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ALGORITHMS TO APPLY FINITE ELEMENT DUAL SINGULAR FUNCTION METHOD FOR THE STOKES EQUATIONS INCLUDING CORNER SINGULARITIES

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ABSTRACT. The dual singular function method [DSFM] is a solver for corner sigulaity problem. We already construct DSFM in previous reserch to solve the Stokes equations including one singulairity at each reentrant corner, but we find out a crucial incorrection in the proof of well-posedness and regularity of dual singular function. The goal of this paper is to prove accuracy and well-posdness of DSFM for Stokes equations including two singulairities at each corner. We also introduce new applicable algorithms to slove multi-singulrarity problems in a complicated domain.

1. INTRODUCTION

Corner singularity occurs a reentrant domain and is a reason losing accuracy. One of the answers for that problem is the dual singular function method [DSFM] which is constructed in [1, 2, 3, 4]. We already proposed DSFM for Stokes equations, but we find out a crucial incorrection in the proof of well-posedness and regularity of dual singular function at the Lemma 4.2 in [3]. We fix the proof at the Lemma 4.2 in this paper and construct new algorithms of DSFM for Stokes equations including 2 singularities at a corner.

The governing equations are

$$\begin{aligned} -\mu\triangle\mathbf{u} + \nabla p &= \mathbf{f}, \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u} &= 0, \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0}, \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

with $\mathbf{f}$ is a given function in $\mathbf{H}^{-1}(\Omega)$, Ω is a computational domain in $\mathbb{R}^2$, and $\mu = Re^{-1}$ is the reciprocal of the Reynolds number. Here the unknowns are the (vector) velocity field $\mathbf{u} \in \mathbf{H}^1_0(\Omega)$ and the (scalar) pressure $p \in L^2_0(\Omega)$.

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It is well known in [5, 6, 3] that 2 singular functions of the solution of (1.1) can be involved in each reentrant corner. It means the solution of (1.1) can be written by the form

$$\begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{w} \\ q \end{pmatrix} + \alpha_1 \begin{pmatrix} \mathbf{u}_1^s \\ p_1^s \end{pmatrix} + \alpha_2 \begin{pmatrix} \mathbf{u}_2^s \\ p_2^s \end{pmatrix}, \tag{1.2}$$

where α_1 and α_2 are the stress intensity factors, $(\mathbf{u}_i^s, p_i^s) \notin \mathbf{H}^2(\Omega) \times H^1(\Omega)$, $i = 1, 2$, are singular functions, and $(\mathbf{w}, q) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$.

Let ω be the internal angle. Without the loss of generality, we assume that the corresponding vertex is at the origin and that the internal angle ω is spanned by the two half-lines $\theta = 0$ and $\theta = \omega$. We denote Γ_{in} for 2 edges on the boundary including the reentrant corner and Γ_{out} for other parts of the boundary. The singular function $(\mathbf{u}_i^s, p_i^s)$, where $\mathbf{u}_i^s = (u_i^s, v_i^s)$, have been computed in [5, 3] with the eigenvalues $\lambda (> 0)$ satisfying

$$\sin^2(\lambda\omega) = \lambda^2 \sin^2(\omega). \tag{1.3}$$

We already know from [3] that (1.3) has only trivial solutions 0 and 1 for the case $\omega \leq \pi$. And (1.3) has a non-trivial unique solution $0.5 < \lambda < 1$ for $\pi < \omega \leq \beta\pi$ and has 2 non-trivial solutions $0.5 < \lambda_1 < \lambda_2 < 1$ for the case $\beta\pi < \omega < 2\pi$, where $\beta :\approx 1.430296653124203$. And (1.3) has a unique solution $\lambda = 0.5$, if $\omega = 2\pi$. Then the singular functions are summarized as, $i = 1, 2,$

$$\begin{aligned} \begin{pmatrix} u_i^s \\ v_i^s \\ p_i^s \end{pmatrix} = C_1 & \begin{pmatrix} \frac{r^{\lambda_i}}{\mu} \lambda_i \sin(\theta) \sin((1-\lambda_i)\theta) \\ \frac{r^{\lambda_i}}{\mu} \left(\sin(\lambda_i\theta) - \lambda_i \sin(\theta)\cos((1-\lambda_i)\theta)\right) \\ -2r^{\lambda_i - 1}\lambda_i \cos((1-\lambda_i)\theta) \end{pmatrix} \\ - C_2 & \begin{pmatrix} \frac{r^{\lambda_i}}{\mu} \left(\sin(\lambda_i\theta) + \lambda_i \sin(\theta)\cos((1-\lambda_i)\theta)\right) \\ \frac{r^{\lambda_i}}{\mu} \lambda_i \sin(\theta) \sin((1-\lambda_i)\theta) \\ 2r^{\lambda_i - 1}\lambda_i \sin((1-\lambda_i)\theta) \end{pmatrix}, \end{aligned} \tag{1.4}$$

where

$$C_1 = \sin(\lambda_i\omega) + \lambda_i \sin(\omega)\cos((1-\lambda_i)\omega) \quad \text{and} \quad C_2 = \lambda_i \sin(\omega)\sin((1-\lambda_i)\omega).$$

We note that the singular function $(\mathbf{u}_i^s, p_i^s)$, $i = 1, 2$, in (1.4) is the solution of homogeneous Stokes equations with vanishing Dirichlet boundary condition at Γ_{in}. And λ_i has to be a positive real number and $(\mathbf{u}_i^s, p_i^s) \in \mathbf{H}^{1+\lambda_i}(\Omega) \times H^{\lambda_i}(\Omega)$, $i = 1, 2$. Let η be a smooth cut-off function which is equal one identically in neighborhood of origin, and the support of η is small enough so that the functions $\eta\mathbf{u}_i^s$, $i = 1, 2$, vanish identically on $\partial\Omega$. Then, in general, the solution $(\mathbf{u}, p)$ including singular parts of (1.1) can be rewritten of the form (1.2) as

$$\begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{w} \\ q \end{pmatrix} + \alpha_1 \begin{pmatrix} \eta_1 \mathbf{u}_1^s \\ \eta_1 p_1^s \end{pmatrix} + \alpha_2 \begin{pmatrix} \eta_2 \mathbf{u}_2^s \\ \eta_2 p_2^s \end{pmatrix}, \tag{1.5}$$

where α_1 and α_2 are the stress intensity factors and $(\mathbf{w}, q) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$.

The strategy of FE-DSFM is to compute the regular solution $(\mathbf{w}, q) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$ and stress intensity factors α_1 and α_2 by applying the standard finite element method. So we need to construct decoupled system by using the following dual singular functions $(\mathbf{u}_i^d, p_i^d)$, where $\mathbf{u}_i^d = (u_i^d, v_i^d)$, $i = 1, 2$, which is derived in [3],

$$\begin{pmatrix} u_i^d \\ v_i^d \\ p_i^d \end{pmatrix} = d_1 \begin{pmatrix} -r^{-\lambda_i} \dfrac{\lambda_i}{\mu} \sin(\theta) \sin((1+\lambda_i)\theta) \\ -r^{-\lambda_i} \dfrac{1}{\mu} \left(\sin(\lambda_i \theta) - \lambda_i \sin(\theta) \cos((1+\lambda_i)\theta)\right) \\ 2r^{-\lambda_i - 1} \lambda_i \cos((1+\lambda_i)\theta) \end{pmatrix}$$

$$+ d_2 \begin{pmatrix} r^{-\lambda_i} \dfrac{1}{\mu} \left(\sin(\lambda_i \theta) + \lambda_i \sin(\theta) \cos((1+\lambda_i)\theta)\right) \\ r^{-\lambda_i} \dfrac{\lambda_i}{\mu} \sin(\theta) \sin((1+\lambda_i)\theta) \\ 2r^{-\lambda_i - 1} \lambda_i \sin((1+\lambda_i)\theta) \end{pmatrix},$$

where

$$d_1 = \sin(\lambda_i \omega) + \lambda_i \sin(\omega) \cos((1+\lambda_i)\omega) \quad \text{and} \quad d_2 = \lambda_i \sin(\omega) \sin((1+\lambda_i)\omega).$$

2. THE FINITE ELEMENT DUAL SINGULAR FUNCTION METHOD

In this section, we build a new variational formulation to find the regular part $(\mathbf{w}, q)$ and the stress intensity factors α_i, $i = 1, 2$ in (1.5) and introduce well-posdness of the system. It will be proved in §3. We now start this section with introducing the following lemma for the properties of the singular and the dual singular functions.

Lemma 2.1 (Properties of singular and dual singular functions)**.** *The singular function* $(\mathbf{u}_i^s, p_i^s) \in \mathbf{H}^{1+\lambda}(\Omega) \times H^{\lambda}(\Omega)$ *and the dual singular function* $(\mathbf{u}_i^d, p_i^d) \notin \mathbf{H}^1(\Omega) \times L^2(\Omega)$, $i = 1, 2$, *satisfy*

$$\begin{aligned} -\mu \triangle \mathbf{u}_i^s + \nabla p_i^s &= \mathbf{0}, \quad \textit{in } \Omega, \\ \nabla \cdot \mathbf{u}_i^s &= 0, \quad \textit{on } \Gamma_{in}, \end{aligned}$$

and

$$\begin{aligned} -\mu \triangle \mathbf{u}_i^d + \nabla p_i^d &= \mathbf{0}, \quad \textit{in } \Omega, \\ \nabla \cdot \mathbf{u}_i^d &= 0, \quad \textit{on } \Gamma_{in}, \end{aligned} \tag{2.1}$$

respectively. The boundary conditions of $\mathbf{u}_i^s$ *and* $\mathbf{u}_i^d$ *vanish on* Γ_{in}*, but the boundary value of* $\mathbf{u}_i^d$ *is not defined at the origin. Both of* $\mathbf{u}_i^s$ *and* $\mathbf{u}_i^d$ *are not* $\mathbf{0}$ *on* Γ_{out}*.*

In order to derive a explicit form of the singular functions, we set

$$B(r_1; r_2) = \{(r, \theta) : r_1 < r < r_2 \text{ and } 0 < \theta < \omega\} \cap \Omega$$

and

$$B(r_1) = B(0; r_1).$$

We define a smooth enough cut-off function of $\eta_\rho(r)$ as follows:

$$\eta_\rho(r) = \begin{cases} 1, & \text{in } B(\frac{1}{2}\rho R), \\ \text{very smooth function}, & \text{in } B(\frac{1}{2}\rho R; \rho R), \\ 0, & \text{in } \Omega \setminus \bar{B}(\rho R), \end{cases}$$

where ρ is a parameter in $(0, 2]$ and R is a fixed real number which will be determined later so that the singular part $\eta_{2\rho}\mathbf{u}_i^s$ has $\mathbf{0}$ on whole $\partial\Omega$. Here and thereafter, we choose that $\eta_1 = \eta_2 = \eta_\rho$ in (1.5) and assume that $0 < \rho < 1$. That is, the singular function representation of the solution of problem (1.1) has the form

$$\begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{w} \\ q \end{pmatrix} + \alpha_1 \begin{pmatrix} \eta_\rho \mathbf{u}_1^s \\ \eta_\rho p_1^s \end{pmatrix} + \alpha_2 \begin{pmatrix} \eta_\rho \mathbf{u}_2^s \\ \eta_\rho p_2^s \end{pmatrix}, \tag{2.2}$$

where $\mathbf{w} \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)$ and $q \in \mathbf{L}_0^2(\Omega) \cap H^1(\Omega)$ satisfying

$$\begin{aligned} -\mu\triangle\mathbf{w} + \nabla q + \sum_{i=1}^{2} \alpha_i \left(-\mu\triangle(\eta_\rho \mathbf{u}_i^s) + \nabla(\eta_\rho p_i^s)\right) = \mathbf{f}, \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{w} + \sum_{i=1}^{2} \alpha_i \nabla \cdot (\eta_\rho \mathbf{u}_i^s) = 0, \quad \text{in } \Omega. \end{aligned} \tag{2.3}$$

For the sake of a clear explanation, we note that the inner product of vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ is

$$\langle \mathbf{a}\,,\, \mathbf{b} \rangle = \langle a_1\,,\, b_1 \rangle + \langle a_2\,,\, b_2 \rangle$$

and also

$$\langle \nabla\mathbf{a}\,,\, \nabla\mathbf{b} \rangle = \langle \partial_x a_1\,,\, \partial_x b_1 \rangle + \langle \partial_x a_2\,,\, \partial_x b_2 \rangle + \langle \partial_y a_1\,,\, \partial_y b_1 \rangle + \langle \partial_y a_2\,,\, \partial_y b_2 \rangle\,.$$

Then we can obtain the weak form of (2.3) by the standard Galerkin finite element technique: find $(\mathbf{w}, q) \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega)$ satisfying, for all $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ and $\phi \in L^2(\Omega)$,

$$\begin{aligned} \mu \langle \nabla\mathbf{w}\,,\, \nabla\mathbf{v} \rangle + \langle \nabla q\,,\, \mathbf{v} \rangle + \sum_{i=1}^{2} \alpha_i c_i(\mathbf{v}) = \langle \mathbf{f}\,,\, \mathbf{v} \rangle\,, \\ \langle \nabla \cdot \mathbf{w}\,,\, \phi \rangle + \sum_{i=1}^{2} \alpha_i d_i(\phi) = 0, \end{aligned} \tag{2.4}$$

where

$$c_i(\mathbf{v}) := \langle -\mu\triangle(\eta_\rho \mathbf{u}_i^s) + \nabla(\eta_\rho p_i^s)\,,\,\mathbf{v}\rangle\,,\quad d_i(\phi) := \langle \nabla\cdot(\eta_\rho \mathbf{u}_i^s)\,,\,\phi\rangle\,. \tag{2.5}$$

Because 4 unknown variables $(\mathbf{w}, q)$ and α_i are coupled in 2 equations of (2.4), we have to build 2 additional equations which are linearly independent with (2.4). Therefore we test $\eta_{2\rho}\mathbf{u}_j^d \notin \mathbf{H}^1(\Omega)$ and $\eta_{2\rho}p_j^d \notin L^2(\Omega)$, $j = 1, 2$, with the first and the second equations in (2.3), respectively. Then we have the additional equations

$$\begin{aligned} \left\langle -\mu\triangle\mathbf{w} + \nabla q\,,\,\eta_{2\rho}\mathbf{u}_j^d \right\rangle + \sum_{i=1}^{2} \alpha_i \beta_{i,j}^m &= \beta_j^{\mathbf{f}}, \\ \left\langle \nabla\cdot\mathbf{w}\,,\,\eta_{2\rho}p_j^d \right\rangle + \sum_{i=1}^{2} \alpha_i \beta_{i,j}^p &= 0, \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} \beta_j^{\mathbf{f}} &:= \left\langle \mathbf{f}\,,\,\eta_{2\rho}\mathbf{u}_j^d \right\rangle, \\ \beta_{i,j}^m &:= \left\langle -\mu\triangle(\eta_\rho \mathbf{u}_i^s) + \nabla(\eta_\rho p_i^s)\,,\,\eta_{2\rho}\mathbf{u}_j^d \right\rangle, \\ \beta_{i,j}^p &:= \left\langle \nabla\cdot(\eta_\rho \mathbf{u}_i^s)\,,\,\eta_{2\rho}p_j^d \right\rangle, \end{aligned}$$

and they are computable. Because the dual singular functions are not smooth enough to apply the integration by parts directly in (2.6), the following lemma is crucial.

Lemma 2.2 (Integration by parts for dual singular functions). *Let* $\mathbf{w} \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)$ *and* $q \in H^1(\Omega)$. *If* $\rho \in (0, 1]$, *then we have that, for* $j = 1, 2$,

$$-\mu \left\langle \triangle\mathbf{w}\,,\,\eta_{2\rho}\mathbf{u}_j^d \right\rangle - \left\langle \nabla\cdot\mathbf{w}\,,\,\eta_{2\rho}p_j^d \right\rangle = \left\langle \mathbf{w}\,,\,-\mu\triangle(\eta_{2\rho}\mathbf{u}_j^d) + \nabla(\eta_{2\rho}p_j^d) \right\rangle \tag{2.7}$$

and

$$\left\langle \nabla q\,,\,\eta_{2\rho}\mathbf{u}_j^d \right\rangle = -\left\langle q\,,\,\nabla\cdot(\eta_{2\rho}\mathbf{u}_j^d) \right\rangle. \tag{2.8}$$

Proof. We can readily obtain (2.7) by integration by parts, if the functions are smooth enough. But the dual singular functions are not smooth enough, so we need to use density argument of Hilbert space. Then it is enough to show boundedness of both sides. Since $\mathbf{w} \in \mathbf{H}_0^1(\Omega)\cap\mathbf{H}^2(\Omega)$ and $\eta_{2\rho}\mathbf{u}_j^d \in \mathbf{L}^2(\Omega)$, we can get the boundedness of the left hand side in (2.7). On the other hand, the right hand side in (2.7) is also bounded, because of (2.1) and the definition of $\eta_{2\rho}$. So we arrive at (2.7). By the same manner, the properties $q \in H^1(\Omega)$, $\eta_{2\rho}\mathbf{u}_j^d \in \mathbf{L}^2(\Omega)$, and $\nabla\cdot(\eta_{2\rho}\mathbf{u}_j^d) = 0$ in $B(\rho)$ yield (2.8). □

We apply Lemma 2.2 after subtraction the second equation from the first equation in (2.6) for each $j = 1, 2$ to obtain

$$\begin{aligned} \beta_j^{\mathbf{f}} - \sum_{i=1}^{2} \alpha_i(\beta_{i,j}^m - \beta_{i,j}^p) &= \left\langle -\mu\triangle\mathbf{w} + \nabla q\,,\,\eta_{2\rho}\mathbf{u}_j^d \right\rangle - \left\langle \nabla\cdot\mathbf{w}\,,\,\eta_{2\rho}p_j^d \right\rangle \\ &= \left\langle \mathbf{w}\,,\,-\mu\triangle(\eta_{2\rho}\mathbf{u}_j^d) + \nabla(\eta_{2\rho}p_j^d) \right\rangle - \left\langle q\,,\,\nabla\cdot\left(\eta_{2\rho}\mathbf{u}_j^d\right) \right\rangle. \end{aligned} \tag{2.9}$$

If we denote the following notations

$$\begin{aligned} \zeta_{i,j} &:= \beta^m_{i,j} - \beta^p_{i,j}, \\ a_j(\mathbf{w}) &:= \left\langle \mathbf{w}\,,\, -\mu\triangle(\eta_{2\rho}\mathbf{u}^d_j) + \nabla(\eta_{2\rho}p^d_j) \right\rangle, \\ b_j(q) &:= \left\langle q\,,\, \nabla\cdot\left(\eta_{2\rho}\mathbf{u}^d_j\right) \right\rangle, \end{aligned} \tag{2.10}$$

then (2.9) can be simply rewritten by

$$\sum_{i=1}^{2} \alpha_i \zeta_{i,j} = \beta^{\mathbf{f}}_j - a_j(\mathbf{w}) + b_j(q)$$

or by matrix form

$$D\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \beta^{\mathbf{f}}_1 - a_1(\mathbf{w}) + b_1(q) \\ \beta^{\mathbf{f}}_2 - a_2(\mathbf{w}) + b_2(q) \end{pmatrix},$$

where $D := \begin{pmatrix} \zeta_{1,1} & \zeta_{2,1} \\ \zeta_{1,2} & \zeta_{2,2} \end{pmatrix}$. So Cramer's rule yields

$$\begin{aligned} \alpha_1 =& \frac{\zeta_{2,2}\left(\beta^{\mathbf{f}}_1 - a_1(\mathbf{w}) + b_1(q)\right) - \zeta_{2,1}\left(\beta^{\mathbf{f}}_2 - a_2(\mathbf{w}) + b_2(q)\right)}{det(D)}, \\ \alpha_2 =& \frac{\zeta_{1,1}\left(\beta^{\mathbf{f}}_2 - a_2(\mathbf{w}) + b_2(q)\right) - \zeta_{1,2}\left(\beta^{\mathbf{f}}_1 - a_1(\mathbf{w}) + b_1(q)\right)}{det(D)}. \end{aligned} \tag{2.11}$$

We now define finite element space to construct fully discrete FE-DSFM. Let $\mathfrak{T} = \{K\}$ be a shape-regular quasi-uniform partition of Ω of meshsize h into closed elements K [7, 8, 9]. The vector and scalar finite element spaces are:

$$\begin{aligned} \mathbb{W}_h &:= \{\mathbf{w}_h \in \mathbf{L}^2(\Omega) : \mathbf{w}_h|_K \in \mathcal{P}(K) \quad \forall K \in \mathfrak{T}\}, \quad \mathbb{V}_h := \mathbb{W}_h \cap \mathbf{H}^1_0(\Omega), \\ \mathbb{P}_h &:= \{q_h \in L^2(\Omega) \cap C^0(\Omega) : q_h|_K \in \mathcal{Q}(K) \quad \forall K \in \mathfrak{T}\}, \end{aligned}$$

where $\mathcal{P}(K)$ and $\mathcal{Q}(K)$ are spaces of polynomials with degree bounded uniformly with respect to $K \in \mathfrak{T}$. We stress that the space $\mathbb{P}_h$ is composed of continuous functions to use integration by parts: for all $q_h \in \mathbb{P}_h$

$$\langle \nabla\cdot\mathbf{v}_h\,,\, q_h\rangle = -\langle \mathbf{v}_h\,,\, \nabla q_h\rangle, \quad \forall \mathbf{v}_h \in \mathbb{V}_h.$$

Then (2.4) becomes, for all $\mathbf{v}_h \in \mathbb{V}_h$ and $\phi_h \in \mathbb{P}_h$,

$$\begin{aligned} \mu\langle \nabla\mathbf{w}_h\,,\, \nabla\mathbf{v}_h\rangle + \langle \nabla q_h\,,\, \mathbf{v}_h\rangle + \sum_{i=1}^{2}\alpha_i c_i(\mathbf{v}_h) &= \langle \mathbf{f}\,,\, \mathbf{v}_h\rangle, \\ \langle \nabla\cdot\mathbf{w}_h\,,\, \phi_h\rangle + \sum_{i=1}^{2}\alpha_i d_i(\phi_h) &= 0. \end{aligned} \tag{2.12}$$

And (2.11) becomes

$$\begin{aligned} \alpha_{1h} =& \frac{\zeta_{2,2}\left(\beta_1^{\mathbf{f}} - a_1(\mathbf{w}_h) + b_1(q_h)\right) - \zeta_{2,1}\left(\beta_2^{\mathbf{f}} - a_2(\mathbf{w}_h) + b_2(q_h)\right)}{det(D)}, \\ \alpha_{2h} =& \frac{\zeta_{1,1}\left(\beta_2^{\mathbf{f}} - a_2(\mathbf{w}_h) + b_2(q_h)\right) - \zeta_{1,2}\left(\beta_1^{\mathbf{f}} - a_1(\mathbf{w}_h) + b_1(q_h)\right)}{det(D)}. \end{aligned} \tag{2.13}$$

In order to solve the system (2.4) and (2.11), we insert α_i in (2.11) into(2.4) to obtain

$$\begin{aligned} \mu\left\langle \nabla\mathbf{w}\,,\,\nabla\mathbf{v}\right\rangle + \left\langle \nabla q\,,\,\mathbf{v}\right\rangle + \sum_{i=1}^{2}\left(A_i(\mathbf{w}) + B_i(q)\right)c_i(\mathbf{v}) \\ = \left\langle \mathbf{f}\,,\,\mathbf{v}\right\rangle - \sum_{i=1}^{2} F_i c_i(\mathbf{v}), \\ \left\langle \nabla\cdot\mathbf{w}\,,\,\phi\right\rangle + \sum_{i=1}^{2}\left(A_i(\mathbf{w}) + B_i(q)\right)d_i(\phi) = -\sum_{i=1}^{2} F_i d_i(\phi), \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} A_1(\mathbf{w}) :=& -\frac{\zeta_{2,2}a_1(\mathbf{w}) - \zeta_{2,1}a_2(\mathbf{w})}{det(D)}, \quad & A_2(\mathbf{w}) :=& -\frac{\zeta_{1,1}a_2(\mathbf{w}) - \zeta_{1,2}a_1(\mathbf{w})}{det(D)}, \\ B_1(q) :=& \frac{\zeta_{2,2}b_1(q) - \zeta_{2,1}b_2(q)}{det(D)}, & B_2(q) :=& \frac{\zeta_{1,1}b_2(q) - \zeta_{1,2}b_1(q)}{det(D)}, \\ F_1 :=& \frac{\zeta_{2,2}\beta_1^{\mathbf{f}} - \zeta_{2,1}\beta_2^{\mathbf{f}}}{det(D)}, & F_2 :=& \frac{\zeta_{1,1}\beta_2^{\mathbf{f}} - \zeta_{1,2}\beta_1^{\mathbf{f}}}{det(D)}. \end{aligned}$$

On the other hand, we insert α_{ih} in (2.13) into (2.12) to get discrete weak form

$$\begin{aligned} \mu\left\langle \nabla\mathbf{w}_h\,,\,\nabla\mathbf{v}_h\right\rangle + \left\langle \nabla q_h\,,\,\mathbf{v}_h\right\rangle + \sum_{i=1}^{2}\left(A_i(\mathbf{w}_h) + B_i(q_h)\right)c_i(\mathbf{v}_h) \\ = \left\langle \mathbf{f}\,,\,\mathbf{v}_h\right\rangle - \sum_{i=1}^{2} F_i c_i(\mathbf{v}_h), \\ \left\langle \nabla\cdot\mathbf{w}_h\,,\,\phi_h\right\rangle + \sum_{i=1}^{2}\left(A_i(\mathbf{w}_h) + B_i(q_h)\right)d_i(\phi_h) = -\sum_{i=1}^{2} F_i d_i(\phi_h). \end{aligned} \tag{2.15}$$

The matrix form of the coupled system (2.15) becomes

$$\left[\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} + \sum_{i=1}^{2}\begin{pmatrix} c_i \\ d_i \end{pmatrix}\begin{pmatrix} A_i, B_i \end{pmatrix}\right]\begin{pmatrix} \mathbf{w} \\ q \end{pmatrix} = \begin{pmatrix} \mathbf{L} \\ l \end{pmatrix}.$$

It is solvable by using the generalized Sherman-Morrison-Woodbury formula in [10]:

$$(M + U_1\cdot V_1^T + U_2\cdot V_2^T)^{-1} = M^{-1} - M^{-1}[U1, U_2]Q^{-1}[V_1^T, V_2^T]^T M^{-1}, \tag{2.16}$$

where Q is 2×2 matrix given by

$$Q = \left[\begin{array}{cc} 1 + V_1^T M^{-1} U_1, & V_1^T M^{-1} U_2 \\ V_2^T M^{-1} U_1, & 1 + V_2^T M^{-1} U_2 \end{array} \right].$$

Finally, we arrive at an implicit FE-DSFM:

Algorithm 1 (Implicit FE-DSFM using Sherman-Morrison-Woodbury formula)**.** Compute $(\mathbf{w}_h, q_h)$ and the stress intensity factors α_{1h} and $\alpha_{2h} \in \mathbb{R}$ by computing

Step 1: Find $(\mathbf{w}_h, q_h)$ as the solution of (2.15) by solving (2.16).
Step 2: Compute α_{1h} and $\alpha_{2h} \in \mathbb{R}$ by (2.13).

3. WELL-POSEDNESS

The goal of FE-DSFM is to compute $(\mathbf{w}, q)$ and the stress intensity factors α_1 and $\alpha_2 \in \mathbb{R}$ by solving the system (2.4) and (2.11). Because it is a coupled problem of 4 variables, we need to construct decoupling system of them. To do this, we first prove well posedness of the system (2.4) and (2.11). The equation (2.4) is a standard saddle point problem and has a unique solution for any given $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ and given $\mathbf{f} \in \mathbf{L}^2(\Omega)$ in [7, 8, 9]. We define mappings $T_{\mathbf{f}}$ from $\mathbb{R}^2$ to $\mathbf{H}^1(\Omega) \times L_0^2(\Omega)$ by the unique solution of (2.4) for any given $\mathbf{f} \in \mathbf{L}^2(\Omega)$. It means that $T_{\mathbf{f}}(\boldsymbol{\alpha}) := (\mathbf{w}_{\boldsymbol{\alpha}}, q_{\boldsymbol{\alpha}})$ is the solution of (2.4) with $\boldsymbol{\alpha} \in \mathbb{R}^2$. Also we define a mapping F from $\mathbf{H}^1(\Omega) \times L_0^2(\Omega)$ to $\mathbb{R}^2$ by using (2.11) as

$$F(\mathbf{w}, q) := D^{-1} \left(\begin{array}{c} \beta_1^{\mathbf{f}} - a_1(\mathbf{w}) + b_1(q) \\ \beta_2^{\mathbf{f}} - a_2(\mathbf{w}) + b_2(q) \end{array} \right). \tag{3.1}$$

Then the composition $F \circ T_{\mathbf{f}}$ is a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$. In order to prove the well-posedness, it is enough to prove existence of the unique fixed point of $F \circ T_{\mathbf{f}}$ and equivalently to prove $\|F \circ T_{\mathbf{f}}\| < 1$ by contraction mapping theorem.

Theorem 1 (Well-posedness)**.** *We have*

$$\|F \circ T_{\mathbf{f}}\| = 0.$$

Proof. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2)$ be arbitrary real vectors. Then we have

$$T_{\mathbf{f}}(\boldsymbol{\alpha}) = (\mathbf{w}_{\boldsymbol{\alpha}}, q_{\boldsymbol{\alpha}}) \quad \text{and} \quad T_{\mathbf{f}}(\boldsymbol{\beta}) = \big(\mathbf{w}_{\boldsymbol{\beta}}, q_{\boldsymbol{\beta}}\big). \tag{3.2}$$

Then we can get from (2.4),

$$\begin{aligned} \mu \left\langle \nabla \left(\mathbf{w}_{\boldsymbol{\alpha}} - \mathbf{w}_{\boldsymbol{\beta}}\right), \nabla \mathbf{v} \right\rangle + \left\langle \nabla \left(q_{\boldsymbol{\alpha}} - q_{\boldsymbol{\beta}}\right), \mathbf{v} \right\rangle + \sum_{i=1}^{2} (\alpha_i - \beta_i)\, c_i(\mathbf{v}) = 0, \\ \left\langle \nabla \cdot \left(\mathbf{w}_{\boldsymbol{\alpha}} - \mathbf{w}_{\boldsymbol{\beta}}\right), \phi \right\rangle + \sum_{i=1}^{2} (\alpha_i - \beta_i)\, d_i(\phi) = 0. \end{aligned} \tag{3.3}$$

And we define that $(\mathbf{x}_i, k_i)$, $i = 1, 2$, is the solution of the Stokes equations

$$\begin{aligned} -\mu\triangle\mathbf{x}_i + \nabla k_i &= -\mu\triangle(\eta_{2\rho}\mathbf{u}_i^d) + \nabla(\eta_{2\rho}p_i^d), && \text{in } \Omega, \\ \nabla\cdot\mathbf{x}_i &= \nabla\cdot(\eta_{2\rho}\mathbf{u}_i^d), && \text{in } \Omega, \\ \mathbf{x}_i &= \mathbf{0}, && \text{on } \partial\Omega. \end{aligned} \tag{3.4}$$

We note here that the right hand side terms are smooth functions which can be readily obtained by Lemma 2.1 and so $(\mathbf{x}_i, k_i) \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega)$. So it is enough to show the compatibility condition to assert the existence and uniqueness of the solution of (3.4). Since we have

$$\int_\Omega \nabla\cdot\mathbf{x}_i dx = \int_{\partial\Omega} \mathbf{x}_i\cdot\boldsymbol{\nu} ds = 0,$$

we need to prove $\int_\Omega \nabla\cdot(\eta_{2\rho}\mathbf{u}_i^d) = 0$. Because $\nabla\cdot(\eta_{2\rho}\mathbf{u}_i^d) = 0$ in $B(\rho)$ and $\eta_{2\rho}\mathbf{u}_i^d = \mathbf{0}$ on $\partial\Omega$ except at the origin, it is clear

$$\int_\Omega \nabla\cdot(\eta_{2\rho}\mathbf{u}_i^d)dx = \int_{\partial\Omega} \eta_{2\rho}\mathbf{u}_i^d\cdot\boldsymbol{\nu} ds = 0 \tag{3.5}$$

and we obtain $\int_\Omega \nabla\cdot\mathbf{x}_i dx = \int_\Omega \nabla\cdot(\eta_{2\rho}\mathbf{u}_i^d)dx = 0$. Thus (3.4) has a unique solution $(\mathbf{x}_i, k_i) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega)$.

In light of (3.2) and (3.1), we can get

$$\begin{aligned} \|F\circ T_{\mathbf{f}}(\boldsymbol{\alpha}) - F\circ T_{\mathbf{f}}(\boldsymbol{\beta})\|_0 &= \left\|F\left(\mathbf{w}_{\boldsymbol{\alpha}}, q_{\boldsymbol{\alpha}}\right) - F\left(\mathbf{w}_{\boldsymbol{\beta}}, q_{\boldsymbol{\beta}}\right)\right\|_0 \\ &= \left\| D^{-1}\begin{pmatrix} -a_1(\mathbf{w}_{\boldsymbol{\alpha}} - \mathbf{w}_{\boldsymbol{\beta}}) + b_1(q_{\boldsymbol{\alpha}} - q_{\boldsymbol{\beta}}) \\ -a_2(\mathbf{w}_{\boldsymbol{\alpha}} - \mathbf{w}_{\boldsymbol{\beta}}) + b_2(q_{\boldsymbol{\alpha}} - q_{\boldsymbol{\beta}}) \end{pmatrix}\right\|_0. \end{aligned}$$

In conjunction with the definitions $a_i(\cdot)$ and $b_i(\cdot)$ in (2.10), the Stokes equations (3.4) and integration by parts yield

$$\begin{aligned} &-a_i(\mathbf{w}_{\boldsymbol{\alpha}} - \mathbf{w}_{\boldsymbol{\beta}}) + b_i(q_{\boldsymbol{\alpha}} - q_{\boldsymbol{\beta}}) = -\left\langle \mathbf{w}_{\boldsymbol{\alpha}} - \mathbf{w}_{\boldsymbol{\beta}}\,,\, -\mu\triangle(\eta_{2\rho}\mathbf{u}_i^d) + \nabla(\eta_{2\rho}p_i^d)\right\rangle \\ &\qquad\qquad + \left\langle q_{\boldsymbol{\alpha}} - q_{\boldsymbol{\beta}}\,,\, \nabla\cdot\left(\eta_{2\rho}\mathbf{u}_i^d\right)\right\rangle \\ &= -\left\langle \mathbf{w}_{\boldsymbol{\alpha}} - \mathbf{w}_{\boldsymbol{\beta}}\,,\, -\mu\triangle\mathbf{x}_i + \nabla k_i\right\rangle + \left\langle q_{\boldsymbol{\alpha}} - q_{\boldsymbol{\beta}}\,,\, \nabla\cdot\mathbf{x}_i\right\rangle \\ &= -\mu\left\langle \nabla\left(\mathbf{w}_{\boldsymbol{\alpha}} - \mathbf{w}_{\boldsymbol{\beta}}\right), \nabla\mathbf{x}_i\right\rangle + \left\langle \nabla\cdot\left(\mathbf{w}_{\boldsymbol{\alpha}} - \mathbf{w}_{\boldsymbol{\beta}}\right), k_i\right\rangle - \left\langle \nabla\left(q_{\boldsymbol{\alpha}} - q_{\boldsymbol{\beta}}\right), \mathbf{x}_i\right\rangle \\ &= \sum_{i=1}^{2}\left(\alpha_i - \beta_i\right)\left(c_i(\mathbf{x}_i) - d_i(k_i)\right), \end{aligned}$$

where the last equality comes from (3.3) by choosing $\mathbf{v} = \mathbf{x}_i$ and $\phi = k_i$. We apply the definitions of $c_i(\cdot)$ and $d_i(\cdot)$ in (2.5) and integration by parts to obtain

$$\begin{aligned}-a_i(\mathbf{w}_{\boldsymbol\alpha} - \mathbf{w}_{\boldsymbol\beta}) + b_i(q_{\boldsymbol\alpha} - q_{\boldsymbol\beta}) &= \sum_{i=1}^{2} (\alpha_i - \beta_i)\,(c_i(\mathbf{x}_i) - d_i(k_i)) \\ &= \sum_{i=1}^{2} (\alpha_i - \beta_i)\,(\langle -\mu\triangle(\eta_\rho \mathbf{u}_i^s) + \nabla(\eta_\rho p_i^s)\,,\, \mathbf{x}_i\rangle - \langle \nabla\cdot(\eta_\rho \mathbf{u}_i^s)\,,\, k_i\rangle) \\ &= \sum_{i=1}^{2} (\alpha_i - \beta_i)\,(\langle \eta_\rho \mathbf{u}_i^s\,,\, -\mu\triangle\mathbf{x}_i + \nabla k_i\rangle - \langle \eta_\rho p_i^s\,,\, \nabla\cdot\mathbf{x}_i\rangle)\,.\end{aligned}$$

If we test $\eta_\rho \mathbf{u}_i^s$ and $\eta_\rho p_i^s$ with the first and the second equations in (3.4) respectively, then we can obtain

$$\begin{aligned}\langle -\mu\triangle\mathbf{x}_i + \nabla k_i\,,\, \eta_\rho \mathbf{u}_i^s\rangle &= \left\langle -\mu\triangle(\eta_{2\rho}\mathbf{u}_i^d) + \nabla(\eta_{2\rho} p_i^d)\,,\, \eta_\rho \mathbf{u}_i^s\right\rangle, \\ \langle \nabla\cdot\mathbf{x}_i\,,\, \eta_\rho p_i^s\rangle &= \left\langle \nabla\cdot(\eta_{2\rho}\mathbf{u}_i^d)\,,\, \eta_\rho p_i^s\right\rangle\end{aligned}$$

and the right hand side terms of above equations are identically zero by Lemma 2.1, because of the distinct supports of η_ρ and $\eta_{2\rho}$. Therefore, we conclude that

$$\|F\circ T_{\mathbf{f}}(\boldsymbol\alpha) - F\circ T_{\mathbf{f}}(\boldsymbol\beta)\|_0 = 0.$$

So the proof is completed. □

From the Theorem 1, we have the following result.

Corollary 1. *Let $\boldsymbol\alpha = (\alpha_1, \alpha_2)$ be the exact solution of the system* (2.4) *and* (2.11). *Then we have $\boldsymbol\alpha = F\circ T_{\mathbf{f}}(\boldsymbol\alpha) = F\circ T_{\mathbf{f}}(\boldsymbol\beta)$, for any $\boldsymbol\beta = (\beta_1, \beta_2) \in \mathbb{R}^2$.*

4. ERROR ESTIMATES

In this section, we will prove Error estimates which are errors of FE-DSFM (2.15) and (2.13) by comparing them with (2.14) and (2.11). In order to introduce a useful lemma for regularities of Stokes equations in [6], we define the Stokes equations:

$$\begin{aligned}-\mu\triangle\mathbf{x} + \nabla k &= \mathbf{f}, \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{x} &= \chi, \quad \text{in } \Omega, \\ \mathbf{x} &= \mathbf{0}, \quad \text{on } \partial\Omega.\end{aligned} \tag{4.1}$$

Lemma 4.1 (Regularities of regular solution). *Let Ω be a polygonal domain with non-convex vertices. If $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ and $\chi \in L^2(\Omega)$, then there exist a unique solution $(\mathbf{x}, k) \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega)$ of* (4.1), *with*

$$\|\mathbf{x}\|_1 + \|k\|_0 \le C\left(\|\mathbf{f}\|_{-1} + \|\chi\|_0\right).$$

Moreover, if $\mathbf{f} \in \mathbf{L}^2(\Omega)$ *and* $\chi \in H^1(\Omega)$*, then the solution* $(\mathbf{x}, k) \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega)$ *can be rewritten in the form of* $\mathbf{x} = \mathbf{x}_R + \alpha_1\eta_1\mathbf{u}_1^s + \alpha_2\eta_2\mathbf{u}_2^s$ *and* $k = k_R + \alpha_1\eta_1 p_1^s + \alpha_2\eta_2 p_2^s$*, with* $(\mathbf{x}_R, k_R) \in \left(\mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)\right) \times \left(L^2(\Omega) \cap H^1(\Omega)\right)$ *satisfying*

$$\|\mathbf{x}_R\|_2 + \|k_R\|_1 + |\alpha_1| + |\alpha_2| \le C\left(\|\mathbf{f}\|_0 + \|\chi\|_1\right),$$

where $(\eta_1\mathbf{u}_1^s, \eta_1 p_1^s)$ *and* $(\eta_2\mathbf{u}_2^s, \eta_2 p_2^s)$ *are singular functions.*

In order to perform error estimate, we also have to have stability assumption on space:

Assumption 1 (Discrete inf-sup)**.** *For given* $p_h \in \mathbb{P}_h$*, there exists a constant* $\gamma > 0$ *such that*

$$\gamma\|p_h\|_0 \le \sup_{\mathbf{v}_h \in \mathbb{V}_h} \frac{\langle \nabla\cdot\mathbf{v}_h\,,\, p_h\rangle}{\|\mathbf{v}_h\|_1}.$$

We evaluate errors under the notations:

$$\begin{aligned} \mathbf{E} &:= \mathbf{w} - \mathbf{w}_h, & \mathbf{E}_h &:= \mathcal{I}_h\mathbf{w} - \mathbf{w}_h, & \mathcal{I}_h\mathbf{E} &:= \mathbf{w} - \mathcal{I}_h\mathbf{w},\\ e &:= q - q_h, & e_h &:= \mathcal{I}_h q - q_h, & \mathcal{I}_h e &:= q - \mathcal{I}_h q, \end{aligned}$$

$$\varepsilon_1 := \alpha_1 - \alpha_{1h} \quad \text{and} \quad \varepsilon_2 := \alpha_2 - \alpha_{2h},$$

where $\mathcal{I}_h$ the Clement interpolant. Because $(\mathbf{w}, q) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$, we can use the well known results

$$\|\mathcal{I}_h\mathbf{E}\|_0 + h\|\mathcal{I}_h\mathbf{E}\|_1 \le Ch^2\|\mathbf{w}\|_2 \quad \text{and } \|\mathcal{I}_h e\|_0 \le Ch\|q\|_1. \tag{4.2}$$

In proof of the main theorem, we will use the solution $(\mathbf{z}, r) \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega)$ of, for all $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ and for all $\phi \in L^2(\Omega)$,

$$\begin{aligned} \mu\langle\nabla\mathbf{z}\,,\,\nabla\mathbf{v}\rangle + \langle\nabla r\,,\,\mathbf{v}\rangle + \sum_{i=1}^{2}\left(c_i(\mathbf{z}) - d_i(r)\right)A_i(\mathbf{v}) &= \langle\mathbf{E}\,,\,\mathbf{v}\rangle\,,\\ \langle\nabla\cdot\mathbf{z}\,,\,\phi\rangle - \sum_{i=1}^{2}\left(c_i(\mathbf{z}) - d_i(r)\right)B_i(\phi) &= 0. \end{aligned} \tag{4.3}$$

In order to use (4.3), we need to check the compatibility condition and it is enough to prove

$$\langle\nabla\cdot\mathbf{z}\,,\,1\rangle = \sum_{i=1}^{2}\left(c_i(\mathbf{z}) - d_i(r)\right)B_i(1). \tag{4.4}$$

We have

$$\int_\Omega \nabla\cdot\mathbf{z}dx = \int_{\partial\Omega}\mathbf{z}\cdot\boldsymbol{\nu}ds = 0,$$

where $\boldsymbol{\nu}$ is the outward unit normal vector. Since we proved $b_j(1) = 0$ in (3.5), we can readily get $B_i(1) = 0$ from the definition of B_i (2.14). So we arrive at (4.4). To prove well-posdness of

(4.3), we rewrite (4.3) using the definitions of $A_i(\mathbf{v})$ and $B_i(\phi)$ in (2.14) and $a_j(\mathbf{v})$ and $b_j(\phi)$ in (2.10) as

$$\begin{aligned} \mu\left\langle \nabla \mathbf{z}\,,\, \nabla \mathbf{v}\right\rangle + \left\langle \nabla r\,,\, \mathbf{v}\right\rangle + \begin{pmatrix} c_1(\mathbf{z}) - d_1(r) \\ c_2(\mathbf{z}) - d_2(r) \end{pmatrix}^T \bar{D} \begin{pmatrix} a_1(\mathbf{v}) \\ a_2(\mathbf{v}) \end{pmatrix} &= \left\langle \mathbf{E}\,,\, \mathbf{v}\right\rangle, \\ \left\langle \nabla\cdot \mathbf{z}\,,\, \phi\right\rangle - \begin{pmatrix} c_1(\mathbf{z}) - d_1(r) \\ c_2(\mathbf{z}) - d_2(r) \end{pmatrix}^T \bar{D} \begin{pmatrix} b_1(\phi) \\ b_2(\phi) \end{pmatrix} &= 0, \end{aligned} \tag{4.5}$$

where $\bar{D} := -D^{-1} = \frac{1}{det(D)} \begin{pmatrix} -\zeta_{2,2} & \zeta_{2,1} \\ \zeta_{1,2} & -\zeta_{1,1} \end{pmatrix}$. We will establish the well-posedness of equations (4.3) by rewriting (4.5) with $\bar{\boldsymbol{\alpha}} := \begin{pmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_2 \end{pmatrix}^T = \begin{pmatrix} c_1(\mathbf{z}) - d_1(r) \\ c_2(\mathbf{z}) - d_2(r) \end{pmatrix}^T \bar{D}$ as

$$\begin{aligned} \mu\left\langle \nabla \mathbf{z}\,,\, \nabla \mathbf{v}\right\rangle + \left\langle \nabla r\,,\, \mathbf{v}\right\rangle + (\bar{\alpha}_1, \bar{\alpha}_2) \begin{pmatrix} a_1(\mathbf{v}) \\ a_2(\mathbf{v}) \end{pmatrix} &= \left\langle \mathbf{E}\,,\, \mathbf{v}\right\rangle, \\ \left\langle \nabla\cdot \mathbf{z}\,,\, \phi\right\rangle - (\bar{\alpha}_1, \bar{\alpha}_2) \begin{pmatrix} b_1(\phi) \\ b_2(\phi) \end{pmatrix} &= 0. \end{aligned} \tag{4.6}$$

In order to prove existence and uniqueness of solution of (4.3), we define a mapping $F_{\mathbf{E}}$ from $\mathbb{R}^2$ to $\mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ by $F_{\mathbf{E}}(\bar{\boldsymbol{\alpha}}) := (\mathbf{z}_{\bar{\boldsymbol{\alpha}}}, r_{\bar{\boldsymbol{\alpha}}})$ to be the solution of (4.6), for any $\bar{\boldsymbol{\alpha}} \in \mathbb{R}^2$. It is well known that (4.6) has unique solution $(\mathbf{z}_{\bar{\boldsymbol{\alpha}}}, r_{\bar{\boldsymbol{\alpha}}})$ for any $\bar{\boldsymbol{\alpha}} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, if $\mathbf{E} \in \mathbf{L}^2(\Omega)$.

And we define $F_{\bar{\boldsymbol{\alpha}}}$ from $\mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ to $\mathbb{R}^2$ by

$$F_{\bar{\boldsymbol{\alpha}}}(\mathbf{z}, r) := (c_1(\mathbf{z}) - d_1(r), c_2(\mathbf{z}) - d_2(r))\, \bar{D}. \tag{4.7}$$

Then $F_{\bar{\boldsymbol{\alpha}}} \circ F_{\mathbf{E}}$ becomes a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$. We will prove that $F_{\bar{\boldsymbol{\alpha}}} \circ F_{\mathbf{E}}$ has a unique fixed point by the contraction mapping theorem [11], and it is equivalent to prove $\|F_{\bar{\boldsymbol{\alpha}}} \circ F_{\mathbf{E}}\| < 1$.

Lemma 4.2 (Well-posedness and regularity of (4.3))**.** *We have*

$$\|F_{\bar{\boldsymbol{\alpha}}} \circ F_{\mathbf{E}}\| = \mathbf{0}. \tag{4.8}$$

Proof. Let $\bar{\boldsymbol{\alpha}}_1$ and $\bar{\boldsymbol{\alpha}}_2$ be arbitrary real numbers in $\mathbb{R}^2$ and let

$$F_{\mathbf{E}}\left(\bar{\boldsymbol{\alpha}}_1\right) = \left(\mathbf{z}_{\bar{\boldsymbol{\alpha}}_1}, r_{\bar{\boldsymbol{\alpha}}_1}\right) \quad \text{ and } \quad F_{\mathbf{E}}\left(\bar{\boldsymbol{\alpha}}_2\right) = \left(\mathbf{z}_{\bar{\boldsymbol{\alpha}}_2}, r_{\bar{\boldsymbol{\alpha}}_2}\right). \tag{4.9}$$

From (4.6), we obtain

$$\begin{aligned} &\mu\left\langle \nabla\left(\mathbf{z}_{\bar{\boldsymbol{\alpha}}_1} - \mathbf{z}_{\bar{\boldsymbol{\alpha}}_2}\right),\, \nabla \mathbf{v}\right\rangle - \left\langle \left(r_{\bar{\boldsymbol{\alpha}}_1} - r_{\bar{\boldsymbol{\alpha}}_2}\right),\, \nabla\cdot \mathbf{v}\right\rangle \\ &\qquad = -\left(\bar{\boldsymbol{\alpha}}_1 - \bar{\boldsymbol{\alpha}}_2\right) \begin{pmatrix} \left\langle -\mu\triangle(\eta_{2\rho}\mathbf{u}_1^d) + \nabla(\eta_{2\rho}p_1^d)\,,\, \mathbf{v}\right\rangle \\ \left\langle -\mu\triangle(\eta_{2\rho}\mathbf{u}_2^d) + \nabla(\eta_{2\rho}p_2^d)\,,\, \mathbf{v}\right\rangle \end{pmatrix}, \\ &\left\langle \nabla\cdot\left(\mathbf{z}_{\bar{\boldsymbol{\alpha}}_1} - \mathbf{z}_{\bar{\boldsymbol{\alpha}}_2}\right),\, \phi\right\rangle = \left(\bar{\boldsymbol{\alpha}}_1 - \bar{\boldsymbol{\alpha}}_2\right) \begin{pmatrix} \left\langle \nabla\cdot(\eta_{2\rho}\mathbf{u}_1^d)\,,\, \phi\right\rangle \\ \left\langle \nabla\cdot(\eta_{2\rho}\mathbf{u}_2^d)\,,\, \phi\right\rangle \end{pmatrix}. \end{aligned} \tag{4.10}$$

In light of Lemma 2.1, (4.9) and (4.7) lead us

$$\begin{aligned}
&\left|F_{\bar{\alpha}} \circ F_{\mathbf{E}}(\bar{\alpha}_1) - F_{\bar{\alpha}} \circ F_{\mathbf{E}}(\bar{\alpha}_2)\right| = \left|F_{\bar{\alpha}}\left(\mathbf{z}_{\bar{\alpha}_1}, r_{\bar{\alpha}_1}\right) - F_{\bar{\alpha}}\left(\mathbf{z}_{\bar{\alpha}_2}, r_{\bar{\alpha}_2}\right)\right| \\
&= \left| \begin{pmatrix} \langle -\mu\triangle(\eta_\rho \mathbf{u}_1^s) + \nabla(\eta_\rho p_1^s)\,,\, \mathbf{z}_{\bar{\alpha}_1} - \mathbf{z}_{\bar{\alpha}_2} \rangle \\ \langle -\mu\triangle(\eta_\rho \mathbf{u}_2^s) + \nabla(\eta_\rho p_2^s)\,,\, \mathbf{z}_{\bar{\alpha}_1} - \mathbf{z}_{\bar{\alpha}_2} \rangle \end{pmatrix}^T \bar{D} - \begin{pmatrix} \langle \nabla\cdot(\eta_\rho \mathbf{u}_1^s)\,,\, r_{\bar{\alpha}_1} - r_{\bar{\alpha}_2} \rangle \\ \langle \nabla\cdot(\eta_\rho \mathbf{u}_2^s)\,,\, r_{\bar{\alpha}_1} - r_{\bar{\alpha}_2} \rangle \end{pmatrix}^T \bar{D} \right| \\
&= \left| \begin{pmatrix} \mu\,\langle \nabla(\eta_\rho \mathbf{u}_1^s)\,,\, \nabla\left(\mathbf{z}_{\bar{\alpha}_1} - \mathbf{z}_{\bar{\alpha}_2}\right) \rangle \\ \mu\,\langle \nabla(\eta_\rho \mathbf{u}_2^s)\,,\, \nabla\left(\mathbf{z}_{\bar{\alpha}_1} - \mathbf{z}_{\bar{\alpha}_2}\right) \rangle \end{pmatrix}^T \bar{D} - \begin{pmatrix} \langle \eta_\rho p_1^s\,,\, \nabla\cdot\left(\mathbf{z}_{\bar{\alpha}_1} - \mathbf{z}_{\bar{\alpha}_2}\right) \rangle \\ \langle \eta_\rho p_2^s\,,\, \nabla\cdot\left(\mathbf{z}_{\bar{\alpha}_1} - \mathbf{z}_{\bar{\alpha}_2}\right) \rangle \end{pmatrix}^T \bar{D} \right. \\
&\qquad \left. - \begin{pmatrix} \langle \nabla\cdot(\eta_\rho \mathbf{u}_1^s)\,,\, r_{\bar{\alpha}_1} - r_{\bar{\alpha}_2} \rangle \\ \langle \nabla\cdot(\eta_\rho \mathbf{u}_2^s)\,,\, r_{\bar{\alpha}_1} - r_{\bar{\alpha}_2} \rangle \end{pmatrix}^T \bar{D} \right|.
\end{aligned}$$

We now choose $\mathbf{v} = \eta_\rho \mathbf{u}_i^s$ and $\phi = \eta_\rho p_i^s$ in (4.10) to derive

$$\begin{aligned}
&\left|F_{\bar{\alpha}} \circ F_{\mathbf{E}}(\bar{\alpha}_1) - F_{\bar{\alpha}} \circ F_{\mathbf{E}}(\bar{\alpha}_2)\right| \\
&= \left| \begin{pmatrix} (\bar{\alpha}_1 - \bar{\alpha}_2) \begin{pmatrix} \left\langle -\mu\triangle(\eta_{2\rho}\mathbf{u}_1^d) + \nabla(\eta_{2\rho}p_1^d)\,,\, \eta_\rho \mathbf{u}_1^s \right\rangle + \left\langle \nabla\cdot(\eta_{2\rho}\mathbf{u}_1^d)\,,\, \eta_\rho p_1^s \right\rangle \\ \left\langle -\mu\triangle(\eta_{2\rho}\mathbf{u}_2^d) + \nabla(\eta_{2\rho}p_2^d)\,,\, \eta_\rho \mathbf{u}_1^s \right\rangle + \left\langle \nabla\cdot(\eta_{2\rho}\mathbf{u}_2^d)\,,\, \eta_\rho p_1^s \right\rangle \end{pmatrix} \\ (\bar{\alpha}_1 - \bar{\alpha}_2) \begin{pmatrix} \left\langle -\mu\triangle(\eta_{2\rho}\mathbf{u}_1^d) + \nabla(\eta_{2\rho}p_1^d)\,,\, \eta_\rho \mathbf{u}_2^s \right\rangle + \left\langle \nabla\cdot(\eta_{2\rho}\mathbf{u}_1^d)\,,\, \eta_\rho p_2^s \right\rangle \\ \left\langle -\mu\triangle(\eta_{2\rho}\mathbf{u}_2^d) + \nabla(\eta_{2\rho}p_2^d)\,,\, \eta_\rho \mathbf{u}_2^s \right\rangle + \left\langle \nabla\cdot(\eta_{2\rho}\mathbf{u}_2^d)\,,\, \eta_\rho p_2^s \right\rangle \end{pmatrix} \end{pmatrix}^T \bar{D} \right|
\end{aligned}$$

and then the right hand side terms of above equation are identically zero by Lemma 2.1, because of the distinct support of η_ρ and $\eta_{2\rho}$. So we conclude

$$\left|F_{\bar{\alpha}} \circ F_{\mathbf{E}}(\bar{\alpha}_1) - F_{\bar{\alpha}} \circ F_{\mathbf{E}}(\bar{\alpha}_2)\right| = \mathbf{0}.$$

and arrive (4.8) and finish this proof. □

Lemma 4.2 intend the existence and uniqueness of solution of (4.3) and we can deduce the following lemma from Lemma 4.1.

Lemma 4.3 (Properties of the solution $(\mathbf{z}, r)$ of (4.3)). *Let $(\mathbf{z}, r)$ be the solutions of (4.3). Then there is a singular function representation*

$$\mathbf{z} = \mathbf{w}_{\mathbf{z}} + \alpha_{\mathbf{z},1}\eta_\rho \mathbf{u}_1^s + \alpha_{\mathbf{z},2}\eta_\rho \mathbf{u}_2^s \quad \textit{and} \quad r = q_{\mathbf{z}} + \alpha_{\mathbf{z},1}\eta_\rho p_1^s + \alpha_{\mathbf{z},2}\eta_\rho p_2^s, \tag{4.11}$$

where $\mathbf{w}_{\mathbf{z}} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$, $q_{\mathbf{z}} \in H^1(\Omega) \cap L_0^2(\Omega)$ and $(\alpha_{\mathbf{z},1}, \alpha_{\mathbf{z},2})$ satisfy the regularity estimate

$$\|\mathbf{w}_{\mathbf{z}}\|_2 + \|q_{\mathbf{z}}\|_1 + |\alpha_{\mathbf{z},1}| + |\alpha_{\mathbf{z},2}| \le C\|\mathbf{E}\|_0. \tag{4.12}$$

We also have

$$\|\mathbf{z}\|_1 + \|r\|_0 \le C_1\|\mathbf{E}\|_0. \tag{4.13}$$

Proof. The equations (4.3) can be rewritten by the form of (4.6) as

$$\begin{aligned}
-\mu\triangle\mathbf{z} + \nabla r &= \mathbf{E} - (\bar{\alpha}_1, \bar{\alpha}_2)\cdot \begin{pmatrix} -\mu\triangle(\eta_{2\rho}\mathbf{u}_1^d) + \nabla(\eta_{2\rho}p_1^d) \\ -\mu\triangle(\eta_{2\rho}\mathbf{u}_2^d) + \nabla(\eta_{2\rho}p_2^d) \end{pmatrix} \\
\nabla\cdot\mathbf{z} &= (\bar{\alpha}_1, \bar{\alpha}_2)\cdot \begin{pmatrix} \nabla\cdot\left(\eta_{2\rho}\mathbf{u}_1^d\right) \\ \nabla\cdot\left(\eta_{2\rho}\mathbf{u}_2^d\right) \end{pmatrix}.
\end{aligned} \tag{4.14}$$

The right hand side terms in (4.14) are in $\mathbf{L}^2(\Omega)$ and $H^1(\Omega) \cap L^2_0(\Omega)$ for the first and the second equations, respectively, which come from (2.1) and (3.5). So the solution $(\mathbf{z}, r)$ can be represented by the form of (4.11) from Lemma 4.1 and

$$\|\mathbf{w_z}\|_2 + \|q_\mathbf{z}\|_1 + |\alpha_{\mathbf{z},1}| + |\alpha_{\mathbf{z},2}| \leq C\|-\mu\triangle\mathbf{z} + \nabla r\|_0 + \|\nabla\cdot\mathbf{z}\|_0.$$

According to (4.14), Lemma 2.1 gives us

$$\begin{aligned}\|\mathbf{w_z}\|_2 + \|q_\mathbf{z}\|_1 + |\alpha_{\mathbf{z},1}| + |\alpha_{\mathbf{z},2}| \leq & C\left\|\mathbf{E} - (\bar\alpha_1, \bar\alpha_2)\cdot\begin{pmatrix} -\mu\triangle(\eta_{2\rho}\mathbf{u}_1^d) + \nabla(\eta_{2\rho}p_1^d) \\ -\mu\triangle(\eta_{2\rho}\mathbf{u}_2^d) + \nabla(\eta_{2\rho}p_2^d)\end{pmatrix}\right\|_0 \\ & + C\left\|(\bar\alpha_1, \bar\alpha_2)\cdot\begin{pmatrix}\nabla\cdot\left(\eta_{2\rho}\mathbf{u}_1^d\right) \\ \nabla\cdot\left(\eta_{2\rho}\mathbf{u}_2^d\right)\end{pmatrix}\right\|_1 \\ \leq & C\left(\|\mathbf{E}\|_0 + |\bar\alpha_1| + |\bar\alpha_2|\right).\end{aligned}$$

Because $|\bar\alpha_i| = |(c_i(\mathbf{z}) - d_i(r))| \leq C\left(\|\mathbf{z}\|_0 + \|r\|_0\right)$, (4.8) leads $|\bar\alpha_i| \leq C\|\mathbf{E}\|_0$ and (4.12). We also readily get (4.13) and it is the proof. □

If we denote

$$\mathbf{G} := \mathbf{z} - \mathcal{I}_h\mathbf{z} \quad \text{and} \quad g := r - \mathcal{I}_h r,$$

then we have, by Lemma 4.3,

$$\|\mathbf{G}\|_1 + \|g\|_0 \leq Ch^\lambda\|\mathbf{E}\|_0, \tag{4.15}$$

because of $\|\eta_\rho\mathbf{u}_s - \mathcal{I}_h\left(\eta_\rho\mathbf{u}_s\right)\|_1 + \|\eta_\rho p_s - \mathcal{I}_h\left(\eta_\rho p_s\right)\|_0 \leq Ch^\lambda$.

Remark 4.4 (The reason of sub-optimality). *The inequality* (4.15) *is the main restriction to get optimal accuracy in the next lemma and the reason of sub-optimality* $|\alpha - \alpha_h| + \|\mathbf{w} - \mathbf{w}_h\|_0 \leq Ch^{1+\lambda}$ *in Theorem 2.*

We start to prove Theorem 2 by the following Lemmas 4.5 and 4.6.

Lemma 4.5 (Estimate $\|\mathbf{E}\|_0$). *Let Assumption 1 hold. Then we have*

$$\|\mathbf{E}\|_0 \leq Ch^\lambda\left(\|\mathbf{E}\|_1 + \|e\|_0\right), \tag{4.16}$$

$$|\varepsilon_i| \leq C\|\mathbf{E}\|_0 \quad \textit{and} \quad \|e\|_0 \leq C\left(\|\mathbf{E}\|_1 + h\|q\|_1\right). \tag{4.17}$$

Proof. We start this proof with constructing error equations by subtracting (2.15) from (2.14) to get

$$\begin{aligned}\mu\left\langle\nabla\mathbf{E}\,,\,\nabla\mathbf{v}_h\right\rangle + \left\langle\nabla e\,,\,\mathbf{v}_h\right\rangle + \sum_{i=1}^2\left(A_i(\mathbf{E}) + B_i(e)\right)c_i(\mathbf{v}_h) = 0, \\ \left\langle\nabla\cdot\mathbf{E}\,,\,\phi_h\right\rangle + \sum_{i=1}^2\left(A_i(\mathbf{E}) + B_i(e)\right)d_i(\phi_h) = 0.\end{aligned} \tag{4.18}$$

We first prove (4.17). From the second equation in (4.18), we have

$$\left|\sum_{i=1}^{2} B_i(e)d_i(\phi_h)\right| = \left|\langle \mathbf{E}\,,\,\nabla\phi_h\rangle - \sum_{i=1}^{2} A_i(\mathbf{E})d_i(\phi_h)\right|.$$

We fix $\phi_h = C_i x$ with $C_i \in \mathbb{R}$, then $\|\nabla\phi_h\|_0 = |C_i||\Omega|^{1/2}$ are bounded numbers, because the space $\mathbb{P}_h$ is composed of continuous functions. So we can readily obtain

$$|B_i(e)| \le C\|\mathbf{E}\|_0. \tag{4.19}$$

Therefore $\varepsilon_i = A_i(\mathbf{E}) + B_i(e)$ which comes from subtracting (2.13) from (2.11) yields

$$|\varepsilon_i| \le C\left(\|\mathbf{E}\|_0 + |B_i(e)|\right) \le C\|\mathbf{E}\|_0,$$

and, in light of (4.18), Assumption 1 leads

$$\begin{aligned}\gamma\|e_h\|_0 &\le \sup_{v_h\in\mathbb{V}_h} \frac{\mu\langle\nabla\mathbf{E}\,,\,\nabla\mathbf{v}_h\rangle + \langle\mathcal{I}_h e\,,\,\nabla\cdot\mathbf{v}_h\rangle + \sum_{i=1}^{2}\left(A_i(\mathbf{E}) + B_i(e)\right)c_i(\mathbf{v}_h)}{\|\mathbf{v}_h\|_1}\\ &\le C\left(\|\mathbf{E}\|_1 + h\|q\|_1 + \|\mathbf{E}\|_0 + \sum_{i=1}^{2}|B_i(e)|\right).\end{aligned}$$

Thus, in conjunction with (4.19), we arrive at (4.17). We now prove (4.16) with choosing $\mathbf{v}_h = \mathcal{I}_h\mathbf{z} = \mathbf{z} - \mathbf{G}$ and $\phi = \mathcal{I}_h r = r - g$ in (4.18):

$$\begin{aligned}\mu\langle\nabla\mathbf{E}\,,\,\nabla(\mathbf{z}-\mathbf{G})\rangle + \langle\nabla e\,,\,\mathbf{z}-\mathbf{G}\rangle + \sum_{i=1}^{2}\left(A_i(\mathbf{E}) + B_i(e)\right)c_i(\mathbf{z}-\mathbf{G}) &= 0,\\ \langle\nabla\cdot\mathbf{E}\,,\,r-g\rangle + \sum_{i=1}^{2}\left(A_i(\mathbf{E}) + B_i(e)\right)d_i(r-g) &= 0,\end{aligned} \tag{4.20}$$

And then we choose $\mathbf{v} = \mathbf{E}$ and $\phi = e$ in (4.3) to get

$$\begin{aligned}\mu\langle\nabla\mathbf{z}\,,\,\nabla\mathbf{E}\rangle + \langle\nabla r\,,\,\mathbf{E}\rangle + \sum_{i=1}^{2}\left(c_i(\mathbf{z}) - d_i(r)\right)A_i(\mathbf{E}) &= \langle\mathbf{E}\,,\,\mathbf{E}\rangle,\\ \langle\nabla\cdot\mathbf{z}\,,\,e\rangle - \sum_{i=1}^{2}\left(c_i(\mathbf{z}) - d_i(r)\right)B_i(e) &= 0.\end{aligned} \tag{4.21}$$

We now replace $\langle\nabla e\,,\,\mathbf{z}\rangle$ at the first equation in (4.20) with the second equation in (4.21) to obtain

$$\begin{aligned}\mu\langle\nabla\mathbf{E}\,,\,\nabla(\mathbf{z}-\mathbf{G})\rangle - \langle\nabla e\,,\,\mathbf{G}\rangle + \sum_{i=1}^{2}\left(A_i(\mathbf{E}) + B_i(e)\right)c_i(\mathbf{z}-\mathbf{G})&\\ - \sum_{i=1}^{2}\left(c_i(\mathbf{z}) - d_i(r)\right)B_i(e) &= 0.\end{aligned} \tag{4.22}$$

By the same manner, we replace $\langle \nabla r \,,\, \mathbf{E} \rangle$ at the first equation in (4.21) with the second equation in (4.20)

$$\begin{aligned} \mu \langle \nabla \mathbf{z} \,,\, \nabla \mathbf{E} \rangle - \langle \nabla \cdot \mathbf{E} \,,\, g \rangle + \sum_{i=1}^{2} (c_i(\mathbf{z}) - d_i(r))\, A_i(\mathbf{E}) \\ + \sum_{i=1}^{2} (A_i(\mathbf{E}) + B_i(e))\, d_i(r - g) = \langle \mathbf{E} \,,\, \mathbf{E} \rangle \,. \end{aligned} \tag{4.23}$$

In light of (4.15), subtracting (4.22) from (4.23) yields

$$\begin{aligned} \|\mathbf{E}\|_0^2 &= \mu \langle \nabla \mathbf{E} \,,\, \nabla \mathbf{G} \rangle - \langle \nabla \cdot \mathbf{E} \,,\, g \rangle - \langle e \,,\, \nabla \cdot \mathbf{G} \rangle + \sum_{i=1}^{2} (A_i(\mathbf{E}) + B_i(e))\, (c_i(\mathbf{G}) - d_i(g)) \\ &\le C h^{\lambda} \left(\|\nabla \mathbf{E}\|_0 + \|\nabla \cdot \mathbf{E}\|_0 + \|e\|_0 \right) \|\mathbf{E}\|_0. \end{aligned}$$

Therefore we arrive at (4.16) and finish the proof of the theorem. □

We now estimate error in $\mathbf{H}_0^1(\Omega)$ space.

Lemma 4.6 (Estimate $\|\mathbf{E}\|_1 + \|e\|_0$)**.** *Let Assumption 1 hold. If the mesh size h be small enough, then we have*

$$\|\mathbf{E}\|_1 + \|e\|_0 \le Ch. \tag{4.24}$$

Proof. We choose $\mathbf{v}_h = \mathbf{E}_h = \mathcal{I}_h \mathbf{w} - \mathbf{w}_h = \mathbf{E} - \mathcal{I}_h \mathbf{E} \in \mathbb{V}_h$ and $\phi_h = e_h = \mathcal{I}_h q - q_h = e - \mathcal{I}_h e \in \mathbb{P}_h$ in (4.18), then we have

$$\begin{aligned} \mu \langle \nabla \mathbf{E} \,,\, \nabla (\mathbf{E} - \mathcal{I}_h \mathbf{E}) \rangle + \langle \nabla e \,,\, \mathbf{E} - \mathcal{I}_h \mathbf{E} \rangle + \sum_{i=1}^{2} (A_i(\mathbf{E}) + B_i(e))\, c_i(\mathbf{E}_h) = 0, \\ \langle \nabla \cdot \mathbf{E} \,,\, e - \mathcal{I}_h e \rangle + \sum_{i=1}^{2} (A_i(\mathbf{E}) + B_i(e))\, d_i(e_h) = 0. \end{aligned} \tag{4.25}$$

And then we replace $\langle \nabla e \,,\, \mathbf{E} \rangle$ in the first equation with the second equation in (4.25) to have

$$\begin{aligned} \mu \langle \nabla \mathbf{E} \,,\, \nabla (\mathbf{E} - \mathcal{I}_h \mathbf{E}) \rangle - \langle \nabla e \,,\, \mathcal{I}_h \mathbf{E} \rangle - \langle \nabla \cdot \mathbf{E} \,,\, \mathcal{I}_h e \rangle \\ + \sum_{i=0}^{2} (A_i(\mathbf{E}) + B_i(e))\, (c_i(\mathbf{E}_h) + d_i(e_h)) = 0. \end{aligned}$$

In conjunction with Lemma 4.5 and $|B_i(e)| \le C\|\mathbf{E}\|_0$ in (4.19), (4.2) yields

$$\begin{aligned}
\mu\|\nabla\mathbf{E}\|_0^2 \le& C\Big(\|\nabla\mathbf{E}\|_0\|\nabla\mathcal{I}_h\mathbf{E}\|_0 + \|e\|_0\|\nabla\cdot\mathcal{I}_h\mathbf{E}\|_0 \\
&+ \|\mathcal{I}_h e\|_0\|\nabla\cdot\mathbf{E}\|_0 + \|\mathbf{E}\|_0\left(\|\mathbf{E}_h\|_0 + \|e_h\|_0\right)\Big) \\
\le& Ch\Big(\left(\|\nabla\mathbf{E}\|_0 + \|e\|_0\right)\|\mathbf{w}\|_2 + \|\nabla\cdot\mathbf{E}\|_0\|q\|_1\Big) \\
&+ Ch^\lambda\left(\|\mathbf{E}\|_1 + \|e\|_0\right)\left(\|\mathbf{E}_h\|_0 + \|e_h\|_0\right) \\
\le& Ch\Big(\left(\|\nabla\mathbf{E}\|_0 + h\|q\|_1\right)\|\mathbf{w}\|_2 + \|\nabla\cdot\mathbf{E}\|_0\|q\|_1\Big) + Ch^\lambda\left(\|\mathbf{E}\|_1 + h\|q\|_1\right)^2 \\
\le& \frac{\mu}{2}\|\nabla\mathbf{E}\|_0^2 + Ch^2\left(\|\mathbf{w}\|_2^2 + \|q\|_1^2\right).
\end{aligned}$$

We note that the last inequality comes from assumption of small enough h. Finally, we arrive at (4.24) by combining with (4.17) and complete this proof. □

Finally we arrive at the following error estimates from the above lemmas.

Theorem 2 (Error estimates). *Let Assumption 1 hold and* $\mathbf{f} \in \mathbf{L}^2(\Omega)$. *If h is small enough, then we have*

$$\sum_{i=1}^{2}|\alpha_i - \alpha_{ih}| + \|\mathbf{w}-\mathbf{w}_h\|_0 \le Ch^{1+\lambda},$$
$$\|\mathbf{w}-\mathbf{w}_h\|_1 + \|q-q_h\|_0 \le Ch.$$

5. SOME OTHER FORMULATIONS OF ALGORITHMS

Algorithm 1 is an applicable solver by using the Sherman-Morrison-Woodbury formula:

$$\left(A + \sum_{k=1}^{N} U_k V_k^T\right)^{-1} = A^{-1} - A^{-1}[U_1, U_2, \cdots, U_N]M^{-1}[V_1^T, V_1^T, \cdots, V_N^T]^T A^{-1},$$

where M is $Nm \times Nm$ matrix given by

$$M = \begin{bmatrix} I_{m\times m} + V_1^T A^{-1}U_1 & V_1 A^{-1}U_2 & \cdots & V_1^T A{-}1U_N \\ V_2^T A{-}1U_1 & I_{m\times m} + V_2^T A^{-1}U_2 & \cdots & V_2 A^{-1}U_N \\ \vdots & \vdots & \ddots & \vdots \\ V_N^T A{-}1U_1 & V_N^T A^{-1}U_2 & \cdots & I_{m\times m} + V_N A^{-1}U_N \end{bmatrix}.$$

This method is a good solver, but a difficulty arise from computation on many reentrant corner domain, because many singular functions are involved for big number N. Corollary 1 says that the stress intensity factor $\boldsymbol{\alpha}$ does not depend on solution $(\mathbf{w}_{\boldsymbol{\alpha}}, q_{\boldsymbol{\alpha}})$ and so we readily obtain following algorithm:

Algorithm 2 (Explicit FE-DSFM). Set α_{1h} and α_{2h} in $\mathbb{R}$, simply $\alpha_{1h} = \alpha_{2h} = 0$. Compute $(\mathbf{w}_h, q_h)$ as the solution of (2.12). Repeat for $N = 1$ or 2,

Step 1: Update α_{1h} and $\alpha_{2h} \in \mathbb{R}$ by computing (2.13),
Step 2: Find $(\mathbf{w}_h, q_h)$ again as the solution of (2.12).

Also we can derive an iteration algorithm to solve the system (2.12) and (2.13) by defining $\left(\mathbf{w}_h^0, q_h^0\right)$ as solution of (2.12) with $\alpha_{1h} = \alpha_{2h} = 0$. It means that $\left(\mathbf{w}_h^0, q_h^0\right)$ is the solution of the standard mixed method of (1.1) and includes corner singularities. In light of (2.2), we can remove these singularities by the following iteration steps:

$$\begin{pmatrix} \mathbf{w}_h^{n+1} \\ q_h^{n+1} \end{pmatrix} = \begin{pmatrix} \mathbf{w}_h^0 \\ q_h^0 \end{pmatrix} + \alpha_{1h}^n \begin{pmatrix} \eta_\rho \mathbf{u}_1^s \\ \eta_\rho p_1^s \end{pmatrix} + \alpha_{2h}^n \begin{pmatrix} \eta_\rho \mathbf{u}_2^s \\ \eta_\rho p_2^s \end{pmatrix}.$$

However, the iterative solution $\left(\mathbf{w}_h^{n+1}, q_h^{n+1}\right)$ still include singularities, provided that α_{1h}^n and α_{2h}^n are not exact solutions. So we hire the Stokes projection $\left(\mathbf{z}_h^i, \kappa_h^i\right)$ where $i = 1, 2$ of the singular functions:

$$\begin{aligned} \mu \left\langle \nabla \mathbf{z}_h^i \,, \nabla \mathbf{v} \right\rangle + \left\langle \nabla \kappa_h^i \,, \mathbf{v} \right\rangle &= \left\langle -\mu \triangle (\eta_\rho \mathbf{u}_{ih}^s) + \nabla (\eta_\rho p_{ih}^s) \,, \mathbf{v} \right\rangle, \\ \left\langle \nabla \cdot \mathbf{z}_h^i \,, \phi \right\rangle &= \left\langle \nabla \cdot (\eta_\rho \mathbf{u}_{ih}^s) \,, \phi \right\rangle. \end{aligned} \tag{5.1}$$

Then we can get a new algorithm

Algorithm 3 (Iterative explicit FE-DSFM)**.** Compute $(\mathbf{w}_h, q_h)$ and the stress intensity factors α_{1h} and $\alpha_{2h} \in \mathbb{R}$ by calculating following steps:

Step 1: Find discrete singular functions $\left(\mathbf{z}_h^i, \kappa_h^i\right)$ by solving (5.1).
Step 2: Find $\left(\mathbf{w}_h^0, q_h^0\right)$ as the solution of (2.12) with $\alpha_{1h}^0 = \alpha_{2h}^0 = 0$.
Step 3: (iteration step) Iterate for $n = 0, 1, 2, \cdots$ until $|\alpha_{ih}^{n+1} - \alpha_{ih}^n| \leq$tolerance: update α_{1h}^{n+1} and $\alpha_{2h}^{n+1} \in \mathbb{R}$ by using (2.13) with $\left(\mathbf{w}_h^n, q_h^n\right)$ and then calculate $\left(\mathbf{w}_h^{n+1}, q_h^{n+1}\right)$ by addition

$$\left(\mathbf{w}_h^{n+1}, q_h^{n+1}\right) = \left(\mathbf{w}_h^0, q_h^0\right) - \alpha_{1h}^{n+1} \left(\mathbf{z}_h^1, \kappa_h^1\right) - \alpha_{2h}^{n+1} \left(\mathbf{z}_h^2, \kappa_h^2\right).$$

The main advantage of Algorithms 2 and 3 is easier application than Algorithm 1. Because we do not need to solve linear equation at the iteration step 3 in Algorithm 3, computational cost is not higher than other algorithms. Algorithms 2 and 3 also request linear solver totally 3 times.

6. Numerical test

In this section, we document the computational performance of each algorithm within a polygonal domain with reentrant corners.

Example 1. We consider the computational domain is Γ shape $([-1,1] \times [-1,1]) \setminus ([0,1] \times [-1,0])$. So, in this experiment, $\omega = 1.5\pi$ and the solution λ of (1.3) becomes $\lambda_1 = 0.5444837367824639$ and $\lambda_2 = 0.9085291898460987$.

Let the solution be given by

$$u = -\sin^2(\pi x)\sin(2\pi y) + 2u_1^s - 3u_2^s,$$
$$v = \sin(2\pi x)\sin^2(\pi y) + 2v_1^s - 3v_2^s,$$
$$p = (2+\cos(\pi x))(2+\cos(\pi y)) - 4 + 2p_1^s - 3p_2^s.$$

We note that the solution for velocity has not vanished on Γ_{out}. The forcing term $\mathbf{f}$ is determined accordingly for any μ; here $\mu = 1$. In order to impose FE-DSFM, we choose the cut-off function $\eta_\rho \in H^3(\Omega)$ as

$$\eta_\rho = \begin{cases} 1, & \text{in } B(\frac{1}{2}\rho R), \\ \frac{1}{32}\left(16 - 35\psi + 35\psi^3 - 21\psi^5 + 5\psi^7\right), & \text{in } B(\frac{1}{2}\rho R; \rho R), \\ 0, & \text{in } \Omega \setminus \bar{B}(\rho R), \end{cases}$$

with $\psi = \frac{4r}{\rho R} - 3$ with $R = 1$. Then the solution $(\mathbf{u}, p)$ can be rewritten by

$$\mathbf{u} = \mathbf{w} + 2\eta_\rho \mathbf{u}_1^s - 3\eta_\rho \mathbf{u}_2^s,$$
$$p = q + 2\eta_\rho p_1^s - 3\eta_\rho p_2^s,$$

where $(\mathbf{w}, q)$ is the regular part of the solution. We note that the regularities of $\mathbf{w} = \mathbf{u} - 2\eta_\rho \mathbf{u}_1^s + 3\eta_\rho \mathbf{u}_2^s$ and $q = p - 2\eta_\rho p_1^s + 3\eta_\rho p_2^s$ are equal to that of η_ρ, and so $(\mathbf{w}, q) \in \mathbf{H}^3(\Omega) \times H^3(\Omega)$ in this example. Computations are carried out with the Taylor-Hood $(\mathcal{P}^2, \mathcal{P}^1)$ finite element pair on the uniform meshes of size h. In this example, we choose $\mu = 1$ and $\rho = 0.4$. All numerical integration is used by 6 points quadrature rule.

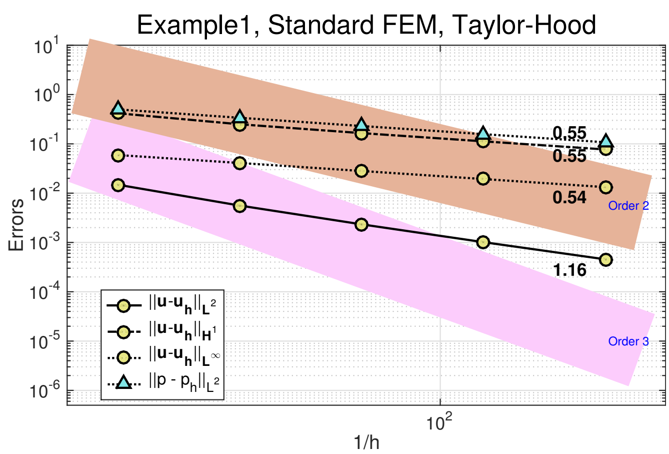

FIGURE 1. Error decay for the standard FEM with Taylor-Hood elements

The Figure 1 is error decays of Example 1 by using standard finite element method with mini element and Talyor-Hood element, respectively. We cannot get optimal accuracy in both tests by the standard technique, because of the corner singularity.

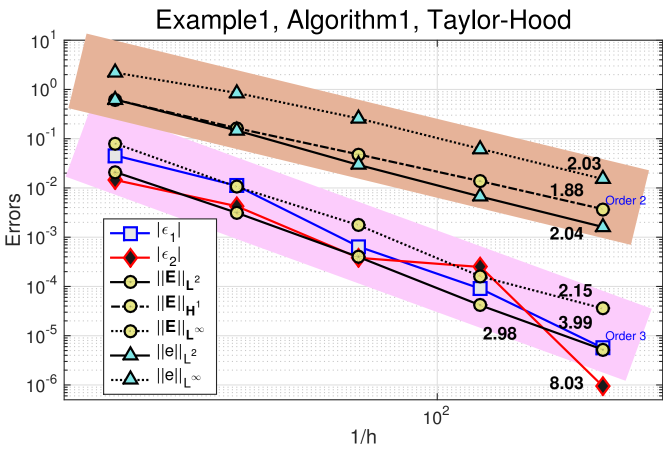

FIGURE 2. Error decay for the Algorithm 1 with Taylor-Hood elements

In contrast, Algorithm 1 using Sherman-Morrison formular displays optimal convergence in Figures 2. We note that the decay behavior of both stress intensity factors ϵ_1 and ϵ_2 are a little bit irregular and these phenomenon due to accuracy of numerical integration of $\mathbf{w}$ to compute α.

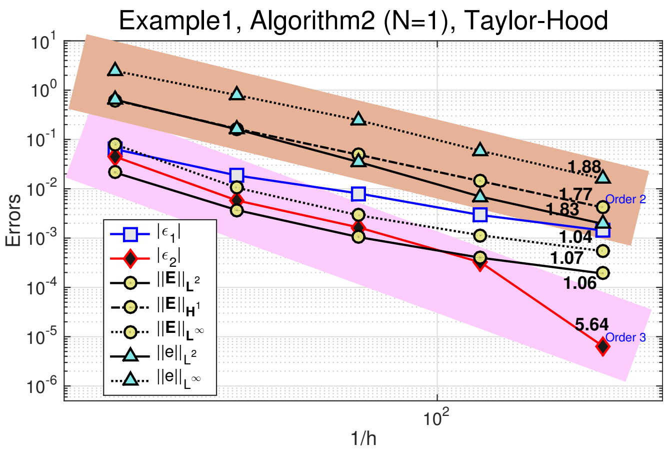

FIGURE 3. Error decay for the Algorithm 2 with $N = 1$ and Taylor-Hood elements

From now, we perform mesh analysis of Algorithm 2. This method has to show theoretically optimal accuracy with $N = 1$, where N is the number of linear solver, but it is not true in numerical experiments, because of losing order of given initial data. So we can not get our desire accuracy for the case $N = 1$ in Figures 3. However we get similar optimal results with those of Algorithm 1 for the case $N = 2$ in Figures 4.

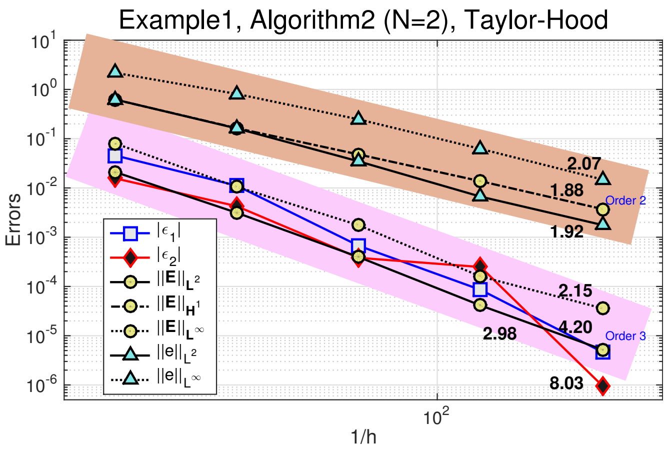

FIGURE 4. Error decay for the Algorithm 2 with $N = 2$ and Taylor-Hood elements

The main difference between Algorithms 1 and 3 is that the former hires iterative solver to compute $\mathbf{w}_h$ using Corollary 1 and the later use Sherman-Morrison formular. We employ tolerance $10e - 08$ in Figures 5. We note that the iteration in Algorithm 3 does not need to

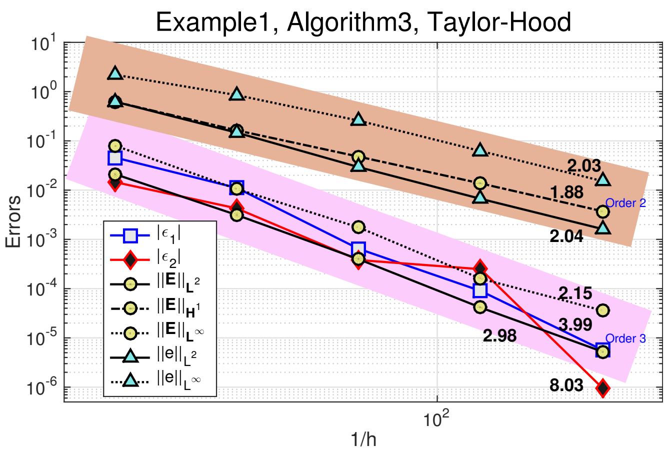

FIGURE 5. Error decay for the Algorithm 3 with Taylor-Hood elements

apply linear solver and both Algorithms 1 and 3 error decay results are almost same.

Figures 6 displays convergence rate for iteration steps in Algorithm 3. Here, $y-$axis value means $\sum_{i=1}^{2} |\alpha_{ih}^{n+1} - \alpha_{ih}^{n}|$. Figure 6 shows that the difference converges to 0 and decay rates are very regular. Moreover if h is smaller, then fewer iteration is required to arrive at the tolerance value.

Example 2. We carry out mesh analysis on a mutiple sigularity domain in Figure 7. In this experiment, we set $\omega = 1.5\pi$, $\lambda_1 = 0.5444837367824639$ and $\lambda_2 = 0.9085291898460987$ in

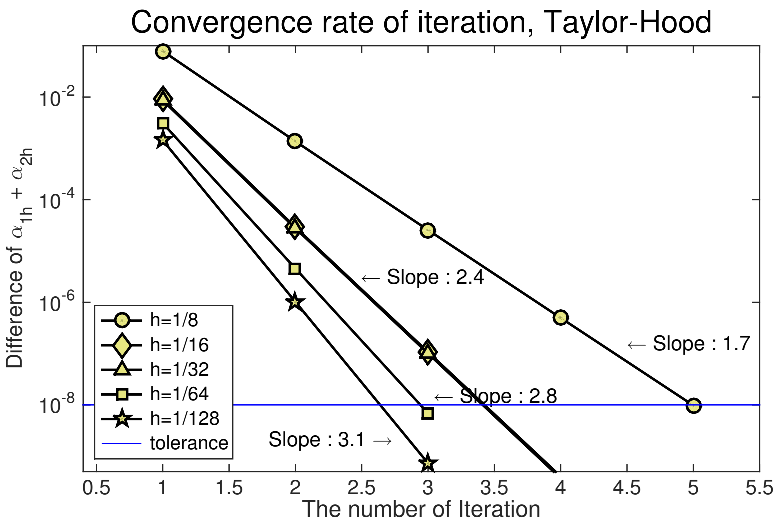

FIGURE 6. Convergence rate by iteration for Algorithm 3 with Taylor-Hood elements

both two two reentrant corners. We choose the smooth part solution as

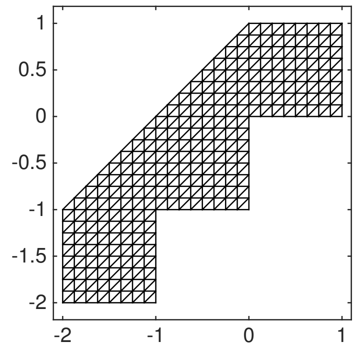

FIGURE 7. Domain and mesh shape of Example 2

$$
\begin{aligned}
w_u &= -\sin^2(\pi x)\sin(2\pi y),\\
w_v &= \sin(2\pi x)\sin^2(\pi y),\\
q &= (2+\cos(\pi x))(2+\cos(\pi y)) - 4.
\end{aligned}
$$

And we choose the solution $(\mathbf{u}, p)$

$$
\begin{aligned}
\mathbf{u} &= \mathbf{w} + \eta_{\rho 1}\mathbf{u}_1^s - 4\eta_{\rho 1}\mathbf{u}_2^s + 3\eta_{\rho 2}\mathbf{u}_3^s - 2\eta_{\rho 2}\mathbf{u}_4^s,\\
p &= q + \eta_{\rho 1}p_1^s - 4\eta_{\rho 1}p_2^s + 3\eta_{\rho 2}p_3^s - 2\eta_{\rho 2}p_4^s,
\end{aligned}
$$

where $\rho 1$ and $\rho 2$ are cut-off functions at each reentrant corner. In order to make empty intersection of supports for $\rho 1$ and $\rho 2$, we fix $\rho = 0.35$. We note $\operatorname{div} \mathbf{u} \neq 0$ in this example. Because of singular functions, we could not find divergence free exact solution with 0 boundary condition near reentrant corners.

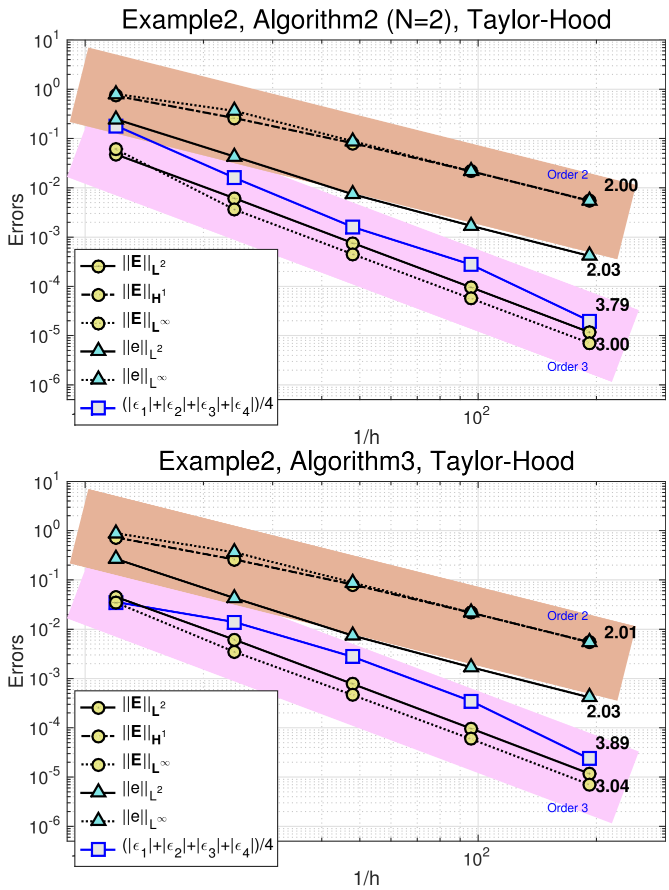

FIGURE 8. Error decay for the Algorithm 2 with $N = 2$ and 3 with Taylor-Hood elements for Example 2

In this example, the weak form of Algorithm 2 becomes as follows: For all $\mathbf{v}_h \in \mathbb{V}_h$ and $\phi_h \in \mathbb{P}_h$, find $\mathbf{v}_H$ and p_h such that

$$\mu \langle \nabla \mathbf{w}_h \,,\, \nabla \mathbf{v}_h \rangle + \langle \nabla q_h \,,\, \mathbf{v}_h \rangle + \sum_{i=1}^{4} \alpha_i c_i(\mathbf{v}_h) = \langle \mathbf{f} \,,\, \mathbf{v}_h \rangle \,,$$

$$\langle \nabla \cdot \mathbf{w}_h \,,\, \phi_h \rangle + \sum_{i=1}^{4} \alpha_i d_i(\phi_h) = 0.$$

Because $\rho 1$ and $\rho 2$ have isolated supports, Step 2 becomes

$$\alpha_{1h} = \frac{\zeta_{2,2}\left(\beta_1^{\mathbf{f}} - a_1(\mathbf{w}_h) + b_1(q_h)\right) - \zeta_{2,1}\left(\beta_2^{\mathbf{f}} - a_2(\mathbf{w}_h) + b_2(q_h)\right)}{det(D)},$$

$$\alpha_{2h} = \frac{\zeta_{1,1}\left(\beta_2^{\mathbf{f}} - a_2(\mathbf{w}_h) + b_2(q_h)\right) - \zeta_{1,2}\left(\beta_1^{\mathbf{f}} - a_1(\mathbf{w}_h) + b_1(q_h)\right)}{det(D)},$$

$$\alpha_{3h} = \frac{\zeta_{4,4}\left(\beta_3^{\mathbf{f}} - a_3(\mathbf{w}_h) + b_3(q_h)\right) - \zeta_{4,3}\left(\beta_4^{\mathbf{f}} - a_4(\mathbf{w}_h) + b_4(q_h)\right)}{det(D)},$$

$$\alpha_{4h} = \frac{\zeta_{3,3}\left(\beta_3^{\mathbf{f}} - a_4(\mathbf{w}_h) + b_4(q_h)\right) - \zeta_{3,4}\left(\beta_3^{\mathbf{f}} - a_3(\mathbf{w}_h) + b_3(q_h)\right)}{det(D)}.$$

By the same manner, we also can apply more complicated problems.

Figures 8 is the result of Example 2 of Algorithm 2 with $N = 2$ and 3. We can get optimal results.

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