebras. Songsaeng and Iampan [31] studied $\mathcal{N}$-fuzzy UP-algebras and their level subsets. Senapati et al. [28, 27] applied cubic set and interval-valued intuitionistic fuzzy structure in UP-algebras. Ansari et al. [3] introduced the concept of graphs associated with commutative UP-algebras. Romano [23] studied proper UP-filters of UP-algebras, etc.

In this paper, the notion of fuzzy proper UP-filters of UP-algebras and prove their generalizations and characteristic fuzzy sets of proper UP-filters. Furthermore, we discuss the relation between fuzzy proper UP-filters and their level subsets.

2. Basic Results on UP-Algebras

Before we begin our study, we will give the definition of a UP-algebra.

Definition 2.1. [7] *An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a UP-algebra where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms:*

(UP-1): $(\forall x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,
(UP-2): $(\forall x \in A)(0 \cdot x = x)$,
(UP-3): $(\forall x \in A)(x \cdot 0 = 0)$, *and*
(UP-4): $(\forall x, y \in A)(x \cdot y = 0, y \cdot x = 0 \Rightarrow x = y)$.

From [7], we know that the notion of UP-algebras is a generalization of KU-algebras.

On a UP-algebra $A = (A, \cdot, 0)$, we define a binary relation $\leq$ on A as follows:

$$(\forall x, y \in A)(x \leq y \Leftrightarrow x \cdot y = 0).$$

Example 2.2. [26] *Let X be a universal set and let $\Omega \in \mathcal{P}(X)$ where $\mathcal{P}(X)$ means the power set of X. Let $\mathcal{P}_\Omega(X) = \{A \in \mathcal{P}(X) \mid \Omega \subseteq A\}$. Define a binary operation $\cdot$ on $\mathcal{P}_\Omega(X)$ by putting $A \cdot B = B \cap (A^C \cup \Omega)$ for all $A, B \in \mathcal{P}_\Omega(X)$ where A^C means the complement of a subset A. Then $(\mathcal{P}_\Omega(X), \cdot, \Omega)$ is a UP-algebra and we shall call it the* generalized power UP-algebra of type 1 with respect to Ω. *Let $\mathcal{P}^\Omega(X) = \{A \in \mathcal{P}(X) \mid A \subseteq \Omega\}$. Define a binary operation $*$ on $\mathcal{P}^\Omega(X)$ by putting $A * B = B \cup (A^C \cap \Omega)$ for all $A, B \in \mathcal{P}^\Omega(X)$. Then $(\mathcal{P}^\Omega(X), *, \Omega)$ is a UP-algebra and we shall call it the* generalized power UP-algebra of type 2 with respect to Ω. *In particular, $(\mathcal{P}(X), \cdot, \emptyset)$ is a UP-algebra and we shall call it the* power UP-algebra of type 1, *and $(\mathcal{P}(X), *, X)$ is a UP-algebra and we shall call it the* power UP-algebra of type 2.

Example 2.3. [4] *Let $\mathbb{N}$ be the set of all natural numbers with two binary operations $\circ$ and $\bullet$ defined by*

$$x \circ y = \begin{cases} y & \text{if } x < y, \\ 0 & \text{otherwise} \end{cases}$$

and

$$x \bullet y = \begin{cases} y & \text{if } x > y \text{ or } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(\mathbb{N}, \circ, 0)$ and $(\mathbb{N}, \bullet, 0)$ are UP-algebras.

For more examples of UP-algebras, see [3, 8, 25, 26].

In a UP-algebra $A = (A, \cdot, 0)$, the following assertions are valid (see [7, 8]).

(2.1) $(\forall x \in A)(x \cdot x = 0)$,

(2.2) $(\forall x, y, z \in A)(x \cdot y = 0, y \cdot z = 0 \Rightarrow x \cdot z = 0)$,

(2.3) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (z \cdot x) \cdot (z \cdot y) = 0)$,

(2.4) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (y \cdot z) \cdot (x \cdot z) = 0)$,

(2.5) $(\forall x, y \in A)(x \cdot (y \cdot x) = 0)$,

(2.6) $(\forall x, y \in A)((y \cdot x) \cdot x = 0 \Leftrightarrow x = y \cdot x)$,

(2.7) $(\forall x, y \in A)(x \cdot (y \cdot y) = 0)$,

(2.8) $(\forall a, x, y, z \in A)((x \cdot (y \cdot z)) \cdot (x \cdot ((a \cdot y) \cdot (a \cdot z))) = 0)$,

(2.9) $(\forall a, x, y, z \in A)((((a \cdot x) \cdot (a \cdot y)) \cdot z) \cdot ((x \cdot y) \cdot z) = 0)$,

(2.10) $(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot (y \cdot z) = 0)$,

(2.11) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow x \cdot (z \cdot y) = 0)$,

(2.12) $(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot (x \cdot (y \cdot z)) = 0)$, and

(2.13) $(\forall a, x, y, z \in A)(((x \cdot y) \cdot z) \cdot (y \cdot (a \cdot z)) = 0)$.

Definition 2.4. [7, 29, 5] *A nonempty subset S of a UP-algebra $(A, \cdot, 0)$ is called*

(1) a UP-subalgebra of A if $(\forall x, y \in S)(x \cdot y \in S)$.
(2) a UP-filter of A if it satisfies the following properties:
 (i) the constant 0 of A is in S, and
 (ii) $(\forall x, y \in A)(x \cdot y \in S, x \in S \Rightarrow y \in S)$.
(3) a UP-ideal of A if it satisfies the following properties:
 (i) the constant 0 of A is in S, and
 (ii) $(\forall x, y, z \in A)(x \cdot (y \cdot z) \in S, y \in S \Rightarrow x \cdot z \in S)$.
(4) a strongly UP-ideal of A if it satisfies the following properties:
 (i) the constant 0 of A is in S, and
 (ii) $(\forall x, y, z \in A)((z \cdot y) \cdot (z \cdot x) \in S, y \in S \Rightarrow x \in S)$.

Guntasow et al. [5] proved the generalization that the notion of UP-subalgebras is a generalization of UP-filters, the notion of UP-filters is a generalization of UP-ideals, and the notion of UP-ideals is a generalization of strongly UP-ideals. Moreover, they also proved that a UP-algebra A is the only one strongly UP-ideal of itself.

In 2018, Romano [23] introduced the notion of proper UP-filters of UP-algebras as the following definition.

Definition 2.5. *A subset S of a UP-algebra $(A, \cdot, 0)$ is called a proper UP-filter of A if it satisfies the following properties:*

(i) the constant 0 of A is not in S, and

(ii) $(\forall x, y, z \in A)(x \cdot (y \cdot z) \notin S, y \notin S \Rightarrow x \cdot z \notin S)$.

In what follows, let A denote UP-algebras unless otherwise specified. The following proposition is very important for the study of UP-algebras.

Proposition 2.6. [23] *A subset S of A is a proper UP-filter if and only if the set S^C is a UP-ideal of A.*

Since $\{0\}$ and A are UP-ideals of A and by Proposition 2.6, we have $\{0\}^C$ and $\emptyset$ are proper UP-filters of A.

Corollary 2.7. [23, 24] *Let S be a proper UP-filter of A. Then*

(1) $(\forall x, y \in A)(x \leq y$ and $y \in S \Rightarrow x \in S)$, and

(2) $(\forall x, y \in A)(x \notin S$ and $y \in S \Rightarrow x \cdot y \in S)$.

Theorem 2.8. *Let $\{B_i\}_{i \in I}$ be a family of proper UP-filter of A. Then $\bigcup_{i \in I} B_i$ is a proper UP-filter of A.*

Proof. If $\{B_i\}_{i \in I} = \emptyset$, then $\bigcup_{i \in I} B_i = \emptyset$ is a proper UP-filter of A. Assume that $\{B_i\}_{i \in I}$ is nonempty. Since $0 \notin B_i$ for all $i \in I$, we have $0 \notin \bigcup_{i \in I} B_i$. Let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \notin \bigcup_{i \in I} B_i$ and $y \notin \bigcup_{i \in I} B_i$. Then $x \cdot (y \cdot z) \notin B_i$ and $y \notin B_i$ for all $i \in I$. Thus $x \cdot z \notin B_i$ for all $i \in I$, so $x \cdot z \notin \bigcup_{i \in I} B_i$. Hence, $\bigcup_{i \in I} B_i$ is a proper UP-filter of A. □

Example 2.9. *Let $A = \{0, 1, 2, 3, 4, 5\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	0	0	2	3	2	3
2	0	1	0	3	1	5
3	0	1	2	0	4	1
4	0	0	0	3	0	3
5	0	0	2	0	2	0

From [16], *we have $(A, \cdot, 0)$ is a UP-algebra, and then $\{1, 3, 4, 5\}$ and $\{2, 3, 4, 5\}$ are proper UP-filters of A. Since $\{1, 3, 4, 5\} \cap \{2, 3, 4, 5\} = \{3, 4, 5\}$, and $3 \cdot (1 \cdot 4) = 3 \cdot 2 = 2 \notin \{3, 4, 5\}$ and $1 \notin \{3, 4, 5\}$ but $3 \cdot 4 = 4 \in \{3, 4, 5\}$, we have $\{1, 3, 4, 5\} \cap \{2, 3, 4, 5\} = \{3, 4, 5\}$ is not a proper UP-filter of A. Hence, the intersection of proper UP-filters of A is not a proper UP-filter in general.*

3. Fuzzy Proper UP-Filters

In this section, we introduce the notions of fuzzy proper UP-filters of UP-algebras, provide the necessary examples and prove their generalizations and characteristic fuzzy sets of proper UP-filters.

Definition 3.1. [36] *A fuzzy set in a nonempty set X (or a fuzzy subset of X) is an arbitrary function $f : X \to [0,1]$, where $[0,1]$ is the unit segment of the real line. If $S \subseteq X$, the characteristic function f_S of X is a function of X into $\{0,1\}$ defined as follows:*

$$f_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of characteristic function, f_S is a function of X into $\{0,1\} \subset [0,1]$. Hence, f_S is a fuzzy set in X.

Definition 3.2. *A fuzzy set f in A is called a fuzzy proper UP-filter of A if it satisfies the following properties:*

(1) $(\exists x \in A)(f(0) < f(x))$, *and*
(2) $(\forall x, y, z \in A)(f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\})$.

If there exists a fuzzy proper UP-filter of a UP-algebra A, then A contains at least two elements.

Example 3.3. *Let $A = \{0,1,2,3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	0	2	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a fuzzy set f in A as follows: $f(0) = 0.7, f(1) = 0.2, f(2) = 0.4$, and $f(3) = 0.9$. Then f is a fuzzy proper UP-filter of A.

We can easily prove the following lemma.

Lemma 3.4. *Let f be a fuzzy set in A which satisfies the following condition:*

$$(\forall x, y \in A)(x \leq y \Leftrightarrow f(x) \geq f(y)). \tag{3.1}$$

Then A is a chain with respect to the UP-ordering.

Example 3.5. *Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	0	0
2	0	1	0	0
3	0	1	2	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a fuzzy set f in A as follows: $f(0) = 0.2, f(1) = 0.8, f(2) = 0.6$, and $f(3) = 0.4$. Then f is a fuzzy proper UP-filter of A with the condition 3.1 and 1 is the minimum element of A with respect to the UP-ordering. Hence, $\{1\}$ is the only one singleton set which is a proper UP-filter of A.

Theorem 3.6. *Let f be a fuzzy proper UP-filter of A which satisfies the condition* (3.1). *Then a is the minimum element of A with respect to the UP-ordering if and only if $\{a\}$ is the only one singleton set which is a proper UP-filter of A.*

Proof. Assume that a is the minimum element of A with respect to the UP-ordering. By (UP-3), we have $x \leq 0$ for all $x \in A$. If $0 \in \{a\}$, then 0 is the minimum element of A. It follows that $x = 0$ for all $x \in A$, so $|A| = 1$ which is a contradiction. Thus $0 \notin \{a\}$. Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \notin \{a\}$ and $y \notin \{a\}$. Then $x \cdot (y \cdot z) \neq a$ and $y \neq a$, so $x \cdot (y \cdot z) \nleq a$ and $y \nleq a$. By (3.1), we have $f(x \cdot (y \cdot z)) \ngeq f(a)$ and $f(y) \ngeq f(a)$. Thus $f(x \cdot (y \cdot z)) < f(a)$ and $f(y) < f(a)$. We will show that $x \cdot z \notin \{a\}$. Assume that $x \cdot z \in \{a\}$. Then $x \cdot z = a$, so $f(x \cdot z) = f(a)$. Since f is a fuzzy proper UP-filter of A, we have $f(a) = f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\} < f(a)$ which is a contradiction. Thus $x \cdot z \notin \{a\}$. Hence, $\{a\}$ is a proper UP-filter of A. Next, assume that there exists $b \neq a$ such that $\{b\}$ is a proper UP-filter of A. Then $b \neq 0$. By Corollary 2.7 (2), we have $a \cdot b \in \{b\}$. Since $a \leq b$, we have $0 = a \cdot b = b$ which is a contradiction. Hence, $\{a\}$ is the only one singleton set which is a proper UP-filter of A.

Conversely, assume that $\{a\}$ is the only one singleton set which is a proper UP-filter of A. We will show that $a \leq x$ for all $x \in A$. Assume that there exists $x \in A$ such that $a > x$. Then $a \neq x$. Since $x \leq a$, $a \in \{a\}$ and by Corollary 2.7 (1), we have $x \in \{a\}$. Thus $x = a$ which is a contradiction. Hence, $a \leq x$ for all $x \in A$, that is, a is the minimum element of A with respect to the UP-ordering. □

Theorem 3.7. *A nonempty subset B of A is a proper UP-filter of A if and only if the characteristic fuzzy set f_B is a fuzzy proper UP-filter of A.*

Proof. Assume that B is a proper UP-filter of A where B is nonempty. Then $0 \notin B$, so $f_B(0) = 0$. Since B is nonempty, there exists $a \in B$ such that $f_B(0) = 0 < 1 = f_B(a)$. Next, let $x, y, z \in A$.

Case 1: $x \cdot (y \cdot z) \notin B$ and $y \notin B$. Since B is a proper UP-filter of A, we have $x \cdot z \notin B$. Thus $f_B(x \cdot (y \cdot z)) = 0, f(y) = 0$ and $f_B(x \cdot z) = 0$, so $\max\{f_B(x \cdot (y \cdot z)), f_B(y)\} = \max\{0, 0\} = 0 \geq 0 = f_B(x \cdot z)$.

Case 2: $x \cdot (y \cdot z) \in B$ or $y \in B$. Then $f_B(x \cdot (y \cdot z)) = 1$ or $f_B(y) = 1$, so $\max\{f_B(x \cdot (y \cdot z)), f_B(y)\} = 1 \geq f_B(x \cdot z)$.

Hence, f_B is a fuzzy proper UP-filter of A.

Conversely, assume that f_B is a fuzzy proper UP-filter of A. If $0 \in B$, then $f_B(0) = 1$. Since $f_B(0) < f_B(x)$ for some $x \in A$, we have $1 < f_B(x)$ which is imposible. Thus $0 \notin B$. Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \notin B$ and $y \notin B$. Then $f_B(x \cdot (y \cdot z)) = 0$ and $f_B(y) = 0$. Thus $f_B(x \cdot z) \leq \max\{f_B(x \cdot (y \cdot z)), f_B(y)\} = \max\{0, 0\} = 0$, so $f_B(x \cdot z) = 0$. That is, $x \cdot z \notin B$. Hence, B is a proper UP-filter of A. □

4. Level Subsets of a Fuzzy Proper UP-Filter

In this section, we discuss the relations between fuzzy proper UP-filters and their level subsets.

Definition 4.1. [29] *Let f be a fuzzy set in A. For any $t \in [0, 1]$, the sets*

$$U(f; t) = \{x \in A \mid f(x) \geq t\} \text{ and } U^+(f; t) = \{x \in A \mid f(x) > t\}$$

are called an upper t-level subset and an upper t-strong level subset of f, respectively. The sets

$$L(f; t) = \{x \in A \mid f(x) \leq t\} \text{ and } L^-(f; t) = \{x \in A \mid f(x) < t\}$$

are called a lower t-level subset and a lower t-strong level subset of f, respectively.

From [21], we have the following lemma.

Lemma 4.2. *Let f be a fuzzy set in A. For any $t \in [0, 1]$, the following properties hold:*

(1) $L(f; t) = U(\overline{f}; 1 - t)$,
(2) $L^-(f; t) = U^+(\overline{f}; 1 - t)$,
(3) $U(f; t) = L(\overline{f}; 1 - t)$, *and*

(4) $U^+(f;t) = L^-(\overline{f}; 1-t)$.

Theorem 4.3. *If f is a fuzzy proper UP-filter of A, then for all $t > f(0)$, $U(f;t)$ is a proper UP-filter of A.*

Proof. Assume that f is a fuzzy proper UP-filter of A and let $t > f(0)$. Then $0 \in L^-(f;t)$, that is, $0 \notin U(f;t)$. If $U(f;t) = \emptyset$, then $U(f;t)$ is a proper UP-filter of A. Assume that $U(f;t) \neq \emptyset$. Let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \notin U(f;t)$ and $y \notin U(f;t)$. Then $f(x \cdot (y \cdot z)) < t$ and $f(y) < t$. Since f is a fuzzy proper UP-filter of A, we have $f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\} < t$. Thus $x \cdot z \in L^-(f;t)$, that is, $x \cdot z \notin U(f;t)$. Hence, $U(f;t)$ is a proper UP-filter of A. □

The following example show that the converse statement of Theorem 4.3 is not true.

Example 4.4. *Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	1	2	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a fuzzy set f in A as follows: $f(0) = 0.8, f(1) = 0.3, f(2) = 0.4$, and $f(3) = 0.6$. If $t > f(0) = 0.8$, then $U(f;t) = \emptyset$ is a proper UP-filter of A. Since $f(0) \geq f(x)$ for all $x \in A$, we have f is not a fuzzy proper UP-filter of A.

Theorem 4.5. *Let f be a fuzzy set in A. If there exists $t > f(0)$ such that $U(f;t) \neq \emptyset$, $U(f; f(0)) = A$ and for all $s > f(0)$, $U(f;s)$ is a proper UP-filter of A, then f is a fuzzy proper UP-filter of A.*

Proof. Assume that there exists $t > f(0)$ such that $U(f;t) \neq \emptyset$, $U(f; f(0)) = A$ and for all $s > f(0)$, $U(f;s)$ is a proper UP-filter of A. Then there exists $a \in U(f;t)$, so $f(a) \geq t > f(0)$. Since $U(f; f(0)) = A$, we have $f(x) \geq f(0)$ for all $x \in A$. Next, let $x, y, z \in A$.

Case 1: Assume that $s := f(x \cdot z) > f(0)$. Clearly, $x \cdot z \in U(f;s)$. By assumption, we have $U(f;s)$ is a proper UP filter of A. This implies that $x \cdot (y \cdot z) \in U(f;s)$ or $y \in U(f;s)$. Thus $f(x \cdot (y \cdot z)) \geq s$ or $f(y) \geq s$. Hence, $f(x \cdot z) = s \leq \max\{f(x \cdot (y \cdot z)), f(y)\}$.

Case 2: Assume that $s := f(x \cdot z) = f(0)$. Then $f(x \cdot z) = f(0) \leq \max\{f(x \cdot (y \cdot z)), f(y)\}$.

Hence, f is a fuzzy proper UP-filter of A. □

The following example shows that the condition $U(f; f(0)) = A$ is necessary.

Example 4.6. *Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	0
2	0	0	0	0
3	0	3	2	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a fuzzy set f in A as follows: $f(0) = 0.5, f(1) = 0.2, f(2) = 0.7$, and $f(3) = 0.1$. Then $U(f; f(0)) = \{0, 2\} \neq A$. Consider, for all $t > f(0) = 0.5$. If $t \in (0.5, 0.7]$, then $U(f; t) = \{2\}$ is a proper UP-filter of A. If $t \in (0.7, 1]$, then $U(f; t) = \emptyset$ is a proper UP-filter of A. Since $f(0 \cdot 1) = 0.2 > 0.1 = \max\{f(0 \cdot (3 \cdot 1), f(3))\}$, we have f is not a fuzzy proper UP-filter of A.

Theorem 4.7. *If f is a fuzzy proper UP-filter of A, then for all $t \geq f(0)$, $U^+(f; t)$ is a proper UP-filter of A.*

Proof. Assume that f is a fuzzy proper UP-filter of A and let $t \geq f(0)$. Then $0 \in L(f; t)$, that is, $0 \notin U^+(f; t)$. If $U^+(f; t) = \emptyset$, then $U^+(f; t)$ is a proper UP-filter of A. Assume that $U^+(f; t) \neq \emptyset$. Let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \notin U^+(f; t)$ and $y \notin U^+(f; t)$. Then $f(x \cdot (y \cdot z)) \leq t$ and $f(y) \leq t$. Since f is a fuzzy proper UP-filter of A, we have $f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\} \leq t$. Thus $x \cdot z \in L(f; t)$, that is, $x \cdot z \notin U^+(f; t)$. Hence, $U^+(f; t)$ is a proper UP-filter of A. □

The following example show that the converse statement of Theorem 4.7 is not true.

Example 4.8. *Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a fuzzy set f in A as follows: $f(0) = 0.9, f(1) = 0.5, f(2) = 0.4$, and $f(3) = 0.8$. If $t \geq f(0) = 0.9$, then $U^+(f; t) = \emptyset$ is a proper UP-filter of A. Since $f(0) \geq f(x)$ for all $x \in A$, we have f is not a fuzzy proper UP-filter of A.

Theorem 4.9. *Let f be a fuzzy set in A. If there exists $t > f(0)$ such that $U^+(f;t) \neq \emptyset$, $U(f;f(0)) = A$ and for all $s \geq f(0)$, $U^+(f;s)$ is a proper UP-filter of A, then f is a fuzzy proper UP-filter of A.*

Proof. Assume that there exists $t > f(0)$ such that $U^+(f;t) \neq \emptyset$, $U(f;f(0)) = A$ and for all $s \geq f(0)$, $U^+(f;s)$ is a proper UP-filter of A. Then there exists $a \in U^+(f;t)$, so $f(a) > t > f(0)$. Since $U(f;f(0)) = A$, we have $f(x) \geq f(0)$ for all $x \in A$. Next, let $x, y, z \in A$. Then $s := \max\{f(x\cdot(y\cdot z)), f(y)\} \geq f(0)$. Thus $f(x\cdot(y\cdot z)) \leq s$ and $f(y) \leq s$, so $x\cdot(y\cdot z) \in L(f;s)$ and $y \in L(f;s)$. That is, $x\cdot(y\cdot z) \notin U^+(f;s)$ and $y \notin U^+(f;s)$. Since $U^+(f;s)$ is a proper UP filter of A, we have $x \cdot z \notin U^+(f;s)$. Thus $f(x\cdot z) \leq s = \max\{f(x\cdot(y\cdot z)), f(y)\}$. Hence, f is a fuzzy proper UP-filter of A. □

The following example shows that the condition $U(f;f(0)) = A$ is necessary.

Example 4.10. *Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	0
2	0	1	0	3
3	0	3	2	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a fuzzy set f in A as follows: $f(0) = 0.7, f(1) = 0.4, f(2) = 0.8$, and $f(3) = 0.3$. Then $U(f;f(0)) = \{0, 2\} \neq A$. Consider, for all $t \geq f(0) = 0.7$. If $t \in [0.7, 0.8)$, then $U^+(f;t) = \{2\}$ is a proper UP-filter of A. If $t \in [0.8, 1]$, then $U^+(f;t) = \emptyset$ is a proper UP-filter of A. Since $f(0\cdot 1) = 0.4 > 0.3 = \max\{f(0\cdot(3\cdot 1), f(3))\}$, we have f is not a fuzzy proper UP-filter of A.

Lemma 4.11. [29] *Let f be a fuzzy set in A. Then the following statements hold:*

(1) $(\forall x, y \in A)(1 - \max\{f(x), f(y)\} = \min\{1 - f(x), 1 - f(y)\})$, *and*
(2) $(\forall x, y \in A)(1 - \min\{f(x), f(y)\} = \max\{1 - f(x), 1 - f(y)\})$.

Theorem 4.12. *If the complement $\overline{f}$ is a fuzzy proper UP-filter of A, then for all $t < f(0)$, $L(f;t)$ is a proper UP-filter of A.*

Proof. Assume that $\overline{f}$ is a fuzzy proper UP-filter of A and let $t < f(0)$. Then $0 \in U^+(f;t)$, that is, $0 \notin L(f;t)$. If $L(f;t) = \emptyset$, then $L(f;t)$ is a proper UP-filter of A. Assume that $L(f;t) \neq \emptyset$. Let $x, y, z \in A$ be such that $x\cdot(y\cdot z) \notin L(f;t)$ and $y \notin L(f;t)$. Then $f(x\cdot(y\cdot z)) > t$ and

$f(y) > t$. Since $\overline{f}$ is a fuzzy proper UP-filter of A, we have $\overline{f}(x \cdot z) \leq \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}$. By Lemma 4.11 (2), we have $1 - f(x \cdot z) \leq \max\{1 - f(x \cdot (y \cdot z)), 1 - f(y)\} = 1 - \min\{f(x \cdot (y \cdot z)), f(y)\}$ and so $f(x \cdot z) \geq \min\{f(x \cdot (y \cdot z)), f(y)\} > t$. Thus $x \cdot z \in U^+(f; t)$, that is, $x \cdot z \notin L(f; t)$. Hence, $L(f; t)$ is a proper UP-filter of A. □

The following example show that the converse statement of Theorem 4.12 is not true.

Example 4.13. *From Example 4.4, we define a fuzzy set f in A as follows: $f(0) = 0.2, f(1) = 0.6, f(2) = 0.9$, and $f(3) = 0.4$. If $t < f(0) = 0.2$, then $L(f; t) = \emptyset$ is a proper UP-filter of A. Since $\overline{f}(0) = 1 - f(0) \geq 1 - f(x) = \overline{f}(x)$ for all $x \in A$, we have $\overline{f}$ is not a fuzzy proper UP-filter of A.*

Theorem 4.14. *Let f be a fuzzy set in A. If there exists $t < f(0)$ such that $L(f; t) \neq \emptyset$, $L(f; f(0)) = A$ and for all $s < f(0)$, $L(f; s)$ is a proper UP-filter of A, then the complement $\overline{f}$ is a fuzzy proper UP-filter of A.*

Proof. Assume that there exists $t < f(0)$ such that $L(f; t) \neq \emptyset$, $L(f; f(0)) = A$ and for all $s < f(0)$, $L(f; s)$ is a proper UP-filter of A. Then there exists $a \in L(f; t)$, so $f(a) \leq t < f(0)$. Thus $\overline{f}(a) = 1 - f(a) > 1 - f(0) = \overline{f}(0)$. Since $L(f; f(0)) = A$, we have $f(x) \leq f(0)$ for all $x \in A$. Next, let $x, y, z \in A$.

Case 1: Assume that $s := f(x \cdot z) < f(0)$. Then $f(x \cdot z) \leq s$, so $x \cdot z \in L(f; s)$. Since $L(f; s)$ is a proper UP filter of A, we have $x \cdot (y \cdot z) \in L(f; s)$ or $y \in L(f; s)$ and so $f(x \cdot (y \cdot z)) \leq s$ or $f(y) \leq s$. Thus $f(x \cdot z) = s \geq \min\{f(x \cdot (y \cdot z)), f(y)\}$. By Lemma 4.11 (2), we have

$$
\begin{aligned}
\overline{f}(x \cdot z) &= 1 - f(x \cdot z) \\
&\leq 1 - \min\{f(x \cdot (y \cdot z)), f(y)\} \\
&= \max\{1 - f(x \cdot (y \cdot z)), 1 - f(y)\} \\
&= \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}.
\end{aligned}
$$

Case 2: Assume that $f(x \cdot z) = f(0)$. Then $f(x \cdot z) = f(0) \geq \min\{f(x \cdot (y \cdot z)), f(y)\}$. By Lemma 4.11 (2), we have

$$\begin{aligned}\overline{f}(x \cdot z) &= 1 - f(x \cdot z)\\ &\leq 1 - \min\{f(x \cdot (y \cdot z)), f(y)\}\\ &= \max\{1 - f(x \cdot (y \cdot z)), 1 - f(y)\}\\ &= \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}.\end{aligned}$$

Hence, f is a fuzzy proper UP-filter of A. □

The following example shows that the condition $L(f; f(0)) = A$ is necessary.

Example 4.15. *From Example 4.6, we define a fuzzy set f in A as follows: $f(0) = 0.5, f(1) = 0.6, f(2) = 0.1$, and $f(3) = 0.7$. Then $L(f; f(0)) = \{0, 2\} \neq A$. Consider, for all $t < f(0) = 0.5$. If $t \in [0, 0.1)$, then $L(f; t) = \emptyset$ is a proper UP-filter of A. If $t \in [0.1, 0.5)$, then $L(f; t) = \{2\}$ is a proper UP-filter of A. Since $\overline{f}(0 \cdot 1) = 1 - f(0 \cdot 1) = 0.4 > 0.3 = \max\{1 - f(0 \cdot (3 \cdot 1)), 1 - f(3)\} = \max\{\overline{f}(0 \cdot (3 \cdot 1)), \overline{f}(3)\}$, we have $\overline{f}$ is not a fuzzy proper UP-filter of A.*

Theorem 4.16. *If the complement $\overline{f}$ is a fuzzy proper UP-filter of A, then for all $t \leq f(0)$, $L^{-}(f; t)$ is a proper UP-filter of A.*

Proof. Assume that $\overline{f}$ is a fuzzy proper UP-filter of A and let $t \leq f(0)$. Then $0 \in U(f; t)$, that is, $0 \notin L^{-}(f; t)$. If $L^{-}(f; t) = \emptyset$, then $L^{-}(f; t)$ is a proper UP-filter of A. Assume that $L^{-}(f; t) \neq \emptyset$. Let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \notin L^{-}(f; t)$ and $y \notin L^{-}(f; t)$. Then $f(x \cdot (y \cdot z)) \geq t$ and $f(y) \geq t$. Since $\overline{f}$ is a fuzzy proper UP-filter of A, we have $\overline{f}(x \cdot z) \leq \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}$. By Lemma 4.11 (2), we have $1 - f(x \cdot z) \leq \max\{1 - f(x \cdot (y \cdot z)), 1 - f(y)\} = 1 - \min\{f(x \cdot (y \cdot z)), f(y)\}$ and so $f(x \cdot z) \geq \min\{f(x \cdot (y \cdot z)), f(y)\} \geq t$. Thus $x \cdot z \in U(f; t)$, that is, $x \cdot z \notin L^{-}(f; t)$. Hence, $L^{-}(f; t)$ is a proper UP-filter of A. □

The following example show that the converse statement of Theorem 4.16 is not true.

Example 4.17. *From Example 4.8, we define a fuzzy set f in A as follows: $f(0) = 0.2, f(1) = 0.9, f(2) = 0.4$, and $f(3) = 0.3$. If $t \leq f(0) = 0.2$, then $L^{-}(f; t) = \emptyset$ is a proper UP-filter of A. Since $\overline{f}(0) = 1 - f(0) \geq 1 - f(x) = \overline{f}(x)$ for all $x \in A$, we have $\overline{f}$ is not a fuzzy proper UP-filter of A.*

Theorem 4.18. *Let f be a fuzzy set in A. If there exists $t < f(0)$ such that $L^-(f;t) \neq \emptyset$, $L(f;f(0)) = A$ and for all $s \leq f(0)$, $L^-(f;s)$ is a proper UP-filter of A, then the complement $\overline{f}$ in A is a fuzzy proper UP-filter of A.*

Proof. Assume that there exists $t < f(0)$ such that $L^-(f;t) \neq \emptyset$, $L(f;f(0)) = A$ and for all $s \leq f(0)$, $L^-(f;s)$ is a proper UP-filter of A. Then there exists $a \in L^-(f;t)$, so $f(a) < t < f(0)$. Thus $\overline{f}(a) = 1 - f(a) > 1 - f(0) = \overline{f}(0)$. Since $L(f;f(0)) = A$, we have $f(x) \leq f(0)$ for all $x \in A$. Next, let $x, y, z \in A$. Then $s := \min\{f(x \cdot (y \cdot z)), f(y)\} \leq f(0)$. Thus $f(x \cdot (y \cdot z)) \geq s$ and $f(y) \geq s$, so $x \cdot (y \cdot z) \in U(f;s)$ and $y \in U(f;s)$. That is, $x \cdot (y \cdot z) \notin L^-(f;s)$ and $y \notin L^-(f;s)$. Since $L^-(f;s)$ is a proper UP filter of A, we have $x \cdot z \notin L^-(f;s)$. Thus $f(x \cdot z) \geq s = \min\{f(x \cdot (y \cdot z)), f(y)\}$. By Lemma 4.11 (2), we have

$$\begin{aligned}\overline{f}(x \cdot z) &= 1 - f(x \cdot z)\\ &\leq 1 - \min\{f(x \cdot (y \cdot z)), f(y)\}\\ &= \max\{1 - f(x \cdot (y \cdot z)), 1 - f(y)\}\\ &= \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}.\end{aligned}$$

Hence, f is a fuzzy proper UP-filter of A. □

The following example shows that the condition $L(f;f(0)) = A$ is necessary.

Example 4.19. *From Example 4.10, we define a fuzzy set f in A as follows: $f(0) = 0.4, f(1) = 0.9, f(2) = 0.2$, and $f(3) = 0.6$. Then $L(f;f(0)) = \{0, 2\}$. Consider, for all $t \leq f(0) = 0.4$. If $t \in [0, 0.2]$, then $L^-(f;t) = \emptyset$ is a proper UP-filter of A. If $t \in (0.2, 0.4]$, then $L^-(f;t) = \{2\}$ is a proper UP-filter of A. Since $\overline{f}(1 \cdot 2) = 1 - f(1 \cdot 2) = 0.8 > 0.6 = \max\{1 - f(1 \cdot (3 \cdot 2)), 1 - f(3)\} = \max\{\overline{f}(1 \cdot (3 \cdot 2)), \overline{f}(3)\}$, we have $\overline{f}$ is not a fuzzy proper UP-filter of A.*

5. Conclusions and Future Work

In this paper, we have introduced the notion of fuzzy proper UP-filters of UP-algebras and investigated some of its important properties.

In our future study of UP-algebras, the following objectives considered:

- To get more results in fuzzy proper UP-filters.
- To define anti-fuzzy proper UP-filters.

- To define operations of fuzzy proper UP-filters and anti-fuzzy proper UP-filters.

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