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SPECTRUM CONVOLUTION OF FULL TRANSFORMATION SEMIGROUP

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Abstract. In this paper, some results are obtained from studying convolution on the spectrum of full transformation semigroup and some of its subsemigroups using Cayley's table. The shift of α determines its eigenvalues and one-dimensional linear convolution is complex in Symmetric group.

1. Introduction

Let X be a finite set of natural numbers and let IDT_n be the identity difference full transformation semigroup of all self - maps of X with the condition

$|w^+(\alpha) - w^-(\alpha)| \leq 1, w^+(\alpha) = \max(Im\alpha)[w^-(\alpha) = \min(Im\alpha)][1, 11]$. It is possible to extend the notion of identity difference transformation semigroup using this condition $|w^+(\alpha) - w^-(\alpha)| \leq n-1$ for each n. This means that, it is a semigroup devoid of the identity element as determined by the upper bound of the condition earlier stated.

For instance, when $n = 3$ in IDT_n then $|w^+(\alpha) - w^-(\alpha)| \leq 1$ implies that IDT_3 has no identity element.

Many semigroup theorists like $[4, 5, 6, 7, 8, 10]$ have vastly worked on full transformation semigroups and its related subsemigroups. IDT_n is a new subsemigroup of full transformation semigroup which is interesting to study.

Howie $[4, 5]$ defined the following subsets associated with an element α of T_n:

$S(\alpha) = \{x \in X : x\alpha \neq x\}$,

$Z(\alpha) = X \backslash X\alpha$ and

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$C(\alpha) = \bigcup\{y\alpha^{-1} : y \in X\alpha \text{ and } \left|y\alpha^{-1}\right| \geq 2\}$.

The numbers $s(\alpha) = |S(\alpha)|, z(\alpha) = |Z(\alpha)|$ and $c(\alpha) = |C(\alpha)|$ are called respectively, the shift, the defect and the collapse of α. The length of image of α is denoted by $\mid im(\alpha) \mid$.

The spectrum of a square matrix A is denoted by $\sigma(A)$ and spectral radius of A, denoted by $\rho(A)$ is defined as:

$\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$.

The spectrum of a matrix was used by Michael and Liancheng, [9] to determine its stability. In this paper, we use spectrum to study the convolutionary behaviour of transformation semigroups.

An eigenvalue of a square matrix A that is larger in absolute value than any other eigenvalue is called the dominant eigenvalue; a corresponding eigenvector is called a dominant eigenvector.

The kernel of $\alpha \in S$ is the equivalence relation, $ker\alpha$ on S given by

$ker\alpha = \{(x, y) \in X \times X : x\alpha = y\alpha\}$.

The matrix representation of each element α in a semigroup is defined in [2] as follows: Let S denote a transformation semigroup and $\phi(\alpha) = (m_i, j)_{i,j=1}^{n}$ represent the nXn matrix such that $m_i, j = \{ {}^{1,\ \alpha(j)=i}_{0,\ otherwise}$.

One dimensional linear convolution of a N - point vector, x and a M - point vector, y has length $N + M - 1$. Each element of $T_{n:r}$ is written as a n - dimensional row vector with the Cayley's table drawn to obtain the linear convolution $C(T_{n:r})$, $1 \leq r \leq n^n, n \geq 2$.

In signal processing, the basic assumption is that signals and filter-impulse responses are time series, with a non-zero constant time-duration, called the sampling rate separating consecutive samples.

A mathematically convenient (and legitimate) way of representing this notion is to write a signal as a polynomial where the time-series represents the coefficients of the polynomial. If the sequence (aka signal or filter) $x = (x_0, x_1, x_2, ...)$ interacts with the sequence $y = (y_0, y_1, y_2, ...)$ in the manner of a 'linear filter', then the resulting sequence, $w = (w_0, w_1, w_2, ...)$ will be such that the effect at the ith index (visualize it as, say, time interval) of w will be the sum of individual 'effects' of all pairs x_k, y_j such that $j + k = i$. This idea is used as an application in transformation semigroup using its spectrum, which yield results that are useful in solving problems related to sequences of numbers.

The aim of this paper is to generalise the spectrum of each element in a semigroup by inspection, using the shift of the element and to study Green's relations on IDT_n.

2. Linear Convolution of Full Transformation Semigroup

Cayley table describes the structure of a finite set of elements by arranging all the possible products in a square table reminding of an addition or multiplication table. In this work, Cayley's table is used to obtain the convolution of spectrum of each element.

The following theorem explains the convolution defined on full transformation semigroup. It describes what to expect in the convolution of spectrum of elements of a transformation semigroup.

Theorem 2.1. *Let $x_0, x_1, \ldots x_r \in T_{n:r}, 1 \le r \le n^n$ such that $x_0 = y_0, x_1 = y_1, \ldots x_r = y_r$. Then the convolution $C(T_{n:r})$ is*

$$w_t = \sum_{k=0}^{t} x_k y_{t-k}, 0 \le t \le 2n-2,$$

where $|w(t)| = N + M - 1$ and $x_0 = 0$ for any $p > k, y_q = 0$ for $q > t - k$.

Proof. From Cayley's table, $w_0 = x_0 y_0, w_1 = x_0 y_1 + x_1 y_0, \ldots, w_{t-1} = x_0 y_{t-1} + x_1 y_{t-2} + \ldots x_{t-1} y_0$. Thus, w_t of $T_{n:r}$ is $\sum_{k=0}^{t} x_k y_{t-k}$. This corresponds to MATLAB command, *conv*(s, s) to obtain the convolution of s, where s is the spectrum of $T_{n:r}$ for each r. $\square$

Examples of convolution on some elements of T_3:

Example 2.2. *Calculate the (w_t) of $T_{3:16}$.*

Solution:
The cardinality of T_3 is 27, that is $r = 1, 2, \ldots 27$.
$\sigma(T_{3:16}) = [-0.5000 + 0.8660i \quad -0.5000 - 0.8660i \quad 1.0000]$ and
w_t of $\sigma(T_{3:16}) = [-0.5000 - 0.8660i \quad 1.9999 \quad -1.5000 + 2.5980i \quad -1.0000 - 1.7320i \quad 1.0000]$.

Example 2.3. *Calculate the (w_t) of $T_{3:8}$.*

Solution:
$\sigma(T_{3:8}) = [-1 \quad 1 \quad 1]$.
The Cayley table yields w_t of $\sigma(T_{3:8}) = [1 \quad -2 \quad -1 \quad 2 \quad 1]$. This can be explained further as follows:

.	-1	1	1
-1	1	-1	-1
1	-1	1	1
1	-1	1	1

Table 1: Cayley's table of $\sigma(T_{3:8})$ with reverse diagonal addition for its w_t

2.1. Some Results on Spectrum of IDT_n

Next theorem describes the relationship between shift of α and its spectrum.

Theorem 2.4. *Let Λ denote the interval of eigenvalues, that is $\Lambda = a \leq \lambda \leq b$, $\alpha \in IDT_n$ and $S(\alpha) = \{x \in X : x\alpha \neq x\}$. Then $|S(\alpha)|$ directly determines Λ.*

Proof. Let $f(\alpha) = \{x \in X : x\alpha = x\}$. If $|f(\alpha)| = n$ then $|S(\alpha)| = 0$ but for $IDT_n, |S(\alpha)| \leq n$ and never zero for $n > 2$. The eigenvalues of α are obtained either by direct calculation or with ease using MATLAB commands. If $S(\alpha) = n$ and α is such that, $\alpha(i) = j$ and $\alpha(j) = i$, then the number of times swapping occurs, determines the number of -1 that are generated as eigenvalues of α. That is, if $i\alpha = j$ and $j\alpha = i$ appears once in α, it follows that -1 is an eigenvalue of α.

If $i\alpha = j$ and $j\alpha = i$ occur twice in α, then -1 occur twice as eigenvalues. The interval of such eigenvalues is given as $\Lambda = [-1, 1]$. If $|S(\alpha)| < n$ then $|f(\alpha)| < n$ and $\Lambda = [0, 1]$. Hence, the nature and interval of eigenvalues can be determined using $|S(\alpha)|$. □

Proposition 2.5. *Let α be an element of IDT_n and let $\rho(\alpha) = \max\{|\lambda| : \lambda \in \sigma(\alpha)\}$. Then $\rho(\alpha) = 1$, for $n \geq 2$.*

Proof. The Identity difference transformation semigroup, IDT_n has $\max|(Im\alpha)| = 2$ and $\min|(Im\alpha)| = 1$.

Let $\lambda \in \sigma(\alpha)$. For each $\alpha \in IDT_n$, the upper bound for $\sigma(\alpha)$ is 1. Hence $\rho(\alpha)$ is 1, for $n \geq 2$. □

Lemma 2.6. *Let $n \geq 2$ and let $H = \sum_{i=1}^{n} \lambda_i$. Then $0 \leq H \leq 2$.*

Proof. IDT_n is a semigroup with $|w^+(\alpha) - w^-(\alpha)| \leq 1$ and from proposition 4, $|S(\alpha)| \leq n$ and never zero for $n > 2$. As $f(\alpha) \uparrow, S(\alpha) \downarrow$ resulting into $H \uparrow$. Here, since $f(\alpha) \neq n$ implies that $H \neq n$ from the fact that $|w^+(\alpha)| = 2$. □

The following theorem derives formula for the sum of spectrum, $H = 0, 1, 2$ in IDTn.

Theorem 2.7. *Let $n \geq 2$ and let $H = \sum_{i=1}^{n} \lambda_i$. When $H = 0, |H| = (n-1)2^{n-2}$; when $H = 1, |H| = n + (n-1)(2^n - 2) - (n-1)2^{n-1}$ and when $H = 2, |H| = (n-1)2^{n-2}$.*

Proof. Lemma 6 shows that H is nonnegative with 2 as upper bound. The cardinality of each H is known by counting and enumeration. □

Next theorem derives formular for idempotent elements when convoluted over a number of times, which is the binomial coefficients of every 2^n - step.

Theorem 2.8. *Let α be idempotent. Then*

$$[\sigma(\alpha)]^{2n} = [conv(\sigma(\alpha), \sigma(\alpha))]^n = \frac{2n!}{k!(2n-k)!}, k = 0, 1, \ldots 2n.$$

Proof. Consider the function $f(y) = (x+y)^{2n}$. The coefficients of $f(y)$ is equivalent to the binomial coefficients of every 2^n - step. Hence the triangle formed has the formula derived as $\frac{2n!}{k!(2n-k)!}, k = 0, 1, \ldots 2n, \equiv 2^{2n}$. This is true for each idempotent element of IDT_n. □

The following figures are examples of convolution of spectrum of idempotents in IDT_n:

Figure 1: Convolution of spectrum for $\lambda = 1, 1$, when $n = 2$.

1 2 1
1 4 6 4 1
1 8 28 56 70 56 28 8 1
1 16 120 560 1820 4368 8008 11440 12870 11440 8008 4368 1820 560 120 16 1

Figure 2: Convolution of spectrum for $\lambda = 1, 0, 1$ when $n = 3$.

1 0 2 0 1
1 0 4 0 6 0 4 0 1
1 0 8 0 28 0 56 0 70 0 56 0 28 0 8 0 1
1 0 16 0 120 0 560 0 1820 0 4368 0 8008 0 11440 0 12870 0 11440 0 8008 0 4368 0 1820 0 560 0 120 0 16 0 1

Figure 3 : Convolution of spectrum for $\lambda = 1, 1, 0$, when $n = 2$.

1 2 1 0 0
1 4 6 4 1 0 0 0 0
1 8 28 56 70 56 28 8 1 0 0 0 0 0 0 0 0
1 16 120 560 1820 4368 8008 11440 12870 11440 8008 4368 1820 560 120 16 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

2.2. Some Results on Spectrum of T_n

The spectrum composition of full transformation semigroup, T_n for $n \geq 2$ is different from that of identity difference transformation semigroup, IDT_n. The following results are obtained on the study of the behaviour of spectrum on full transformation semigroup, T_n.

Lemma 2.9. *Element that has at most one fixed point and is singular has spectrum* $[1, 0, 0, 0, 0, \ldots]$ *with* $n-1$ *zeros.*

Proof. The number of times fix points appear in an element, determines the number of 1's in spectrum balanced up with zeros as eigenvalues. The occurrence of fix points $n-$ times in an element implies $n-$times occurrence of $1's$ in spectrum and no zero eigenvalue. If an element has $n-1$ fixed points in a singular transformation semigroup, then such an element has $n-1$, $1's$ in spectrum and only one zero. Deductively, element with one fixed point has only 1 in the spectrum with $n-1$ zeros. □

The complexity of the convolution of spectrum of elements of a symmetric group is in a theorem as follows:

Theorem 2.10. *Elements in Symmetric group have complex convolution.*

Proof. The spectrum of symmetric group mixes both the real and complex numbers being a permutation group. Hence the convolution also mixes both real and complex numbers. □

Theorem 2.11. *Conjugacy classes of a symmetric group have the same spectrum each.*

Proof. Conjugacy preserves spectrum and the corresponding convolution. □

Theorem 2.12. *The convolution of identity elements forms symmetrical pyramid of numbers.*

Proof. The implication of Lemma 6 is that an identity element has $1's$ only in the spectrum. $C(T_{n:r})$ for each n where r is an identity is obtained as explained in theorem 1 above. Since each n has one identity element, the pyramid has n as its line of symmetry. This is seen in the figure below:

Figure 4: Convolution of identity elements in symmetric group for $n = 2, 3, \ldots 10$.

$$
\begin{array}{c}
1\ 2\ 1 \\
1\ 2\ 3\ 2\ 1 \\
1\ 2\ 3\ 4\ 3\ 2\ 1 \\
1\ 2\ 3\ 4\ 5\ 4\ 3\ 2\ 1 \\
1\ 2\ 3\ 4\ 5\ 6\ 5\ 4\ 3\ 2\ 1 \\
1\ 2\ 3\ 4\ 5\ 6\ 7\ 6\ 5\ 4\ 3\ 2\ 1 \\
1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1 \\
1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1 \\
1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1
\end{array}
$$

□

The sum of each step in figure 4 is $n^2, n = 2, 3, \ldots$.

Other results are as follows:

- Sum of convolution of positive - valued spectrum is a square of the sum of spectrum.
- Sum of convolution of convolution always give the square of the sum of preceeding convolution.
- Sum of spectrum with integer values is equivalent to sum of its convolution.

  Example: An element of IDT_n has the spectrum $[-1, 0, 1]$, sum of which is 0.Its first convolution is $-1, -1, 2, 1, -1$, whose sum is also 0. The second convolution is $1, 2, -3, -6, 4, 6, -3, -2, 1$ with sum equal to zero.

2.3. Green's Relation on IDT_n

The relations L, R, H, D and J on the semigroup S are called Green's relations as introduced in [3]. An equivalence L on S is defined by the rule that aLb if and only if a and b generate the same principal left ideal, that is, if and only if $S^1a = S^1b$. Similarly, the equivalence R is defined by the rule that, aRb if and only if $aS^1 = bS^1$.

The minimum equivalence relation on S which contains both R and L is called the $D-$ relation. The intersection of two equivalence relations is always an equivalence relation. $H-$relation is defined to be the intersection of R and L. $H(a)$ denotes the $H-$class of an element a. Elements a and b are $J-$ equivalent provided that they generate the same principal two - sided ideal, that is, $S^1aS^1 = S^1bS^1$. Also, $J(a)$ denotes the $J-$class of an element a[2].

Composition of mapping on elements in each $L-$related class naturally yields elements whose $\mid im(\alpha) \mid = 1$.

The following results are obtained on Green's relation in IDT_n:

Theorem 2.13. *Let $X = 1, 2, 3, \ldots n$. The semigroup IDT_n contains $2n - 1$ different $L-$classes, 2^{n-1} different $R-$classes and $n-$different J-classes.*

Proof. Let R, L and $J-$ denote the $R-$class, $L-$class and $J-$class of IDT_n. Each $L-$class is a structure of the elements with the same image as subset of X. That is, $\alpha L \beta$ if $im(\alpha) = im(\beta) \subseteq X$. For each $\alpha \in IDT_n, im(\alpha) = \{i, i+1\}, i = 0, 1, 2, \ldots n - 1$. In this semigroup, image of an element α, can only assume two consecutive numbers. Hence, there are $2n - 1, L-$classes. $R-$classes follow the same argument of proof with $L-$classes, only that the structure of $R-$related elements is the kernel of α.

From Green's definition of $J-$classes, in this study $J-$classes are the $L-$related elements for elements with $\mid im(\alpha) \mid > 1$. Elements with $\mid im(\alpha) \mid = 1$ form a $J-$class and hence the proof. □

Theorem 2.14. *Let D_k denote each of the $D-$classes in IDT_n for each $k, n \in N$ Then IDT_n has only two $D-$classes.*

Proof. A rectangular table shown below conveniently shows that the rows correspond to $R-$classes and the column correspond to $L-$classes. The length of image of α, $\mid im(\alpha) \mid$ can assume either 1 or 2 since $im(\alpha) = \{i, i+1\}, i = 0, 1, 2, \ldots n-1$. Hence there are two $D-$classes in IDT_n for all n. □

$Table1 : TheD - ClassesofIDT_n$

$D_1 =$

$L \backslash R$	{1, 2}	{2, 3}	{3, 4}
123/4	(1112), (2221)	(2223), (3332)	(3334), (4443)
124/3	(1121), (2212)	(2232), (3323)	(3343), (4434)
134/2	(1211), (2122)	(2322), (3233)	(3433), (4344)
1/234	(1222), (2111)	(2333), (3222)	(3444), (4333)
13/24	(1212), (2121)	(2323), (3232)	(3434), (4343)
12/34	(1122), (2211)	(2233), (3322)	(3344), (4433)
14/23	(1221), (2112)	(2332), (3223)	(3443), (4334)

$D_2 =$

$L \backslash R$	{1}	{2}	{3}	{4}
1234	(1111)	(2222)	(3333)	(4444)

. $H-$classes are the intersection of L and R.

The following results are obtained on the $L-$related elements of IDT_n :

Lemma 2.15. *Each $L-$related elements is a subsemigroup.*

Proof. Let L_a denotes each group of elements that are $L-$related for each $a \in N$. Since IDT_n is

a semigroup satisfying the condition $|w^+(\alpha) - w^-(\alpha)| \leq 1$

$\forall \alpha, \beta \in IDT_n$ and for $x, y, z \in L_a, x * (y * z) = (x * y) * z \in L_a$, then $L-$related elements is a subsemigorup for each $im(\alpha) = im(\beta)$. □

Theorem 2.16. *Let L_a denotes each group of elements that are $L-$related and R_a denotes each group of elements that are $R-$related for each $a \in N$, then L_a contains a left zero semigroup and R_a contains a right zero semigroup.*

Proof. Let $a, b \in L_a$ such that $a^2 = a$ and $b^2 = b$. It is obtained that the idempotent elements a, b implies $ab = a$. Hence the idempotent elements in L_a is a left zero semigroup. Also, let $a, b \in R_a$ such that $a^2 = a$ and $b^2 = b$. It is obtained that the idempotent elements a, b implies $ab = b$. Hence the idempotent elements in R_a is a right zero semigroup. $\square$

Conclusion

It has been shown that the convolution of spectrum of a transformation semigroup is a concept that generates new sequences that can be used to solve problems related to sequences of numbers and supports existing ones like the 2^n- step binomial coeffients. The study can be applied to other transformation semigroups and series of numbers.

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