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CLASSIFICATION OF THREE-DIMENSIONAL CONFORMALLY FLAT QUASI-PARA-SASAKIAN MANIFOLDS

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Abstract. The aim of this paper is to study three-dimensional conformally flat quasi-para-Sasakian manifolds. First, the necessary and sufficient conditions are provided for three-dimensional quasi-para-Sasakian manifolds to be conformally flat. Next, a characterization of three-dimensional conformally flat quasi-para-Sasakian manifold is given. Finally, a method for constructing examples of three-dimensional conformally flat quasi-para-Sasakian manifolds is presented.

1. Introduction

Almost paracontact geometry was first introduced and studied by Kaneyuki and Williams in [9] and then many other authors continued to study. Zamkovoy studied almost paracontact metric manifolds in [20]. Because of there are lots of studies on almost contact geometry, it seems there should be new studies about almost paracontact geometry. Therefore, paracontact metric manifolds have been studied in recent years by many authors, emphasizing similarities and differences with respect to the most well known contact case. Interesting papers connecting these fields are, for example, [6], [4], [18], [20], and references therein.

Z. Olszak studied normal almost contact metric manifolds of dimension three [15]. He derived certain necessary and sufficient conditions for an almost contact metric structure on manifold to be normal. He found curvature properties of such structures and he considered normal

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almost contact metric manifolds of constant curvature. Curvature and torsion of Frenet-Legendre curves in three-dimensional normal almost paracontact metric manifolds were investigated in [19] and then normal almost paracontact metric manifolds were studied in [1], [10], [11].

The notion of quasi-Sasakian manifolds, introduced by D. E. Blair in [2], unifies Sasakian and cosymplectic manifolds. A quasi-Sasakian manifold is a normal almost contact metric manifold whose fundamental 2-form $\Phi := g(\cdot, \phi\cdot)$ is closed. Quasi-Sasakian manifolds can be viewed as an odd-dimensional counterpart of Kaehler structures. These manifolds studied by several authors (e.g. [8], [14], [16], [17]).

Quasi-Sasakian manifolds were studied by many different authors and are considered a well-established topic in contact Riemannian geometry. But to the author's knowledge, there do not exist any comprehensive study about quasi-para-Sasakian manifolds.

Motivated by these considerations, in [13], the author makes the first contribution to investigate basic properties and general curvature identities of quasi-para-Sasakian manifolds.

In this paper, we study three-dimensional conformally flat quasi-para-Sasakian manifolds.

Section 2 is preliminary section, where we recall the definition of almost paracontact metric manifold and quasi-para-Sasakian manifolds.

In Section 3, we mainly proved that for a three-dimensional quasi-para-Sasakian manifold M , the followings are equivalent.

$i)$ M is locally symmetric.

$ii)$ M is conformally flat and its scalar curvature τ is const.,

$iii)$ M is conformally flat and β is const.,

$iv)$• If $\beta = 0$, then M is a paracosymplectic manifold which is locally a product of the real line R and a 2-dimensional para-Kaehlerian manifold or

•If $\beta \neq 0$, then M is of constant negative curvature and the quasi-para-Sasakian structure can be obtained by a homothetic deformation of a para-Sasakian structure.

Finally, we gave a theorem which gives a method for constructing examples of three-dimensional conformally flat quasi-para-Sasakian manifolds.

2. Preliminaries

A $(2n+1)$-dimensional differentiable manifold M has an *almost paracontact structure* (ϕ, ξ, η) if it admits a $(1,1)$ tensor field ϕ, a vector field ξ and a one-form η satisfying followings

(i) $\phi^2 = Id - \eta \otimes \xi, \quad \eta(\xi) = 1,$

(ii) distribution

$$D : p \in M \to D_p \subset T_pM : D_p = Ker\eta = \{X \in T_pM : \eta(X) = 0\}$$

is called *paracontact distribution* generated by η.

The manifold M is said to be an *almost paracontact manifold* if it is endowed with an almost paracontact structure [20].

If an almost paracontact manifold admits a pseudo-Riemannian metric g of a signature $(n+1, n)$, i.e.

$$g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y), \tag{1}$$

then the manifold will be called an *almost paracontact metric manifold* and g is *compatible.*

For such manifold, we have

$$\eta(X) = g(X, \xi),\ \ \phi(\xi) = 0,\ \ \eta \circ \phi = 0. \tag{2}$$

Moreover, we can define a skew-symmetric tensor field (a 2-form) Φ by

$$\Phi(X,Y) = g(X, \phi Y), \tag{3}$$

usually called *fundamental form.*

For an almost paracontact manifold, there exists an orthogonal basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi\}$ such that $g(X_i, X_j) = \delta_{ij}$, $g(Y_i, Y_j) = -\delta_{ij}$ and $Y_i = \phi X_i$, for any $i, j \in \{1, \ldots, n\}$. Such basis is called a *ϕ-basis.*

On an almost paracontact manifold, one defines the $(1,2)$-tensor field $N^{(1)}$ by

$$N^{(1)}(X,Y) = [\phi, \phi](X,Y) - 2d\eta(X,Y)\xi, \tag{4}$$

where $[\phi, \phi]$ is the *Nijenhuis torsion* of ϕ

$$[\phi, \phi](X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y].$$

The almost paracontact manifold (structure) is said to be *normal* when $N^{(1)} = 0$ [20]. The normality condition implies that the almost paracomplex structure J is integrable which is defined by

$$J(X, \lambda\frac{d}{dt}) = (\phi X + \lambda\xi, \eta(X)\frac{d}{dt}),$$

on $M \times \mathbb{R}$.

If $d\eta(X,Y) = g(X,\phi Y)$, then η is a *paracontact form* and the almost paracontact metric manifold (M,ϕ,ξ,η,g) is said to be *paracontact metric manifold.* In a paracontact metric manifold one defines a symmetric, trace-free operator $h = \frac{1}{2}\mathcal{L}_\xi\phi$, where $\mathcal{L}_\xi$, denotes the Lie derivative. In [20], it is proved that the operator h satisfies the followings: $h\xi = 0$, tr$h$ =tr$h\phi = 0$ and $\nabla\xi = -\phi + \phi h$, where ∇ is the Levi-Civita connection of the pseudo-Riemannian manifold (M,g). Also h anti-commutes with ϕ .

Moreover $h = 0$ if and only if ξ is Killing vector field. In this case (M,ϕ,ξ,η,g) is said to be a *K-paracontact manifold.* Similarly as in the class of almost contact metric manifolds [3], a normal almost paracontact metric manifold will be called *para-Sasakian* if $\Phi = d\eta$ [7]. The para-Sasakian condition implies the K-paracontact condition and the converse holds only in dimension three. A paracontact metric manifold will be called *paracosymplectic* if $d\Phi = 0$, $d\eta = 0$ [6].

Now, we will give some results about three-dimensional quasi-para-Sasakian manifolds that we will use next sections.

Proposition 2.1. [19] *For a three-dimensional almost paracontact metric manifold M the following three conditions are mutually equivalent*

(a) *M is normal,*

(b) *there exist functions α, β on M such that*

$$(\nabla_X\phi)Y = \beta(g(X,Y)\xi - \eta(Y)X) + \alpha(g(\phi X,Y)\xi - \eta(Y)\phi X), \tag{5}$$

(c) *there exist functions α, β on M such that*

$$\nabla_X\xi = \alpha(X - \eta(X)\xi) + \beta\phi X. \tag{6}$$

Corollary 2.2. [10] *For a normal almost paracontact metric structure (ϕ,ξ,η,g) on M, we have $\nabla_\xi\xi = 0$ and $d\eta = -\beta\Phi$. The functions α,β realizing (5) as well as (6) are given by* [19]

$$2\alpha = \text{ Trace } \{X \longrightarrow \nabla_X\xi\}, \quad 2\beta = \text{Trace } \{X \longrightarrow \phi\nabla_X\xi\}. \tag{7}$$

Proposition 2.3. [19] *For a three-dimensional almost paracontact metric manifold M, the following three conditions are mutually equivalent*

(a) *M is quasi-para-Sasakian,*

(b) *there exists a function β on M such that*

$$(\nabla_X\phi)Y = \beta(g(X,Y)\xi - \eta(Y)X), \tag{8}$$

(*c*) *there exists a function* β *on* M *such that*

$$\nabla_X \xi = \beta \phi X. \tag{9}$$

A three-dimensional normal almost paracontact metric manifold is

• quasi-para-Sasakian if and only if $\alpha = 0$ and β is certain function [7], [19], in particular, para-Sasakian if $\beta = -1$ [19], [20],

• paracosymplectic if $\alpha = \beta = 0$ [6],

• α-para-Kenmotsu if $\alpha \neq 0$ and α is constant and $\beta = 0$ [12].

Namely, the class of para-Sasakian and paracosymplectic manifolds are contained in the class of quasi-para-Sasakian manifolds.

Theorem 2.4. [10]*Let* (M, ϕ, ξ, η, g) *be a three-dimensional normal almost paracontact metric manifold. Then the following curvature identities hold*

$$\begin{aligned}
& R(X,Y)Z \\
= \ & (2(\xi(\alpha) + \alpha^2 + \beta^2) + \frac{1}{2}\tau)(g(Y,Z)X - g(X,Z)Y) \\
& -(\xi(\alpha) + 3(\alpha^2 + \beta^2) + \frac{1}{2}\tau)((g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\
& +\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y) + (\phi Z(\beta) - Z(\alpha))(\eta(Y)X - \eta(X)Y) \\
& +(\phi Y(\beta) - Y(\alpha))(\eta(Z)X - g(X,Z)\xi) \\
& -(\phi X(\beta) - X(\alpha))(\eta(Z)Y - g(Y,Z)\xi) \\
& +(\phi \mathrm{grad}\beta + \mathrm{grad}\alpha)(\eta(Y)g(X,Z) - \eta(X)g(Y,Z)).
\end{aligned} \tag{10}$$

$$\begin{aligned}
S(Y,Z) \ = \ & -(\xi(\alpha) + \alpha^2 + \beta^2 + \frac{1}{2}\tau)g(\phi Y, \phi Z) \\
& +\eta(Z)(\phi Y(\beta) - Y(\alpha)) \\
& +\eta(Y)(\phi Z(\beta) - Z(\alpha)) - 2(\alpha^2 + \beta^2)\eta(Y)\eta(Z),
\end{aligned} \tag{11}$$

where R*,* S *and* τ *are resp. Riemannian curvature, Ricci tensor and scalar curvature of* M*.*

If we take $\alpha = 0$ in Theorem *2.4*, we get following

Theorem 2.5. *Let (M,ϕ,ξ,η,g) be a three-dimensional quasi-para-Sasakian manifold. Then the following curvature identities hold*

$$\begin{aligned} & R(X,Y)Z \\ = \; & (2\beta^2+\frac{1}{2}\tau)(g(Y,Z)X-g(X,Z)Y) \\ & -(3\beta^2+\frac{1}{2}\tau)((g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi \\ & +\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y)+\phi Z(\beta)(\eta(Y)X-\eta(X)Y) \\ & +\phi Y(\beta)(\eta(Z)X-g(X,Z)\xi) \\ & -\phi X(\beta)(\eta(Z)Y-g(Y,Z)\xi) \\ & +(\phi\mathrm{grad}\beta)(\eta(Y)g(X,Z)-\eta(X)g(Y,Z)). \end{aligned} \tag{12}$$

$$\begin{aligned} S(Y,Z) \; = \; & (\beta^2+\frac{1}{2}\tau)g(Y,Z)-(3\beta^2+\frac{1}{2}\tau)\eta(Y)\eta(Z) \\ & +\eta(Y)\phi Z(\beta)+\eta(Z)\phi Y(\beta). \end{aligned} \tag{13}$$

where R, S and τ are resp. Riemannian curvature, Ricci tensor and scalar curvature of M.

Remark 2.6. *In the proof of Theorem 2.4, the author showed that $\xi(\beta)+2\alpha\beta=0$. Namely, for three-dimensional quasi-para-Sasakian manifolds,*

$$\xi(\beta)=0. \tag{14}$$

Theorem 2.7. [13]*Let $(M^{2n+1},\phi,\xi,\eta,g)$ be a quasi-para-Sasakian manifold of constant curvature K. Then $K\leq 0$. Furthermore,*

- *If $K=0$, the manifold is paracosymplectic,*
- *If $K<0$, the structure (ϕ,ξ,η,g) is obtained by a homothetic deformation of a para-Sasakian structure on M^{2n+1}.*

3. Three-dimensional conformally flat quasi-Para-Sasakian manifolds

For the conformal flatness of three dimensional semi-Riemannian manifold, we will use linear $(1,1)$-tensor field (Weyl-Schouten tensor) L which is defined by

$$L=Q-\frac{\tau}{4}Id, \tag{15}$$

where $S(X,Y)=g(QX,Y)$[5].

From now on, we will use the notion $df(X)$ instead of $g(gradf,X)$.

Lemma 3.1. *The linear operator L of a three-dimensional quasi-para-Sasakian manifold is given by*

$$LY = \left(\frac{\tau}{4} + \beta^2\right) Y - \left(3\beta^2 + \frac{\tau}{2}\right) \eta(Y)\xi - \eta(Y)\phi grad\beta + d\beta(\phi Y)\xi. \tag{16}$$

Proof. By (13), we obtain

$$QY = \left(\frac{\tau}{2} + \beta^2\right) Y - \left(3\beta^2 + \frac{\tau}{2}\right) \eta(Y)\xi - \eta(Y)\phi grad\beta + d\beta(\phi Y)\xi. \tag{17}$$

The requested equation comes from combining (15) and the above last equation. □

From (16), we have

$$\begin{aligned}
(\nabla_X L)Y &= \nabla_X LY - L\nabla_X Y \\
&= \left(\frac{d\tau(X)}{4} + 2\beta d\beta(X)\right) Y - \left(6\beta d\beta(X) + \frac{d\tau(X)}{2}\right) \eta(Y)\xi \\
&\quad - \left(3\beta^2 + \frac{\tau}{2}\right) ((\nabla_X \eta)(Y)\xi + \eta(Y)\nabla_X \xi) \\
&\quad -(\nabla_X \eta)(Y)\phi grad\beta - \eta(Y)(\nabla_X \phi) grad\beta - \eta(Y)\phi\nabla_X grad\beta \\
&\quad +(\nabla_X d\beta)(\phi Y)\xi + d\beta((\nabla_X \phi)Y)\xi + d\beta(\phi Y)\nabla_X \xi.
\end{aligned}$$

If we use (8), (9) and (14) in the last equation, we can state following:

Lemma 3.2. *For a three-dimensional quasi-para-Sasakian manifold, the following formula is valid for the covariant derivative of the linear operator L*

$$\begin{aligned}
(\nabla_X L)Y &= \left(\frac{d\tau(X)}{4} + 2\beta d\beta(X)\right) Y - \left(6\beta d\beta(X) + \frac{d\tau(X)}{2}\right) \eta(Y)\xi \\
&\quad -\beta \left(3\beta^2 + \frac{\tau}{2}\right) (g(\phi X, Y)\xi + \eta(Y)\phi X) - \beta g(\phi X, Y)\phi grad\beta \\
&\quad -\beta d\beta(X)\eta(Y)\xi - \eta(Y)\phi\nabla_X grad\beta + (\nabla_X d\beta)(\phi Y)\xi \\
&\quad -\beta\eta(Y)d\beta(X)\xi + \beta d\beta(\phi Y)\phi X.
\end{aligned} \tag{18}$$

Lemma 3.3. *For the the function β of three-dimensional quasi-para-Sasakian manifold, the following equality holds*

$$\nabla_\xi grad\beta = \beta\phi grad\beta. \tag{19}$$

Proof. By virtue of (14), we have

$$[\xi, X](\beta) = \xi(X(\beta)) - X(\xi(\beta)) = g(\nabla_\xi grad\beta, X) + g(grad\beta, \nabla_\xi X). \tag{20}$$

By (9), we get

$$[\xi, X](\beta) = g(grad\beta, [\xi, X]) = g(grad\beta, \nabla_\xi X) + \beta g(\phi grad\beta, X). \tag{21}$$

The proof comes from (20) and (21). $\square$

There exists a local orthonormal ϕ-basis $\{e_1=\phi e_2,\ e_2=\phi e_1,\ e_3=\xi\}$, such that $g(e_1,e_1)=-g(e_2,e_2)=g(e_3,e_3)=1$, for any point $p\in U\subset M$.

For the sake of shortness, we will give followings

$$\begin{aligned}
\tau_i &= d\tau(e_i),\\
\beta_i &= d\beta(e_i)\\
\beta_{ij} &= (\nabla_{e_i} d\beta)(e_j),\\
L_{ij} &= (\nabla_{e_i} L)e_j\\
grad\beta &= \beta_1 e_1-\beta_2 e_2+\beta_3 e_3,\\
\nabla_{e_i} grad\beta &= \beta_{i1} e_1-\beta_{i2} e_2+\beta_{i3} e_3
\end{aligned}$$

for $1\le i,j\le 3$, where τ_i, β_i, β_{ij} are the functions and L_{ij} are the vector fields on U. Also we can write,

$$\begin{aligned}
(\nabla_{e_i} d\beta)(e_j)-(\nabla_{e_j} d\beta)(e_i) &= \nabla_{e_i} d\beta(e_j)-d\beta\nabla_{e_i} e_j-\nabla_{e_j} d\beta(e_i)+d\beta\nabla_{e_j} e_i\\
&= e_i(e_j(\beta))-(\nabla_{e_i} e_j)(\beta)-e_j(e_i(\beta))+(\nabla_{e_j} e_i)(\beta)\\
&= [e_i,e_j]\,\beta-\left[\nabla_{e_i} e_j-\nabla_{e_j} e_i\right]\beta\\
&= 0. \qquad (22)
\end{aligned}$$

From (22) we have $\beta_{ij}=\beta_{ji}$. Moreover, we obtain

$$\begin{aligned}
(\nabla_{e_i} d\beta)(e_j) &= \nabla_{e_i} d\beta(e_j)-d\beta(\nabla_{e_i} e_j)\\
&= \nabla_{e_i}\langle grad\beta, e_j\rangle-\langle grad\beta,\nabla_{e_i} e_j\rangle\\
&= \langle \nabla_{e_i} grad\beta, e_j\rangle \qquad (23)
\end{aligned}$$

and

$$(\nabla_{e_j} d\beta)(e_i)=\left\langle \nabla_{e_j} grad\beta, e_i\right\rangle. \qquad (24)$$

Namely

$$\langle \nabla_{e_i} grad\beta, e_j\rangle=\left\langle \nabla_{e_j} grad\beta, e_i\right\rangle. \qquad (25)$$

We will use the well known formula for semi-Riemannian manifolds

$$trace\,\{Y\to(\nabla_Y Q)X\}=\frac{1}{2}\nabla_X\tau.$$

If we put $X=\xi$ in the above formula and use (17) and (9), we have

$$(\nabla_Y Q)\xi=-5\beta Y(\beta)\xi+(-3\beta^3-\beta\frac{\tau}{2})\phi Y-\nabla_Y\phi grad\beta. \qquad (26)$$

Using (26), we get

$$\begin{aligned}\frac{1}{2}\xi(\tau) &= \sum_{i=1}^{3}\varepsilon_i g((\nabla_{e_i}Q)\xi, e_i)\\ &= -g(\nabla_{e_1}\phi grad\beta, e_1) + g(\nabla_{e_2}\phi grad\beta, e_2) - g(\nabla_{e_3}\phi grad\beta, \xi)\end{aligned}$$

where $1 \leq i \leq 3$.

By the help of (5), (14) and (25) we find that

$$\xi(\tau) = \tau_3 = 0. \tag{27}$$

From (14) and (27), we obtain

$$\beta_3 = 0. \tag{28}$$

(19) implies that

$$\beta_{13} = \beta_{31} = -\beta\beta_2,\ \beta_{23} = \beta_{32} = -\beta\beta_1,\ \beta_{33} = 0. \tag{29}$$

Lemma 3.4. *For a three-dimensional quasi-para-Sasakian manifold, the following is valid*

$$\begin{aligned}L_{ij} - L_{ji} &= 0 \quad \textit{for } 1 \leq i, j \leq 3 \Leftrightarrow\\ \tau_1 &= -20\beta\beta_1,\ \tau_2 = -20\beta\beta_2,\ \beta_{12} = \beta_{21} = 0,\\ \beta_{22} &= -\beta_{11} = \beta\left(3\beta^2 + \frac{\tau}{2}\right).\end{aligned} \tag{30}$$

Proof. By direct computations, using (18), (19), (28) and (29), we derive

$$\begin{aligned}L_{12} - L_{21} &= -\left(\frac{\tau_2}{4} + 5\beta\beta_2\right)e_1 + \left(\frac{\tau_1}{4} + 5\beta\beta_1\right)e_2\\ &\quad +(\beta_{11} - \beta_{22} + \beta(\tau + 6\beta^2))\xi.\\ L_{13} - L_{31} &= \beta_{12}e_1 + \left(-\beta_{11} - \beta\left(3\beta^2 + \frac{\tau}{2}\right)\right)e_2\\ &\quad + \left(-\frac{\tau_1}{4} - 5\beta\beta_1\right)\xi.\\ L_{23} - L_{32} &= \left(\beta_{22} - \beta\left(3\beta^2 + \frac{\tau}{2}\right)\right)e_1 - \beta_{12}e_2\\ &\quad + \left(-\frac{\tau_2}{4} - 5\beta\beta_2\right)\xi.\end{aligned} \tag{31}$$

The proof follows from (31). □

We know that a semi-Riemannian manifold is conformally flat$\Leftrightarrow$ $(\nabla_X L)Y - (\nabla_Y L)X = 0$, for any vector fields X and Y. Hence, we can say that a three-dimensional quasi-para-Sasakian manifold is conformally flat if and only if (31) holds. By (31), we can give following result.

Theorem 3.5. *A three-dimensional quasi-para-Sasakian manifold is conformally flat if and only if the function β satisfies the followings*

$$\begin{aligned} \tau + 10\beta^2 &= const., \\ (\nabla_X d\beta)(Y) &= -\beta\left(3\beta^2 + \frac{\tau}{2}\right)(g(X,Y) - \eta(X)\eta(Y)) \\ &\quad -\beta\eta(X)d\beta(\phi Y) - \beta\eta(Y)d\beta(\phi X). \end{aligned} \tag{32}$$

Theorem 3.6. *For a three-dimensional quasi-para-Sasakian manifold M, the following assertions are equivalent to each other:*

i) M is locally symmetric.
ii) M is conformally flat and its scalar curvature τ is const.,
iii) M is conformally flat and β is const.,
iv)• If $\beta = 0$, then M is a paracosymplectic manifold which is locally a product of the real line R and a 2-dimensional para-Kaehlerian manifold or
• If $\beta \neq 0$, then M is of constant negative curvature and the quasi-para-Sasakian structure can be obtained by a homothetic deformation of a para-Sasakian structure.

Proof. First of all, (i) implies (ii) because of the $\dim M = 3$. From (32), one can see $(ii) \Leftrightarrow (iii)$. Now, we will show (iii) implies (iv). Using (32), we get $\beta\left(3\beta^2 + \frac{\tau}{2}\right) = 0$ and τ is const. Now there are two possibilities. If $\beta = 0$, then M is a paracosymplectic manifold which is locally a product of the real line R and a 2-dimensional para-Kaehlerian manifold [6]. If $\beta \neq 0$, then $\tau = -6\beta^2$, namely M has constant negative curvature. By using $\tau = -6\beta^2$ in (13), we get M is Einstein since $S = \frac{\tau}{3}g$. Using Theorem 2.7, one can say that the quasi-para-Sasakian structure can be obtained by a homothetic deformation of a para-Sasakian structure. One can easily deduce that $(iv) \Rightarrow (i)$. □

Theorem 3.7. [21]*(a) The classes of the 3-dimensional normal almost paracontact metric manifolds are G_5, G_6 and $G_5 \oplus G_6$;*
(b) The classes of the 3-dimensional paracontact metric manifolds are $\bar{G}_5$ and $\bar{G}_5 \oplus G_{10}$;
(c) The class of the 3-dimensional para-Sasakian manifolds is $\bar{G}_5$;
(d) The class of the 3-dimensional K-paracontact metric manifolds is $\bar{G}_5$;
(e) The class of the 3-dimensional quasi-para-Sasakian manifolds is G_5 .

Let $\mathcal{L}$ be a three-dimensional real connected Lie group and $\mathfrak{g}$ be its Lie algebra with a basis $\{E_1, E_2, E_3\}$ of left invariant vector fields. An almost paracontact structure (ϕ, ξ, η) and a pseudo-Riemannian metric g defined by

$$\begin{aligned} \phi E_1 &= E_2,\ \phi E_2 = E_1,\ \phi E_3 = 0, \\ \xi &= E_3,\ \eta(E_3) = 1,\ \eta(E_1) = \eta(E_2) = 0, \\ g(E_1, E_1) &= g(E_3, E_3) = -g(E_2, E_2) = 1, \\ g(E_i, E_j) &= 0,\ i \neq j \in \{1, 2, 3\}\,. \end{aligned} \tag{33}$$

Then $(\mathcal{L}, \phi, \xi, \eta, g)$ is a three-dimensional almost paracontact metric manifold. Because of the metric g is left invariant, one can write Koszul equality by following

$$2g(\nabla_x y, z) = g([x, y], z) + g([z, x], y) + g([z, y], x), \tag{34}$$

where ∇ is the Levi-Civita connection of g.

Let the commutators of $\mathfrak{g}$ be defined by $[E_i, E_j] = C_{ij}^k E_k$, where the structure constants C_{ij}^k are real numbers and $C_{ij}^k = -C_{ji}^k$.

Theorem 3.8. [21] *The manifold $(\mathcal{L}, \phi, \xi, \eta, g)$ belongs to the class $G_i (i \in \{5, 6, 10, 12\})$ if and only if the corresponding Lie algebra $\mathfrak{g}$ is determined by the following commutators:*

$$\begin{aligned} G_5 &: [E_1, E_2] = C_{12}^1 E_1 + C_{12}^2 E_2 + C_{12}^3 E_3, \\ &: [E_1, E_3] = C_{13}^2 E_2, \\ [E_2, E_3] &= C_{13}^2 E_1 :\ C_{12}^3 \neq 0,\ C_{12}^1 C_{13}^2 = 0,\ C_{12}^2 C_{13}^2 = 0; \end{aligned} \tag{35}$$

$$\begin{aligned} G_6 &: [E_1, E_2] = C_{12}^1 E_1 + C_{12}^2 E_2, \\ &: [E_1, E_3] = C_{13}^1 E_1 + C_{13}^2 E_2, \\ [E_2, E_3] &= C_{13}^2 E_1 + C_{13}^1 E_2 :\ -2C_{13}^1 \neq 0, \\ C_{12}^2 C_{13}^2 - C_{13}^1 C_{12}^1 &= 0,\ C_{12}^1 C_{13}^2 - C_{13}^1 C_{12}^2 = 0; \end{aligned} \tag{36}$$

$$\begin{aligned} G_{10} &: [E_1, E_2] = C_{12}^1 E_1 + C_{12}^2 E_2, \\ &: [E_1, E_3] = C_{13}^1 E_1 + C_{13}^2 E_2, \\ [E_2, E_3] &= C_{23}^1 E_1 - C_{13}^1 E_2 :\ C_{13}^2 \neq C_{23}^1 \text{ or } C_{13}^1 \neq 0, \\ C_{12}^1 C_{13}^1 + C_{12}^2 C_{23}^1 &= 0,\ C_{12}^1 C_{13}^2 - C_{12}^2 C_{13}^1 = 0; \end{aligned} \tag{37}$$

$$\begin{aligned} G_{12} &: [E_1, E_2] = C_{12}^1 E_1 + C_{12}^2 E_2, \\ &: [E_1, E_3] = C_{13}^2 E_2 + C_{13}^3 E_3, \\ [E_2, E_3] &= C_{13}^2 E_1 + C_{23}^3 E_3 :\ C_{13}^3 \neq 0 \text{ or } C_{23}^3 \neq 0, \\ (C_{12}^1 - C_{23}^3) C_{13}^2 &= 0,\ (C_{12}^2 + C_{13}^3) C_{13}^2 = 0, \\ (C_{12}^1 - C_{23}^3) C_{13}^3 &+ (C_{12}^2 + C_{13}^3) C_{23}^3 = 0 \end{aligned} \tag{38}$$

Theorem 3.9. *A three-dimensional quasi-para-Sasakian manifold is conformally flat if and only if the corresponding Lie algebra* $\mathfrak{g}$ *is determined by the following commutator* G_5

$$G_5 : [E_1, E_2] = C^3_{12}E_3,\ [E_1, E_3] = C^3_{12}E_2,\ [E_2, E_3] = C^3_{12}E_1. \tag{39}$$

Proof. Assume that the three-dimensional quasi-para-Sasakian manifold is conformally flat. Using (33), (34) and (36) we have

$$\begin{aligned}
\nabla_{E_1}E_1 &= C^1_{12}E_2, & \nabla_{E_2}E_1 &= -C^2_{12}E_2 - \frac{1}{2}C^3_{12}E_3,\\
& & \nabla_{E_3}E_1 &= (-C^2_{13} + \frac{1}{2}C^3_{12})E_2,\\
\nabla_{E_1}E_2 &= C^1_{12}E_1 + \frac{1}{2}C^3_{12}E_3, & \nabla_{E_2}E_2 &= -C^1_{12}E_1,\\
& & \nabla_{E_3}E_2 &= (-C^2_{13} + \frac{1}{2}C^3_{12})E_1,\\
\nabla_{E_1}E_3 &= \frac{1}{2}C^3_{12}E_2, & \nabla_{E_2}E_3 &= \frac{1}{2}C^3_{12}E_1,\\
& & \nabla_{E_3}E_3 &= 0.
\end{aligned}$$

From (9), we get $\beta = \frac{1}{2}C^3_{12}$ is a constant function. Using the above covariant derivatives, we obtain

$$\begin{aligned}
R(E_1, E_2)E_3 &= (\frac{1}{2}C^3_{12}C^1_{12} - \frac{1}{2}C^2_{12}C^3_{12})E_1 + (\frac{1}{2}C^3_{12}C^1_{12} - \frac{1}{2}C^1_{12}C^3_{12})E_2,\\
R(E_1, E_2)E_2 &= (-\frac{3}{4}(C^3_{12})^2 - (C^1_{12})^2 + C^2_{12}C^1_{12} + C^3_{12}C^2_{13})E_1\\
&\qquad + (-(C^1_{12})^2 + C^1_{12}C^2_{12})E_2,\\
R(E_1, E_2)E_1 &= (-C^2_{12}C^1_{12} + (C^1_{12})^2)E_1\\
&\qquad + (-\frac{3}{4}(C^3_{12})^2 - (C^1_{12})^2 + (C^2_{12})^2 + C^3_{12}C^2_{13})E_2,\\
R(E_2, E_3)E_3 &= (-\frac{1}{4}(C^3_{12})^2)E_2,\\
R(E_2, E_3)E_2 &= (-C^2_{13}C^1_{12})E_1\\
&\qquad + (C^2_{13}C^2_{12} - \frac{1}{2}C^3_{12}C^2_{12} - C^1_{12}C^2_{13} + \frac{1}{2}C^1_{12}C^3_{12})E_2\\
&\qquad + \left(-\frac{1}{4}(C^3_{12})^2E_3\right),\\
R(E_2, E_3)E_1 &= C^2_{13}(C^1_{12} - C^2_{12})E_1 + \frac{1}{2}C^3_{12}(C^2_{12} - C^1_{12})E_1 - C^2_{13}C^1_{12}E_2,
\end{aligned}$$

$$\begin{aligned} R(E_1,E_3)E_3 &= -\frac{1}{4}(C_{12}^3)^2E_1,\\ R(E_1,E_3)E_2 &= C_{13}^2C_{12}^1E_1,\\ R(E_1,E_3)E_1 &= (C_{13}^2C_{12}^2)E_2+\frac{1}{4}(C_{12}^3)^2E_3. \end{aligned}$$

Using above equations, we have constant scalar curvature as follows,

$$\begin{aligned} (40)\quad S(E_1,E_1) &= \frac{1}{2}(C_{12}^3)^2+(C_{12}^1)^2-C_{12}^2C_{12}^1-C_{12}^3C_{13}^2,\\ (41)\quad S(E_2,E_2) &= -\frac{1}{2}(C_{12}^3)^2-(C_{12}^1)^2+(C_{12}^2)^2+C_{12}^3C_{13}^2,\\ (42)\quad S(E_3,E_3) &= -\frac{1}{2}(C_{12}^3)^2,\\ (43)\quad \tau &= S(E_1,E_1)-S(E_2,E_2)+S(E_3,E_3),\\ (44)\quad \tau &= \frac{1}{2}(C_{12}^3)^2+2(C_{12}^1)^2-(C_{12}^2)^2-2C_{12}^3C_{13}^2-C_{12}^2C_{12}^1. \end{aligned}$$

Using the fact that our manifold is conformally flat, if we use Theorem 3.5 in (13) and by the equations (40), (41), we have

$$\begin{aligned} (45)\quad -\frac{1}{2}(C_{12}^3)^2 &= \frac{1}{2}(C_{12}^3)^2+(C_{12}^1)^2-C_{12}^2C_{12}^1-C_{12}^3C_{13}^2.\\ (46)\quad \frac{1}{2}(C_{12}^3)^2 &= -\frac{1}{2}(C_{12}^3)^2-(C_{12}^1)^2+(C_{12}^2)^2+C_{12}^3C_{13}^2. \end{aligned}$$

If we sum (45) and (46), we have

$$C_{12}^2(C_{12}^2-C_{12}^1)=0.$$

By virtue of the last equation, following cases occurs.

Case I: Accept $C_{12}^2=0$. If we subtract (45) from (46) and use $C_{12}^2=0$, we obtain

$$(47)\qquad 2(C_{12}^3)^2=-2(C_{12}^1)^2+2C_{12}^3C_{13}^2.$$

Taking into account $\tau=-6\beta^2=-\frac{3}{2}(C_{12}^3)^2$ in (44), we get

$$(48)\qquad C_{12}^3(C_{13}^2-C_{12}^3)=(C_{12}^1)^2.$$

If we act C_{13}^2 on both sides of the last equation and using the fact that $C_{12}^3\neq 0$ and $C_{12}^1C_{13}^2=0$ in G_5, we obtain

$$C_{13}^2=0 \text{ or } C_{13}^2=C_{12}^3.$$

If we take $C_{13}^2=0$, by (48) we get $-(C_{12}^3)^2=(C_{12}^1)^2$. So C_{13}^2 shold be different from zero. If we take $C_{13}^2=C_{12}^3$ in (47), we get $C_{12}^1=0$.

Case II: Assume $C_{12}^2=C_{12}^1$. By virtue of (45), we get

$$(49)\qquad C_{12}^3(C_{13}^2-C_{12}^3)=0.$$

From (49), there are two possibilities. The first one is $C_{12}^3 = 0$. But this contradicts with the $C_{12}^3 \neq 0$ in G_5. So the second one is $C_{13}^2 = C_{12}^3$. If we use $C_{13}^2 = C_{12}^3$ in the fact that $C_{12}^3 \neq 0$, $C_{12}^1 C_{13}^2 = 0$, $C_{12}^2 C_{13}^2 = 0$ in G_5, we obtain $C_{12}^1 = C_{12}^2 = 0$.

Namely, by Case I and Case II, we get $C_{12}^1 = C_{12}^2 = 0$, $C_{13}^2 = C_{12}^3$ which gives us (39). The proof of converse side is obvious. □

Remark 3.10. *Using Theorem 3.9, one can construct several examples of three-dimensional conformally flat quasi-para-Sasakian manifolds. For example,*

•*For $\beta = 2$, using the commutators $[E_1, E_2] = 4E_3$, $[E_1, E_3] = 4E_2$, $[E_2, E_3] = 4E_1$, one can get a 3-dimensional conformally flat proper quasi-para-Sasakian manifold with $\tau = -24$ which is neither the para-cosymplectic manifold nor the para-Sasakian manifold.*

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