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QUASI AND BI IDEALS IN LEFT ALMOST RINGS

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Abstract. The aim of this paper is to extend the concept of quasi and bi-ideals from left almost semigroups to left almost rings which are the generalization of one sided ideals. Further, we discuss quasi and bi-ideals in regular left almost rings and intra regular left almost rings. We then explore many interesting and elegant properties of quasi and bi-ideals.

1. Introduction

In ternary operations the commutative law is given by $abc = cba$. In 1972, M. A. Kazim and M. Naseeruddin [6] introduced braces on the left of this equation to get a new pseudo associative law, that is $(ab)c = (cb)a$. It is called left invertive law. A groupoid is called left almost semigroup, abbreviated as LA-semigroup, if its elements satisfy the left invertive law. It corresponds to a semigroup and is basically the generalization of a commutative semigroup. In [10], LA-semigroup is also known as an Abel-Grassmann's groupiod (AG-groupiod) after the name of Abel-Grassmann. An LA-semigroup is an algebraic structure midway between a groupoid and a commutative semigroup. Despite the fact that the structure is non-associative and non-commutative, it nevertheless possesses many interesting properties which we usually find in commutative and associative algebraic structures. In 1993, M. S. Kamran [8] extended the notion of an LA-semigroup to a left almost group, abbreviated as LA-group. LA-group corresponds to a group. It is a non-associative structure and the generalization of a commutative group. In

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[8], the author proved some most useful and elegant results about LA-groups. Particularly the author discussed substructures of an LA-group and then quotient structures. In 2006, S. M. Yusuf extended [12] the notion of an LA-group to a non-associative structure called left almost ring, abbreviated as LA-ring. An LA-ring basically corresponds to a ring. Further different peoples in [1], [2], [3], [4], [9], [13] and [14] worked on LA-rings and explored many interesting properties of LA-rings. The concept of quasi and bi-ideals in an LA-semigroup was introduced by M. Khan, V. Amjad and Faisal in [7] which is the generalization of one sided ideals. In this study, we extend the notion of quasi and bi-ideals from LA-semigroups to LA-rings. Further, we explore some interesting and elegant properties of quasi and bi-ideals in different type of LA-rings.

2. Preliminaries

In this section, we give some definitions and results which will be used in later sections. These definitions and results have been taken from the sources: [1], [5], [6], [8], [11] and [12]. Firstly we are going to define LA-semigroups.

Definition 2.1. *A groupoid* $(\boldsymbol{S}, *)$ *is called a left almost semigroup; abbreviated as an LA- semigroup; if it satisfies left invertive law, i.e.* $(x * y) * z = (z * y) * x$ *for all* $x, y, z \in \boldsymbol{S}$.

To understand the above concept, we give an example. The following example has been taken from the paper [11].

Example 2.2. *Let* $\boldsymbol{Z}$ *denotes the set of integers. Let the binary operation '*$*$*' in* $\boldsymbol{Z}$ *is defined in the following manner:*

$l * m = m - l$ *for all* $l, m \in \boldsymbol{Z}$,

where '$-$*' denotes the ordinary subtraction. Then* $(\boldsymbol{Z}, *)$ *is an LA-semigroup.*

Let us present some properties of LA-semigroups which have been taken from [5] and will be used later.

Lemma 2.3. *Let* $(\boldsymbol{S}, *)$ *be an LA-semigroup. Then the following law holds.*

$(x * y) * (z * w) = (x * z) * (y * w)$ *for all* $x, y, z, w \in \boldsymbol{S}$.

The above law is called medial law.

Lemma 2.4. *Let ($\boldsymbol{S}$, $*$) be an LA-semigroup with left identity 'e'. Then the following law holds.*

$(x * y) * (z * w) = (w * y) * (z * x)$ *for all* $x, y, z, w \in \boldsymbol{S}$.

The above law is called paramedial law.

Lemma 2.5. *If* ($\boldsymbol{S}$, $*$) *is an LA-semigroup with left identity 'e'; then,* $x * (y * z) = y * (x * z)$ *for all* $x, y, z \in \boldsymbol{S}$.

In [8], Kamran extended the concept of an LA-semigroup to a left almost group (LA-group) which is a non-associative structure. We are now giving a proper definition of an LA-group.

Definition 2.6. *A groupoid* $\boldsymbol{G}$ *with the binary operation '$*$' is said to be an LA-group if the following conditions are satisfied:*

(i) There exists an element $e \in \boldsymbol{G}$ *such that* $e * a = a \; \forall \; a \in \boldsymbol{G}$,

(ii) For $a \in \boldsymbol{G}$ *there exists* $a^{-1} \in \boldsymbol{G}$ *such that* $a^{-1} * a = a * a^{-1} = e$, *i.e. left inverse of each element of* $\boldsymbol{G}$ *exits in* $\boldsymbol{G}$,

(iii) Left invertive law holds in $\boldsymbol{G}$.

It follows from the definition that Lemma 2.3, Lemma 2.4 and Lemma 2.5 are true for LA-groups as well. To understand the above notion of LA-groups, we give an example. The following example has been taken from the thesis [8].

Example 2.7. *Let* $\boldsymbol{G} = \{l, m, n, w, q\}$ *and '$*$' be the binary operation defined in the following table:*

$\cdot$	l	m	n	w	q
l	l	m	n	w	q
m	q	l	m	n	w
n	w	q	l	m	n
w	n	w	q	l	m
q	m	n	w	q	l

The operation '$*$' defined in the above table is non-associative, i.e. $(l* m)*n \neq l* (m*n)$. But $\boldsymbol{G}$ satisfies all the properties of an LA-group. It can be easily seen that l is the left identity and each element is the left inverse of itself. Thus $\boldsymbol{G}$ is an LA-group.

We are now going to discuss substructure of an LA-group $\boldsymbol{G}$. The following definition has been taken from the source [8].

Definition 2.8. *Let* $\boldsymbol{G}$ *be an LA-group and let* $\emptyset \neq \boldsymbol{H} \subseteq \boldsymbol{G}$. *Then* $\boldsymbol{H}$ *is called an LA-subgroup of* $\boldsymbol{G}$ *if* $\boldsymbol{H}$ *itself is an LA-group under the same binary operation as defined in* $\boldsymbol{G}$. *If* $\boldsymbol{H}$ *is an LA-subgroup of* $\boldsymbol{G}$, *then we write* $\boldsymbol{H} \leq \boldsymbol{G}$.

Let us describes some properties which have been taken from [8]. The following result gives us equivalent conditions for LA-subgroups.

Theorem 2.9. *Let* $\boldsymbol{G}$ *be an LA-group and let* $\emptyset \neq \boldsymbol{H} \subseteq \boldsymbol{G}$. *Then* $\boldsymbol{H}$ *is an LA-subgroup of* $\boldsymbol{G}$ *if and only if* $ab^{-1} \in \boldsymbol{H}$ *for all* $a, b \in \boldsymbol{H}$.

Theorem 2.10. *Intersection of any family of LA-subgroups of an LA-group is again an LA-subgroup.*

In [12], S. M. Yusuf extended the notion of an LA-group to a left almost ring (LA-ring), the non-associative structure with two binary operations '+' and '·'. We are now giving a proper definition of LA-rings.

Definition 2.11. *A left almost ring is a non-empty set* $\boldsymbol{R}$ *together with two binary operations '+' and '·' satisfying the following:*

(i) $(\boldsymbol{R}, +)$ *is an LA-group,*

(ii) $(\boldsymbol{R}, \cdot)$ *is an LA-semigroup,*

(iii) Both left and right distributive laws hold. That is for all $l, m, n \in \boldsymbol{R}$

$l\cdot(m + n) = l\cdot m + l\cdot n$ and $(l + m)\cdot n = l\cdot n + m\cdot n$.

It should be noted that Lemma 2.3, Lemma 2.4 and Lemma 2.5 are true for the binary operation '+' of an LA-ring $\boldsymbol{R}$. Lemma 2.3, Lemma 2.4 and Lemma 2.5 are also for the binary '·' of an LA-ring $\boldsymbol{R}$ if $(\boldsymbol{R}, \cdot)$ contains left identity, i.e. it contains an element $e \in \boldsymbol{R}$ such that $e * a = a \ \forall \ a \in \boldsymbol{R}$.

To understand LA-rings we give an example. The following examples have been taken from the source [12] and [9] respectively.

Example 2.12. *Let* $(\boldsymbol{R}, +, \cdot)$ *be a commutative ring, then we can always get an LA-ring* $(\boldsymbol{R}, \oplus, \cdot)$ *by defining for* $m, n \in \boldsymbol{R}$, $m \oplus n = n - m$ *and* $m \cdot n$ *is the same as in the ring* $(\boldsymbol{R}, +, \cdot)$.

Example 2.13. *Let* $\boldsymbol{R} = \{0, 1, 2, 3, 4\}$ *and the binary operations of '+' and '·' are defined in the following table:*

+	0	1	2	3	4
0	0	1	2	3	4
1	4	0	1	2	3
2	3	4	0	1	2
3	2	3	4	0	1
4	1	2	3	3	4

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Then one can easily verified that ($\boldsymbol{R}$, $+$, $\cdot$) *is an LA-ring.*

We are now going to discuss substructures of an LA-ring $\boldsymbol{R}$. Firstly we are going to define its substructure which is called an LA-subring. The following definition has been taken from the paper [12].

Definition 2.14. *Let* ($\boldsymbol{R}$, $+$, $\cdot$) *be an LA-ring. If* $\boldsymbol{B}$ *is a non-empty subset of* $\boldsymbol{R}$ *and* $\boldsymbol{B}$ *is itself an LA-ring under the same binary operation as defined in* $\boldsymbol{R}$*, then* $\boldsymbol{B}$ *is called an LA-subring of* $\boldsymbol{R}$.

Let us describe some properties which have been taken from [12]. The following result gives us equivalent conditions for LA-subrings.

Lemma 2.15. *If* $\boldsymbol{B}$ *is non-empty subset of an LA-ring* ($\boldsymbol{R}$, $+$, $\cdot$)*, then* $\boldsymbol{B}$ *is an LA-subring of* $\boldsymbol{R}$ *if and only if* $a - b$, $a \cdot b \in \boldsymbol{B}$ *for all* a, $b \in \boldsymbol{B}$.

Theorem 2.16. *The intersection of any family of LA-subrings of an LA-ring* $\boldsymbol{R}$ *is again an LA-subring.*

It follows from the above theorem that if $\boldsymbol{A}$ and $\boldsymbol{B}$ are two LA-subrings of an LA-ring $\boldsymbol{R}$, then the intersection of $\boldsymbol{A}$ and $\boldsymbol{B}$ is again an LA-subring of $\boldsymbol{R}$. We are now going to define the second substructure of an LA-ring $\boldsymbol{R}$ which is called an ideal. The following definition has been taken from the source [12].

Definition 2.17. *Let* ($\boldsymbol{R}$, $+$, $\cdot$) *be an LA-ring and* $\boldsymbol{I}$ *an LA-subring of* $\boldsymbol{R}$*. Then* $\boldsymbol{I}$ *is said to be a left ideal of* $\boldsymbol{R}$ *if* $\boldsymbol{RI} \subseteq \boldsymbol{I}$ *and* $\boldsymbol{I}$ *is called a right ideal of* $\boldsymbol{R}$ *if* $\boldsymbol{IR} \subseteq \boldsymbol{I}$. $\boldsymbol{I}$ *is said to be a two sided ideal or simply an ideal of* $\boldsymbol{R}$ *if it is both left and right ideal of* $\boldsymbol{R}$.

Let us present some properties which have been taken from [1].

Theorem 2.18. *Let* ($\boldsymbol{R}$, $+$, $\cdot$) *be an LA-ring with left identity 'e', then every right ideal is a left ideal.*

Theorem 2.19. *Intersection of two left(right) ideals of an LA-ring is again a left(right) ideal.*

Corollary 2.20. *The intersection of any family of left(right) ideals of an LA-ring is a left(right) ideal.*

Proof. The proof follows from the above theorem by induction. □

We are now going to define sum of two ideals of an LA-ring. The following definition has been taken from [1].

Definition 2.21. *Let* $(\boldsymbol{R}, +, \cdot)$ *be an LA-ring. Let* $\boldsymbol{I}$, $\boldsymbol{J}$ *be ideals of* $\boldsymbol{R}$*. The sum of* $\boldsymbol{I}$ and $\boldsymbol{J}$ *is defined as:*
$\boldsymbol{I} + \boldsymbol{J} = \{i + j : i \in \boldsymbol{I} \text{ and } j \in \boldsymbol{J}\}$.

It follows from the above definition that $\boldsymbol{I} + \boldsymbol{J} \subseteq \boldsymbol{R}$. Let us describe some properties, which have been taken from [1].

Theorem 2.22. *Let* $(\boldsymbol{R}, +, \cdot)$ *be an LA-ring. Then sum of two left(right) ideals of* $\boldsymbol{R}$ *is again a left (right) ideal of* $\boldsymbol{R}$.

Corollary 2.23. *The sum of one left and one right ideal of an LA-ring with left identity 'e' is a left ideal.*

Proof. Follow from Theorem 2.18 and above theorem. □

We are now going to define product of two ideals of an LA-ring. The definition has been taken from [1].

Definition 2.24. *Let* $(\boldsymbol{R}, +, \cdot)$ *be an LA-ring and* $\boldsymbol{I}$ *and* $\boldsymbol{J}$ *be two ideals of* $\boldsymbol{R}$*. Then the product of* $\boldsymbol{I}$ and $\boldsymbol{J}$ *is denoted by* $\boldsymbol{IJ}$ *and is defined as:*
$\boldsymbol{IJ} = \{\sum_{i=1}^{n} r_i\ s_i\colon\ r_i \in \boldsymbol{I} \text{ and } s_i \in \boldsymbol{J}\} = \{(...(((r_1 s_1 + r_2 s_2) + r_3 s_3) + ... + r_{n-1} s_{n-1}) + r_n s_n)\colon r_i \in \boldsymbol{I} \text{ and } s_i \in \boldsymbol{J}\}$.

Let us describe some properties which have been taken from the source [1].

Theorem 2.25. *Let* $(\boldsymbol{R}, +, \cdot)$ *be an LA-ring with left identity 'e'. Then the product of two left(right) ideal is again a left(right) ideal of* $\boldsymbol{R}$.

The following result is a direct consequence of the above theorem.

Corollary 2.26. *If* $\boldsymbol{I}$ *is a right ideal of an LA-ring* $\boldsymbol{R}$ *with left identity 'e' then* $\boldsymbol{I}^2$ *is an ideal of* $\boldsymbol{R}$.

3. Quasi and Bi-Ideals

Corresponding to quasi and bi-ideals of rings, in this section, we define quasi and bi-ideals in LA-rings. We give some properties of quasi and bi-ideals. We show that under some given condition every quasi-ideal is a bi-ideal.

Definition 3.1. *Let* $(\boldsymbol{R}, +, \cdot)$ *be an LA-ring. A non-empty subset* $\boldsymbol{Q}$ *of* $\boldsymbol{R}$ *is said to be a quasi-ideal of* $\boldsymbol{R}$*, if* $(\boldsymbol{Q}, +)$ *is an LA-subgroup of* $(\boldsymbol{R}, +)$ *such that* $\boldsymbol{RQ} \cap \boldsymbol{QR} \subseteq \boldsymbol{Q}$.

It is clear that every one-sided ideal of an LA-ring $(\boldsymbol{R}, +, \cdot)$ is a quasi-ideal of $\boldsymbol{R}$. To understand quasi ideals, we give an example.

Example 3.2. *Let* $\boldsymbol{R} = \{0, 1, 2, 3\}$ *and the binary operation of addition and multiplication are defined in the following table:*

+	0	1	2	3
0	0	1	2	3
1	3	0	1	2
2	2	3	0	1
3	1	2	3	0

·	0	1	2	3
0	0	0	0	0
1	0	0	3	3
2	0	2	2	0
3	0	2	1	3

Then one can easily verified that $(\boldsymbol{R}, +, \cdot)$ *is an LA-ring. Let* $\boldsymbol{Q} = \{0, 1\}$*. Now* $\boldsymbol{RQ} = \{0, 2\}$ *and* $\boldsymbol{QR} = \{0, 3\}$*.* $\boldsymbol{Q}$ *is a quasi-ideal of* $\boldsymbol{R}$ *because* $\boldsymbol{RQ} \cap \boldsymbol{QR} = \{0, 2\} \cap \{0, 3\} = \{0\} \subseteq \boldsymbol{Q}$*, i.e.* $\boldsymbol{RQ} \cap \boldsymbol{QR} \subseteq \boldsymbol{Q}$.

Let us state and prove some properties of quasi-ideals. The following results are true in case of rings. Here we prove them for LA-rings.

Proposition 3.3. *Each quasi-ideal of an LA-ring* $(\boldsymbol{R}, +, \cdot)$ *is an LA-subring of* $(\boldsymbol{R}, +, \cdot)$.

Proof. Let $\boldsymbol{Q}$ be a quasi-ideal of an LA-ring $(\boldsymbol{R}, +, \cdot)$, then by definition $(\boldsymbol{Q}, +)$ is an LA-subgroup of $(\boldsymbol{R}, +)$. Now
$\boldsymbol{Q}^2 = \boldsymbol{QQ} \subseteq \boldsymbol{RQ}$, i.e. $\boldsymbol{Q}^2 \subseteq \boldsymbol{RQ}$
and
$\boldsymbol{Q}^2 = \boldsymbol{QQ} \subseteq \boldsymbol{QR}$, i.e. $\boldsymbol{Q}^2 \subseteq \boldsymbol{QR}$.

So,
$\boldsymbol{Q}^2 \subseteq \boldsymbol{RQ} \cap \boldsymbol{QR} \subseteq \boldsymbol{Q}$, i.e. $\boldsymbol{Q}^2 \subseteq \boldsymbol{Q}$.
Thus $\boldsymbol{Q}$ is an LA-subring of $(\boldsymbol{R}, +, \cdot)$. □

Proposition 3.4. *The intersection of any family of quasi-ideals of an LA-ring $(\boldsymbol{R}, +, \cdot)$ is a quasi-ideal of $(\boldsymbol{R}, +, \cdot)$.*

Proof. Let $\{\boldsymbol{Q}_i : i \in \Omega\}$ be a family of quasi-ideals of an LA-ring $(\boldsymbol{R}, +, \cdot)$. Then clearly $\bigcap_{i\in\Omega} Q_i$ is an LA-subgroup of $(\boldsymbol{R}, +)$. Now
$\boldsymbol{R}(\bigcap_{i\in\Omega} Q_i) \cap (\bigcap_{i\in\Omega} Q_i)\boldsymbol{R} \subseteq \boldsymbol{RQ}_i \cap \boldsymbol{Q}_i\boldsymbol{R} \subseteq \boldsymbol{Q}_i \ \forall\, i \in \Omega$.
This gives
$\boldsymbol{R}(\bigcap_{i\in\Omega} Q_i) \cap (\bigcap_{i\in\Omega} Q_i)\boldsymbol{R} \subseteq \bigcap_{i\in\Omega} Q_i$.
Thus $\bigcap_{i\in\Omega} Q_i$ is a quasi-ideal of $(\boldsymbol{R}, +, \cdot)$. □

We are now going to state and prove a result which is based on the above theorem.

Corollary 3.5. *The intersection of a right ideal $\boldsymbol{I}$ and a left ideal $\boldsymbol{J}$ of an LA-ring $(\boldsymbol{R}, +, \cdot)$ is a quasi ideal of $\boldsymbol{R}$.*

Proof. The right ideal $\boldsymbol{I}$ and the left ideal $\boldsymbol{J}$ of $(\boldsymbol{R}, +, \cdot)$ being one-sided ideals are quasi-ideal of $(\boldsymbol{R}, +, \cdot)$. Thus by the above proposition, $\boldsymbol{I} \cap \boldsymbol{J}$ is a quasi ideal of $\boldsymbol{R}$. □

We are now going to define bi-ideals in LA-rings.

Definition 3.6. *Let $(\boldsymbol{R}, +, \cdot)$ be an LA-ring and $\boldsymbol{B}$ an LA-subring of $\boldsymbol{R}$, then $\boldsymbol{B}$ is called a bi-ideal of $\boldsymbol{R}$ if $(\boldsymbol{BR})\boldsymbol{B} \subseteq \boldsymbol{B}$.*

It is easy to see that every one sided ideal is a bi-ideal. Let us state and prove some properties of bi-ideals.

Theorem 3.7. *Let $(\boldsymbol{R}, +, \cdot)$ be an LA-ring with left identity 'e' such that $(xe)\boldsymbol{R} = x\boldsymbol{R}$ for all $x \in \boldsymbol{R}$, then every quasi-ideal of $\boldsymbol{R}$ is a bi-ideal of $\boldsymbol{R}$.*

Proof. Let $\boldsymbol{Q}$ be a quasi-ideal of $\boldsymbol{R}$. Then by Proposition 3.3, $\boldsymbol{Q}$ is an LA-subring of $\boldsymbol{R}$. Now $(\boldsymbol{QR})\boldsymbol{Q} \subseteq \boldsymbol{RQ}$ and $(\boldsymbol{QR})\boldsymbol{Q} \subseteq (\boldsymbol{QR})\boldsymbol{R} =$
$(\boldsymbol{QR})(e\boldsymbol{R}) = (\boldsymbol{Q}e)(\boldsymbol{RR})$ $\because$ by medial law
$= (\boldsymbol{Q}e)\boldsymbol{R} = \boldsymbol{QR}$ $\because (xe)\boldsymbol{R} = x\boldsymbol{R}$
Hence it follows that $(\boldsymbol{QR})\boldsymbol{Q} \subseteq \boldsymbol{QR} \cap \boldsymbol{RQ} \Longrightarrow (\boldsymbol{QR})\boldsymbol{Q} \subseteq \boldsymbol{Q}$. □

The following result is true in case of rings. Here we prove it for LA-rings.

Theorem 3.8. *The intersection of any family of bi-ideals of an LA-ring $(\boldsymbol{R}, +, \cdot)$ is a bi-ideal of $\boldsymbol{R}$.*

Proof. Let $\{\boldsymbol{B}_i\colon\ i \in \Omega\}$ be a family of bi-ideals of the LA-ring $\boldsymbol{R}$. Then $\boldsymbol{B} = \bigcap_{i\in\Omega} B_i$ being the intersection of LA-subrings of $\boldsymbol{R}$ is also an LA-subring of $\boldsymbol{R}$. Now $(\boldsymbol{B}_i\boldsymbol{R})\boldsymbol{B}_i \subseteq \boldsymbol{B}_i$ for all $i \in \Omega$. Also $\boldsymbol{B} \subseteq \boldsymbol{B}_i$ for all $i \in \Omega$. Therefore, $(\boldsymbol{BR})\boldsymbol{B} \subseteq (\boldsymbol{B}_i\boldsymbol{R})\boldsymbol{B}_i \subseteq \boldsymbol{B}_i$ for all $i \in \Omega$. Thus $(\boldsymbol{BR})\boldsymbol{B} \subseteq \boldsymbol{B}_i$ for all $i \in \Omega \Longrightarrow (\boldsymbol{BR})\boldsymbol{B} \subseteq \bigcap_{i\in\Omega} B_i = \boldsymbol{B}$. □

It follows from the above result that intersection of a right ideal $\boldsymbol{I}$ and a left ideal $\boldsymbol{J}$ of an LA-ring $\boldsymbol{R}$ is a bi-ideal of $\boldsymbol{R}$.

4. Regular and Intra-Regular LA-Rings

Corresponding to regular and intra-regular rings in this section, we define regular and intra-regular LA-rings. Firstly we are going to define regular LA-rings.

Definition 4.1. *Let $(\boldsymbol{R}, +, \cdot)$ be an LA-ring and let c be an element of $\boldsymbol{R}$, then c is called a regular element with respect to the binary operation '$\cdot$' of $\boldsymbol{R}$, if and only if $(cu)c = c$ for some $u \in \boldsymbol{R}$.*

If every element of an LA-ring $\boldsymbol{R}$ is regular with respect to the binary operation '$\cdot$', then the LA-ring $\boldsymbol{R}$ is called regular with respect to the binary operation '$\cdot$'.

Note that, we may define regular element and regular LA-ring with respect to the binary operation '+' as well. Let us describe some properties. The following results are true in case of rings. Here we prove them for LA-rings.

Theorem 4.2. *Let $(\boldsymbol{R}, +, \cdot)$ be a regular LA-ring with respect to the binary operation '$\cdot$' with left identity 'e', then $\boldsymbol{I} \cap \boldsymbol{B} = (\boldsymbol{BI})\boldsymbol{B}$ for every ideal $\boldsymbol{I}$ of $\boldsymbol{R}$ and every bi-ideal $\boldsymbol{B}$ of $\boldsymbol{R}$.*

Proof. Given that $(\boldsymbol{R}, +, \cdot)$ is a regular LA-ring with respect to the binary operation '$\cdot$' with left identity 'e' and $\boldsymbol{I}$ an ideal of $\boldsymbol{R}$ and $\boldsymbol{B}$ a bi-ideal of $\boldsymbol{R}$. Now

$(\boldsymbol{BI})\boldsymbol{B} \subseteq \boldsymbol{IB} \subseteq \boldsymbol{I}$

and

$(\boldsymbol{BI})\boldsymbol{B} \subseteq (\boldsymbol{BR})\boldsymbol{B} \subseteq \boldsymbol{B}$.

Thus, $(\boldsymbol{BI})\boldsymbol{B} \subseteq \boldsymbol{I} \cap \boldsymbol{B}$.

Conversely let $u \in \boldsymbol{I} \cap \boldsymbol{B}$, then $u = (uv)u$ for some $v \in \boldsymbol{R}$. Now

$u = (uv)u = (((uv)u)v)u = ((vu)(uv))u = (u((vu)v))u \in (\boldsymbol{BI})\boldsymbol{B}$.

Thus, $\boldsymbol{I} \cap \boldsymbol{B} \subseteq (\boldsymbol{BI})\boldsymbol{B}$ and so $\boldsymbol{I} \cap \boldsymbol{B} = (\boldsymbol{BI})\boldsymbol{B}$. □

Theorem 4.3. *Let ($\boldsymbol{R}$, $+$, $\cdot$) be a regular LA-ring with respect to the binary operation '$\cdot$', then $\boldsymbol{IJ} = \boldsymbol{I} \cap \boldsymbol{J}$ for every right ideal $\boldsymbol{I}$ and left ideal $\boldsymbol{J}$ of $\boldsymbol{R}$.*

Proof. Given that $\boldsymbol{I}$ is a right ideal and $\boldsymbol{J}$ a left ideal of ($\boldsymbol{R}$, $+$, $\cdot$). Then obviously $\boldsymbol{IJ} \subseteq \boldsymbol{I} \cap \boldsymbol{J}$. Now let $u \in \boldsymbol{I} \cap \boldsymbol{J}$, then $u \in \boldsymbol{I}$ and $u \in \boldsymbol{J}$. As $\boldsymbol{R}$ is regular with respect to the binary operation '$\cdot$', so there exists $v \in \boldsymbol{R}$ such that $u = (uv)u$ and $(uv)u \in \boldsymbol{IJ}$.

It follows that $\boldsymbol{I} \cap \boldsymbol{J} \subseteq \boldsymbol{IJ}$. This completes the proof. □

We are now going to define intra-regular LA-rings with respect to the binary operation '$\cdot$'.

Definition 4.4. *Let ($\boldsymbol{R}$, $+$, $\cdot$) be an LA-ring, then an element a of $\boldsymbol{R}$ is called an intra-regular with respect to the binary operation '$\cdot$' if there exists u, $v \in \boldsymbol{R}$ such that $a = (ua^2)v$.*

If every element of $\boldsymbol{R}$ is intra-regular with respect to the binary operation '$\cdot$', then the LA-ring $\boldsymbol{R}$ is called intra-regular with respect to the binary operation '$\cdot$' .

Note that, we may define intra-regular element and intra-regular LA-ring with respect to the binary operation '$+$' as well. To understand the above notion we give an example.

Example 4.5. *Let $\boldsymbol{R} = \{v, w, x, y, z\}$. Define addition and multiplication in the following tables:*

$+$	v	w	x	y	z
v	v	w	x	y	z
w	z	v	w	x	y
x	y	z	v	w	x
y	x	y	z	v	w
z	w	x	y	z	v

$\cdot$	v	w	x	y	z
v	v	v	v	v	v
w	v	w	x	y	z
x	v	z	w	x	y
y	v	y	z	w	x
z	v	x	y	z	w

Then ($\boldsymbol{R}$, $+$, $\cdot$) is an LA-ring. Now from the above table $v = (vv^2)v$, $w = (xw^2)z$, $x = (yx^2)z$, $y = (xy^2)x$ and $z = (wz^2)z$, so it follows that ($\boldsymbol{R}$, $+$, $\cdot$) is an intra-regular LA-ring with respect to the binary operation '$\cdot$'.

Let us state and prove some properties of intra-regular LA-rings with respect to the binary operation '$\cdot$'. The idea of these properties have came from the paper [7] in which the authors do similar calculations for LA-semigroups. We extend these properties to LA-rings.

Theorem 4.6. *If $(\boldsymbol{R}, +, \cdot)$ is an intra-regular LA-ring with respect to the binary operation '$\cdot$' with left identity 'e', then $(\boldsymbol{BR})\boldsymbol{B} = \boldsymbol{B} \cap \boldsymbol{R}$, for every bi-ideal $\boldsymbol{B}$ of $(\boldsymbol{R}, +, \cdot)$.*

Proof. See the proof of Theorem 3 in [7]. □

We are now going to state a result which is based on the above theorem.

Corollary 4.7. *If $(\boldsymbol{R}, +, \cdot)$ is an intra-regular LA-ring with respect to the binary operation '$\cdot$' with left identity 'e', then $(\boldsymbol{BR})\boldsymbol{B} = \boldsymbol{B}$ for every bi-ideal $\boldsymbol{B}$ of $(\boldsymbol{R}, +, \cdot)$.*

Proof. Straight forward. □

The following result gives us equivalent conditions for bi-ideals in intra-regular LA-rings with respect to the binary operation '$\cdot$'.

Theorem 4.8. *If $(\boldsymbol{R}, +, \cdot)$ is an intra-regular LA-ring with respect to the binary operation '$\cdot$' with left identity 'e' and $\boldsymbol{B}$ a non-empty subset of $\boldsymbol{R}$, then the following conditions are equivalent:*
(i) $\boldsymbol{B}$ is a bi-ideal of $\boldsymbol{R}$,
(ii) $(\boldsymbol{BR})\boldsymbol{B} = \boldsymbol{B}$ and $\boldsymbol{B}^2 = \boldsymbol{B}$.

Proof. See the proof of Theorem 8 in [7]. □

The following result gives us equivalent conditions for quasi-ideals in intra-regular LA-rings with respect to the binary operation '$\cdot$' with left identity 'e'.

Theorem 4.9. *Let $(\boldsymbol{R}, +, \cdot)$ be an intra-regular LA-ring with respect to the binary operation '$\cdot$' with left identity 'e' and $\boldsymbol{Q}$ a non-empty subset of $\boldsymbol{R}$, then the following conditions are equivalent:*
(i) $\boldsymbol{Q}$ is a quasi-ideal of $\boldsymbol{R}$,
(ii) $\boldsymbol{RQ} \cap \boldsymbol{QR} = \boldsymbol{Q}$.

Proof. See the proof of Theorem 9 in [7]. □

References

[1] I. Rehman, *On Generalized Commutative Ring and Related Structures*, PhD. Thesis., Quaid-i-Azam University, Islamabad, 2012.

[2] I. Rehman, M. Shah, T. Shah and A. Razzaque, *On Existence of Nonassociative LA-Ring.*, An. St. Univ. Ovidius Constanta. **21(3)** (2013), 223-228.

[3] F. Hussain and S. Firdous, *Direct Product of Left Almost Rings.*, International Journal of Sciences: Basic and Applied Research. **5(2)** (2016), 113-127.

[4] F. Hussain and W. Khan, *Congruences on Left Almost Rings.*, International Journal of Algebra and Statistics. **4(1)** (2015), 1-6.

[5] J. R. Cho, Pusan, J. Jezek, T. Kepka and Praha, *Paramedial Groupoids*, Czechoslovak Mathematical Journal, Praha. **49(124)** (1996).

[6] M. A. Kazim and M. Naseeruddin, *On Almost Semigroups*, Alig. Bull. Math. **2** (1972), 1-7.

[7] M. Khan, V. Amjid and Faisal, *Ideals in Intra-Regular Left Almost Semigroups*, arXiv preprint arXiv. (2010), 1012-5598.

[8] M. S. Kamran, *Conditions for LA-Semigroups to Resemble Associative Structures*, PhD. Thesis., Quaid-i-Azam University, Islamabad, 1993. Available at http://eprints.hec.gov.pk/2370/1/2225.htm.

[9] M. Shah and T. Shah, *Some Basic Properties of LA-Rings.*, International Mathematical Forum. **6(44)** (2011), 2195-2199.

[10] P.V.Protic and N. Stevanovi, *AG-test and Some General Properties of Abel-Grassmann's Groupoid*, P.U.M.A. **6(4)** (1995), 371-383.

[11] Q. Mushtaq and S.M. Yousaf, *On LA-Semigroups* , Alig-Bull Math. **8** (1978), 65-70.

[12] S.M.Yusuf, *On Left Almost Ring*, Proc. of 7th International Pure Math. Conference, 2006.

[13] T. Shah and I. Rehman, *On LA-Rings of Finitely Non-Zero Functions*, Int. J. Contemp.Math. Sciences. **5(5)** (2010), 209-222.

[14] T. Shah and I. Rehman, *On Characterization of LA-Rings Through Some Properties of Their Ideals.*, Southeast Asian Bulletin of Mathematics. **36** (2012), 695-705.

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