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AN INTRODUCTION TO ϵ_0-DENSITY AND ϵ_0-DENSE ACE

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Abstract. In this paper, we introduce a concept of the ϵ_0- limits of vector and multiple valued sequences in R^m. Using this concept, we study about the concept of the ϵ_0-dense subset and of the points of ϵ_0-dense ace in the open subset of R^m. We also investigate the properties and the characteristics of the ϵ_0-dense subsets and of the points of ϵ_0-dense ace.

1. Introduction

In this section, we introduce a concept of the ϵ_0-limits of vector and multiple valued sequences in R^m. And we study some properties of this ϵ_0-limit which we need later.

Definition 1.1. Let $\{x_n\}$ be a vector-valued and multi-valued infinite sequence of elements of R^m. And let $\epsilon_0 \geq 0$ be any, but fixed, non-negative real number. For a set S, if the following condition is satisfied, we call that the ϵ_0- limit of $\{x_n\}$ as n converges to ∞ is S, and we denote it by $\boxed{\epsilon_0 - \lim_{n\longrightarrow\infty}} x_n = S$: S is the set of all vectors $\alpha \in R^m$ such that

$$\forall\epsilon > \epsilon_0, \exists K \in N \ s.t.(\forall n \in N)n \geq K, (\forall x_n) \Rightarrow \|x_n - \alpha\| < \epsilon.$$

Definition 1.2. For a multi-valued infinite sequence $\{x_n\}$ of vectors in R^m, we call that $\{x_n\}$ is ultimately bounded if and only if there exist two real numbers K and M such that $(\forall n \in N)n \geq K, \forall x_n \Rightarrow \|x_n\| \leq M$.

For the ϵ_0- limit, we have the following representation lemma.

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Lemma 1.3. *(Representation)* *Let $\{x_n\}$ be a vector-valued and multi-valued infinite sequence of elements of R^m. And let $\epsilon_0 \geq 0$ be any, but fixed, non-negative real number. Suppose that $\{x_n\}$ is ultimately bounded. If $\boxed{\epsilon_0 - \lim_{n \longrightarrow \infty}} x_n = S \neq \emptyset$ then S is a compact and convex subset of R^m such that $S = \bigcap_{\alpha \in SSL} \overline{B}(\alpha, \epsilon_0)$. Here $\overline{B}(\alpha, \epsilon_0)$ denotes the closed ball $\overline{B}(\alpha, \epsilon_0) = \{x \in R^m | \|x - \alpha\| \leq \epsilon_0\}$ and*

$$SSL = \{\alpha \in R^m | \exists \{x_{n_k}\} \leq \{x_n\} \ s.t. \lim_{k \to \infty} x_{n_k} = \alpha\}$$

and $\{x_{n_k}\} \leq \{x_n\}$ means that $\{x_{n_k}\}$ is a single-valued subsequence of $\{x_n\}$.

Proof. ($\subseteq$) Let any element $\beta \in S$ and any member $\alpha \in SSL$ be given. Then we have

$$\forall \epsilon > \epsilon_0, \exists K_1 \in N \ s.t. (\forall n \in N) n \geq K_1, (\forall x_n) \Rightarrow \|x_n - \beta\| < \epsilon_0 + \frac{\epsilon - \epsilon_0}{2}.$$

Since $\alpha \in SSL$, there is a single-valued and convergent subsequence $\{x_{n_k}\}$ such that $\lim_{k \to \infty} x_{n_k} = \alpha$. Hence we have

$$\forall \epsilon > \epsilon_0, \exists K_2 \in N \ s.t. \ (\forall k \in N) \ k \geq K_2 \Rightarrow \|x_{n_k} - \alpha\| < \frac{\epsilon - \epsilon_0}{2}.$$

If we choose a natural number $K = \max(K_1, K_2)$ then we have

$$\begin{aligned} \|\beta - \alpha\| &= \|\beta - x_{n_K} + x_{n_K} - \alpha\| \\ &\leq \|\beta - x_{n_K}\| + \|x_{n_K} - \alpha\| \\ &< \epsilon_0 + \frac{\epsilon - \epsilon_0}{2} + \frac{\epsilon - \epsilon_0}{2} = \epsilon. \end{aligned}$$

Since $\epsilon > \epsilon_0$ was arbitrary, we have $\|\beta - \alpha\| \leq \epsilon_0$. That is, $\beta \in \overline{B}(\alpha, \epsilon_0)$. Since $\alpha \in SSL$ was arbitrary, we have $\beta \in \bigcap_{\alpha \in SSL} \overline{B}(\alpha, \epsilon_0)$. Since $\beta \in S$ was also arbitrary, S is a subset of $\bigcap_{\alpha \in SSL} \overline{B}(\alpha, \epsilon_0)$. ($\supseteq$) In order to show the opposite inclusion, let $\beta \notin S$ be any element of $R^m - S$. Then we have

$$\exists \epsilon_1 > \epsilon_0 \ s.t. (\forall k \in N, \exists n_k \in N, \exists x_{n_k} \ \ s.t. \|x_{n_k} - \beta\| \geq \epsilon_1).$$

Since $\{x_n\}$ is ultimately bounded, $\{x_{n_k}\}$ is a bounded sequence in R^m. Thus there exists a convergent subsequence $\{x_{n_{k_p}}\}$ of $\{x_{n_k}\}$ by the Bolzano-Weierstrass theorem. Hence we may assume that $\lim_{p \to \infty} x_{n_{k_p}} = \alpha$

for some vector $\alpha \in R^m$. Then we have, for such an $\epsilon_1 > \epsilon_0$,

$$\exists K \in N \ s.t.p \geq K \Rightarrow \|x_{n_{k_p}} - \alpha\| < \frac{\epsilon_1 - \epsilon_0}{2}.$$

Thus we have

$$\begin{aligned} \|\beta - \alpha\| &= \|\beta - x_{n_{k_K}} + x_{n_{k_K}} - \alpha\| \\ &\geq \|\beta - x_{n_{k_K}}\| - \|x_{n_{k_K}} - \alpha\| \\ &> \epsilon_1 - \frac{\epsilon_1 - \epsilon_0}{2} = \frac{\epsilon_1 + \epsilon_0}{2}. \end{aligned}$$

Since $\frac{\epsilon_1+\epsilon_0}{2} > \epsilon_0$, this implies that $\beta \notin \overline{B}(\alpha, \epsilon_0)$. Since $\alpha \in SSL$, this implies that $\beta \notin \bigcap_{\alpha \in SSL} \overline{B}(\alpha, \epsilon_0)$. Consequently, we have $S = \bigcap_{\alpha \in SSL} \overline{B}(\alpha, \epsilon_0)$. On the other hand, since S is the intersection of the closed balls $\overline{B}(\alpha, \epsilon_0)$ which are bounded, closed and convex, S is compact and convex in R^m. □

Note that if $m = 1$ in the above lemma then we have

$$\boxed{\epsilon_0 - \lim_{n \longrightarrow \infty}} x_n = [A - B, A + B]$$

for some A and $0 \leq B \leq \epsilon_0$, since the compact and convex subset of R is just a closed and bounded interval.

Moreover, we have the following corollary.

Corollary 1.4. *Let $\{x_n\}$ be a single-valued sequence of vectors in R^m which converges to some vector $a \in R^m$. Then we have*

$$\boxed{\epsilon_0 - \lim_{n \longrightarrow \infty}} x_n = \overline{B}(a, \epsilon_0).$$

for all $\epsilon_0 \geq 0$.

Proof. Since the subsequential limit a of $\{x_n\}$ is unique, this corollary follows from the above lemma 1.3. □

2. Epsilon zero density in R^m

In this section, we investigate about the concept of the ϵ_0-dense subset in R^m and research the shape of this set. Throughout this section, $\epsilon_0 \geq 0$ denotes any, but fixed, non-negative real number. We denote the open and closed balls in R^m by $B(\alpha, \epsilon) = \{x \in R^m | \|x - \alpha\| < \epsilon\}$ and $\overline{B}(\alpha, \epsilon) = \{x \in R^m | \|x - \alpha\| \leq \epsilon\}$.

Definition 2.1. For a given subset S of R^m, a point $a \in R^m$ is an ϵ_0-accumulation point of S if and only if $B(a,\epsilon) \cap (S - \{a\}) \neq \emptyset$ for any positive real number $\epsilon > \epsilon_0$. And a point $a \in S$ is an ϵ_0-isolated point of S if and only if $B(a,\epsilon_1) \cap (S - \{a\}) = \emptyset$ for some positive real number $\epsilon_1 > \epsilon_0$.

Note that $0-$accumulation point of S is the usual accumulation point of S.

Definition 2.2. If S is a subset of R^m, then we define the ϵ_0-derived set as the set of all the ϵ_0-accumulation points of S and denote it by $S'_{(\epsilon_0)}$.

Note that $0-$derived set is the derived set in the usual sense.

Definition 2.3. Let E be any non-empty and open subset of R^m and $\epsilon_0 > 0$. We define that a subset D of E is an ϵ_0-dense subset of E in E if and only if $E \subseteq D'_{(\epsilon_0)} \cup D$. In this case, we say that D is ϵ_0-dense in E.

Note that E can be a proper subset of $D'_{(\epsilon_0)} \cup D$ in the above definition.

Lemma 2.4. *Let D be any non-empty subset of R^m. Then $a \in D'_{(\epsilon_0)}$ if and only if there exists a single-valued sequence $\{b_n\}$ in $D - \{a\}$ such that $a \in \underset{n\longrightarrow\infty}{\boxed{\epsilon_0 - \lim}} b_n$.*

Proof. ($\Rightarrow$) If $a \in D'_{(\epsilon_0)}$ then we have $\forall \epsilon > \epsilon_0, B(a,\epsilon) \cap (D - \{a\}) \neq \emptyset$. Choosing $\epsilon = \epsilon_0 + \frac{1}{n}$ for each natural number $n \in N$, we have

$$B(a, \epsilon_0 + \frac{1}{n}) \cap (D - \{a\}) \neq \emptyset.$$

Thus there is a single-valued vector sequence $\{b_n\}$ in $D - \{a\}$ such that $b_n \in B(a, \epsilon_0 + \frac{1}{n})$ for each $n \in N$. For any given positive real number $\epsilon > \epsilon_0$, choosing a natural number $K \in N$ so large that $\epsilon_0 + \frac{1}{K} < \epsilon$, we have a statement

$$\forall \epsilon > \epsilon_0, \exists K \in N \ s.t. n \geq K \Rightarrow \|b_n - a\| < \epsilon_0 + \frac{1}{n} \leq \epsilon_0 + \frac{1}{K} < \epsilon$$

which implies that $a \in \underset{n\longrightarrow\infty}{\boxed{\epsilon_0 - \lim}} b_n$. ($\Leftarrow$) Suppose that there exists a single-valued sequence $\{b_n\}$ in $D - \{a\}$ such that $a \in \underset{n\longrightarrow\infty}{\boxed{\epsilon_0 - \lim}} b_n$. And let any positive real number $\epsilon > \epsilon_0$ be given. Then we have

$$\forall \epsilon > \epsilon_0, \exists K \in N \ s.t. n \geq K \Rightarrow \|b_n - a\| < \epsilon.$$

Since $b_K \neq a$, this implies that $b_K \in B(a,\epsilon) \cap (D - \{a\}) \neq \emptyset$ which completes the proof. $\square$

LEMMA 2.5. *Let E be any non-empty and open subset of R^m. Let D be a subset of E and $\epsilon_0 \geq 0$ be any, but fixed, non-negative real number. Then D is ϵ_0-dense in E if and only if for each element $a \in E$, there exists a sequence $\{b_n\}$ in D such that $a \in \boxed{\epsilon_0 - \lim}_{n \longrightarrow \infty} b_n$.*

Proof. ($\Rightarrow$) Let any element $a \in E$ be given. If $a \in D$ then we need only to choose a sequence $\{b_n\}$ so that $b_n = a$ for each natural number $n \in N$. On the other hand, if $a \in E - D$ then $a \in D'_{(\epsilon_0)}$. Thus, by lemma 2.4, there exists a single-valued sequence $\{b_n\}$ in $D - \{a\}$ such that $a \in \boxed{\epsilon_0 - \lim}_{n \longrightarrow \infty} b_n$. ($\Leftarrow$) Let any element $a \in E$ be given. If $a \in D$ then we are done. Suppose that $a \in E - D$. Since $a \in \boxed{\epsilon_0 - \lim}_{n \longrightarrow \infty} b_n$ for the sequence $\{b_n\}$ of the assumption in this lemma, we have

$$\forall \epsilon > \epsilon_0, \exists K \in N \ s.t. n \geq K \Rightarrow \|b_n - a\| < \epsilon.$$

But $b_K \neq a$ since $a \in E - D$ and $b_K \in D$. Hence we have $b_K \in B(a,\epsilon) \cap (D - \{a\}) \neq \emptyset$ which implies that $a \in D'_{(\epsilon_0)}$. Therefore, D is ϵ_0-dense in E. $\square$

THEOREM 2.6. *Let D be a bounded, non-empty subset of R^m and $\epsilon_0 \geq 0$ be any, but fixed, non-negative real number. Let $\{x_n\}$ be the multi-valued sequence in R^m such that $x_n = D$ for each natural number $n \in N$. Then $SSL(\{x_n\}) \subseteq D'_{(\epsilon_0)} \cup D$. Here $SSL(\{x_n\})$ is the set of all the single-valued subsequential limits of $\{x_n\}$ which was introduced in lemma 1.3.*

Proof. Let any element $a \in SSL(\{x_n\})$ be given. Then there exists a single-valued subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{n \to \infty} x_{n_k} = a$. Hence

$$a \in \boxed{\epsilon_0 - \lim}_{n \longrightarrow \infty} x_{n_k} = \overline{B}(a, \epsilon_0)$$

by the corollary 1.4. If $a \in D$ then we are done. On the other hand, if $a \notin D$ then $x_{n_k} \neq a$ for each natural numbers $k \in N$. Hence $\{x_{n_k}\}$ is a single-valued sequence in $D - \{a\}$. Then $a \in D'_{(\epsilon_0)}$ by lemma 2.4. $\square$

It can be easily proved that, for any subsets C and D of an open subset E,

$$C \subseteq D \Rightarrow C'_{(\epsilon_0)} \subseteq D'_{(\epsilon_0)} \text{ and } \epsilon_1 < \epsilon_2 \Rightarrow D'_{(\epsilon_1)} \subseteq D'_{(\epsilon_2)}.$$

Moreover, we have

THEOREM 2.7. *$C'_{(\epsilon_0)} \cup D'_{(\epsilon_0)} = (C \cup D)'_{(\epsilon_0)}$ for any subsets C and D of E.*

Proof. Clearly, $C'_{(\epsilon_0)} \subseteq (C \cup D)'_{(\epsilon_0)}$ and $D'_{(\epsilon_0)} \subseteq (C \cup D)'_{(\epsilon_0)}$. Hence we have $C'_{(\epsilon_0)} \cup D'_{(\epsilon_0)} \subseteq (C \cup D)'_{(\epsilon_0)}$. Conversely, let any element $a \in (C \cup D)'_{(\epsilon_0)}$ be given. By the above lemma 2.4, there exists a sequence $\{b_n\}$ in $(C \cup D) - \{a\}$ such that $a \in \boxed{\epsilon_0 - \lim}_{n \longrightarrow \infty} b_n$. Since $(C \cup D) - \{a\}$ contains infinitely many terms of $\{b_n\}$, either C or D contains infinitely many terms of $\{b_n\}$. Thus $a \in \boxed{\epsilon_0 - \lim}_{n \longrightarrow \infty} b_{n_k}$ for some subsequence $\{b_{n_k}\}$ of elements of $C - \{a\}$ or of elements of $D - \{a\}$. Therefore, $a \in C'_{(\epsilon_0)}$ or $a \in D'_{(\epsilon_0)}$ by lemma 2.4. □

Note that if D is ϵ_1-dense in E then D is also ϵ_2-dense in E for each positive real number $\epsilon_2 \geq \epsilon_1$.

LEMMA 2.8. *Let a subset D of R^m be given. Then D is $0-$dense in R^m if and only if $D'_{(0)} = R^m$.*

Proof. ($\Leftarrow$) Since $D \subseteq R^m$, we have $D \cup D'_{(0)} = D \cup R^m = R^m$. ($\Rightarrow$) Suppose that D is a $0-$dense subset of R^m. Then $D \cup D'_{(0)} = R^m$. Hence we need only to show that $D \subseteq D'_{(0)}$. Suppose that this is not true. Then there is a point $a \in D$ such that $a \notin D'_{(0)}$. Thus we have

$$\exists \epsilon_1 > 0 \ s.t. B(a, \epsilon_1) \cap (D - \{a\}) = \emptyset.$$

Now set $b = a + \frac{1}{2}(\epsilon_1, 0, \cdots, 0)$. Then $b \notin D$ and $B(b, \frac{\epsilon_1}{4}) \cap (D - \{b\}) = \emptyset$. Hence b is not a $0-$accumulation point of D . Thus we have $b \notin D \cup D'_{(0)} = R^m$. This is a contradiction which completes the proof. □

The following example shows that the above lemma 2.8 is not true for a positive real number $\epsilon_0 > 0$ in general.

EXAMPLE 2.9. *Let $D = \{0\} \cup \{R^m - \overline{B}(0, \frac{6}{5})\}$. Then D is $1-$dense in R^m, but $D'_{(1)} \neq R^m$.*

Proof. Clearly, we have $R^m - B(0, \frac{6}{5}) \subseteq D'_{(0)} \subseteq D'_{(1)}$. And if $0 < \|a\| \leq 1$ then $0 \in B(a, \epsilon) \cap (D - \{a\}) \neq \emptyset$ for any positive real number $\epsilon > 1$. Hence we have $\overline{B}(0, 1) - \{0\} \subseteq D'_{(1)}$. Moreover, if $1 < \|a\| \leq \frac{6}{5}$

then, choosing an element $b \in R^m$ such that $b = \frac{7a}{5\|a\|}$, we have $\|b\| = \frac{7}{5}$ and

$$\|b-a\| = \|\frac{7a}{5\|a\|} - a\| = \|\frac{7a - 5\|a\|a}{5\|a\|}\| = \frac{7-5\|a\|}{5} \leq \frac{2}{5}.$$

Thus we have $b \in B(a,\epsilon) \cap (D - \{a\}) \neq \emptyset$ for any positive real number $\epsilon > 1$. Therefore, we must have $R^m - \{0\} \subseteq D'_{(1)}$. But

$$B(0, \frac{11}{10}) \cap (D - \{0\}) = \emptyset \text{ with } \frac{11}{10} > 1.$$

Hence $D'_{(1)} \neq R^m$. But D is $1-$dense in R^m since $D \cup D'_{(1)} = R^m$. □

Lemma 2.10. *Let E be any non-empty and open subset of R^m. If a subset D of E satisfies the relation $E \subseteq \underset{b\in D}{\cup} \overline{B}(b,\epsilon_0)$ then D is ϵ_0-dense in E. But the converse is not true in general.*

Proof. Suppose that $E \subseteq \underset{b\in D}{\cup} \overline{B}(b,\epsilon_0)$ and any vector $a \in E$ be given. If $a \in D$ then we are done. Now suppose that $a \in E - D$. Then there is an element $b \in D$ such that $a \in \overline{B}(b,\epsilon_0)$. Hence $\|b-a\| \leq \epsilon_0$. Now let any positive real number $\epsilon > \epsilon_0$ be given. Then we have $\|b-a\| \leq \epsilon_0 < \epsilon$. Thus $b \in B(a,\epsilon)$. Hence we have $B(a,\epsilon) \cap (D - \{a\}) \neq \emptyset$ since $b \in D$ and $a \neq b$. Thus $a \in D'_{(\epsilon_0)}$. Hence D is ϵ_0-dense in E.

Finally, Let $D = R^m - \overline{B}(0,1)$. Then $\underset{b\in D}{\cup} \overline{B}(b,1) \neq R^m$ since $0 \notin \underset{b\in D}{\cup} \overline{B}(b,1)$. But, for any positive real number $\epsilon > 1$, we have $B(0,\epsilon) \cap (D - \{0\}) \neq \emptyset$ which implies that 0 is a $1-$accumulation point of D. Since we can prove by the similar method that any point of $\overline{B}(0,1) - \{0\}$ is $1-$accumulation point of D, D is a $1-$dense subset of R^m. □

Theorem 2.11. *Let D be a nonempty subset of an open subset E of R^m and $\overline{D} = D'_{(0)} \cup D$. Then $E \subseteq \underset{b\in \overline{D}}{\cup} \overline{B}(b,\epsilon_0)$ if and only if D is ϵ_0-dense in E.*

Proof. ($\Rightarrow$) By lemma 2.10. $\overline{D}$ is ϵ_0-dense in E. In order to show that D is ϵ_0-dense in E, let any element $a \in E$ and any positive real number $\epsilon > \epsilon_0$ be given. Since $\frac{\epsilon+\epsilon_0}{2} > \epsilon_0$ and $\overline{D}$ is ϵ_0-dense in E, we have $B(a, \frac{\epsilon+\epsilon_0}{2}) \cap (\overline{D} - \{a\}) \neq \emptyset$. Hence there is an element $b \in \overline{D} - \{a\}$ such that $\|b-a\| < \frac{\epsilon+\epsilon_0}{2}$. Since $b \in \overline{D} - \{a\}$, we have $b \in D - \{a\}$ or $b \in D'_{(0)} - \{a\}$. If $b \in D - \{a\}$ then we have

$$b \in B(a, \frac{\epsilon+\epsilon_0}{2}) \cap (D - \{a\}) \subseteq B(a,\epsilon) \cap (D - \{a\}) \neq \emptyset$$

which implies that $a \in D'_{(\epsilon_0)} \cup D$. On the other hand, if $b \in D'_{(0)} - \{a\}$ then there exists an element $c \in D - \{a\}$ such that $\|c-b\| < \frac{\epsilon-\epsilon_0}{2}$. Hence we have

$$\|c-a\| \leq \|c-b\| + \|b-a\| < \frac{\epsilon-\epsilon_0}{2} + \frac{\epsilon+\epsilon_0}{2} = \epsilon.$$

Thus $c \in B(a,\epsilon) \cap (D - \{a\}) \neq \emptyset$ which also implies that $a \in D'_{(\epsilon_0)} \cup D$. ($\Leftarrow$) Let any element $a \in E$ be given. If $a \in D$ then we are done since $a \in \overline{B}(a,\epsilon_0)$. Now suppose that $a \notin D$. Then $a \in D'_{(\epsilon_0)}$. Since $\epsilon_0 + \frac{1}{n} > \epsilon_0$ for each natural number $n \in N$, we have

$$B(a, \epsilon_0 + \frac{1}{n}) \cap (D - \{a\}) \neq \emptyset.$$

Hence there exists a single-valued sequence $\{b_n\}$ of the elements of D such that

$$b_n \in B(a, \epsilon_0 + \frac{1}{n}) \cap (D - \{a\}).$$

Since $\{b_n\}$ is a bounded sequence of elements of R^m, by applying the Bolzano-Weierstrass theorem, there is a convergent subsequence $\{b_{n_k}\}$ of $\{b_n\}$ such that $\lim_{k\to\infty} b_{n_k} = b_0$ for some vector $b_0 \in R^m$. Since $\overline{D}$ is closed in R^m, we have $b_0 \in \overline{D}$. Moreover, since $\|b_{n_k} - a\| < \epsilon_0 + \frac{1}{n_k}$, by taking the limit as $k \to \infty$, we have $\|b_0 - a\| \leq \epsilon_0$. Thus $a \in B(b_0,\epsilon_0) \subseteq \bigcup_{b\in\overline{D}} \overline{B}(b,\epsilon_0)$. This completes the proof. □

For example, consider the cartesian product Z^m of the set Z of all the integers. Since the length of the diagonal line of the unit $m-$dimensional cube in Z^m is $\sqrt{1^2 + \cdots + 1^2} = \sqrt{m}$, we have $R^m = \bigcup_{a\in Z^m} \overline{B}(a, \frac{\sqrt{m}}{2})$. Hence Z^m is $\frac{\sqrt{m}}{2}-$dense in R^m since $Z^m = \overline{Z^m}$. But the closed set Z^m is not ϵ_0-dense in R^m for each $0 \leq \epsilon_0 < \frac{\sqrt{m}}{2}$ since $R^m \neq \bigcup_{a\in Z^m} \overline{B}(a,\epsilon_0)$ in this case.

THEOREM 2.12. *Let D be a subset of an open subset E of R^m and $\epsilon_0 \geq 0$ be any, but fixed, non-negative real number. Then D is ϵ_0-dense in E if and only if for each positive real number $\epsilon > \epsilon_0$, we have $E \subseteq \bigcup_{b\in D} \overline{B}(b,\epsilon)$.*

Proof. ($\Rightarrow$) Suppose that D is ϵ_0-dense in E and let any positive real number $\epsilon > \epsilon_0$ be given. For a vector $a \in E$, if $a \in D$ then we are done since $a \in \overline{B}(a,\epsilon)$. Now suppose that $a \in E - D$. Since D is ϵ_0-dense in E and $\epsilon > \epsilon_0$, we have $B(a,\epsilon) \cap (D - \{a\}) \neq \emptyset$. Thus there exists an element $b \in D$ such that $b \in B(a, \epsilon)$. Then we also have $a \in B(b, \epsilon)$. Hence we have

$$a \in B(b, \epsilon) \subseteq \overline{B}(b, \epsilon) \subseteq \underset{b \in D}{\cup} \overline{B}(b, \epsilon).$$

($\Leftarrow$) Let any element $a \in E$ be given. And let any positive real number $\epsilon > \epsilon_0$ be given. If $a \in D$ then we are done since $a \in D \cup D'_{(\epsilon_0)}$. Suppose that $a \in E - D$. Since $E \subseteq \underset{b \in D}{\cup} \overline{B}(b, \epsilon_0 + \frac{\epsilon - \epsilon_0}{2})$, we have $a \in \overline{B}(b_\epsilon, \epsilon_0 + \frac{\epsilon - \epsilon_0}{2})$ for some element $b_\epsilon \in D$ since $\epsilon_0 + \frac{\epsilon - \epsilon_0}{2} > \epsilon_0$. Hence we have $b_\epsilon \in \overline{B}(a, \epsilon_0 + \frac{\epsilon - \epsilon_0}{2})$. Since $\epsilon_0 + \frac{\epsilon - \epsilon_0}{2} < \epsilon_0 + \epsilon - \epsilon_0 = \epsilon$, we have $b_\epsilon \in \overline{B}(a, \epsilon)$ which implies that $B(a, \epsilon) \cap (D - \{a\}) \neq \emptyset$ since this set contains an element $b_\epsilon \in D$ and $a \neq b_\epsilon$. Therefore, we have $a \in D'_{(\epsilon_0)}$ which completes the proof. □

Corollary 2.13. *Let D be a subset of an open subset E of R^m and $\epsilon_0 \geq 0$ be any, but fixed, non-negative real number. Then D is not ϵ_0-dense in E if and only if we have $B(a_1, \epsilon_1) \cap D = \emptyset$ for some positive real number $\epsilon_1 > \epsilon_0$ and some vector $a_1 \in E$.*

Proof. ($\Rightarrow$)Suppose that D is not ϵ_0-dense in E. Then E is not a subset of the union $\underset{b \in D}{\cup} \overline{B}(b, \epsilon_1)$ for some positive real number $\epsilon_1 > \epsilon_0$ by theorem 2.12. Hence there is an element $a_1 \in E$ such that $a_1 \notin \overline{B}(a, \epsilon_1)$ for all $a \in D$. And $a_1 \notin D$ since $a \in \overline{B}(a, \epsilon_1)$ for all $a \in D$. Now we have $B(a_1, \epsilon_1) \cap D = \emptyset$, for if $a \in B(a_1, \epsilon_1) \cap D = \emptyset$ for some $a \in D$ then $a_1 \in B(a, \epsilon_1) \subseteq \overline{B}(a, \epsilon_1)$ which is a contradiction. ($\Leftarrow$) Conversely, suppose that $B(a_1, \epsilon_1) \cap D = \emptyset$ for some positive real number $\epsilon > \epsilon_0$ and some vector $a_1 \in E$. Then we have, for each $a \in D$,

$$\|a_1 - a\| \geq \epsilon_1 > \frac{\epsilon_1 + \epsilon_0}{2}.$$

Thus we have

$$a_1 \notin \underset{b \in D}{\cup} \overline{B}(b, \frac{\epsilon_1 + \epsilon_0}{2}) \text{ and } E \not\subseteq \underset{b \in D}{\cup} \overline{B}(b, \frac{\epsilon_1 + \epsilon_0}{2}).$$

Since $\frac{\epsilon_1 + \epsilon_0}{2} > \epsilon_0$, D is not ϵ_0-dense in E by theorem 2.12. □

Theorem 2.14. *Let D be a subset of an open subset E of R^m and ϵ_0 be any, but fixed, positive real number. Then D is ϵ_0-dense in E if and only if D is ϵ_1-dense in E for each positive real number $\epsilon_1 > \epsilon_0$.*

Proof. ($\Rightarrow$) This follows immediately from the fact that $\epsilon > \epsilon_1 \Rightarrow \epsilon > \epsilon_0$. ($\Leftarrow$) Suppose that D is not ϵ_0-dense in E. Then, by corollary 2.13, there exists a positive real number $\epsilon_1 > \epsilon_0$ and a vector $a_1 \in E$ such that D is disjoint from $B(a_1, \epsilon_1)$. Now consider the positive real number $\frac{\epsilon_1+\epsilon_0}{2}$. Then we have

$$\exists \epsilon_1 > \frac{\epsilon_1 + \epsilon_0}{2} \text{ and } \exists a_1 \in E \text{ s.t. } B(a_1, \epsilon_1) \cap D = \emptyset.$$

Thus, by corollary 2.13 again, D is not $\frac{\epsilon_1+\epsilon_0}{2}-$dense in E. Since $\frac{\epsilon_1+\epsilon_0}{2} > \epsilon_0$, this contradicts to the fact that D is $\epsilon-$dense in E for each positive real number $\epsilon > \epsilon_0$. Hence D is ϵ_0-dense in E. □

COROLLARY 2.15. *Let D be a closed subset of an open subset E of R^m and $\epsilon_0 \geq 0$ be any, but fixed, non-negative real number. Then $E \subseteq \bigcup_{b \in D} \overline{B}(b, \epsilon)$ for each positive real number $\epsilon > \epsilon_0$ if and only if $E \subseteq \bigcup_{b \in D} \overline{B}(b, \epsilon_0)$.*

Proof. ($\Rightarrow$) By theorem 2.12, D is ϵ_0-dense in E. Then, since $D = \overline{D}$ is a closed subset of R^m, we have $E \subseteq \bigcup_{b \in D} \overline{B}(b, \epsilon_0)$ by theorem 2.11. ($\Leftarrow$) This follows immediately from the inclusion $\overline{B}(b, \epsilon_0) \subseteq \overline{B}(b, \epsilon)$ for each positive real number $\epsilon > \epsilon_0$ and each element $b \in E$. □

Note that if $E \subseteq \bigcup_{b \in D} \overline{B}(b, \epsilon_2)$ for some positive real number $\epsilon_2 < \epsilon_0$, then D is ϵ_0-dense in E since D is ϵ_2-dense in E and $\epsilon_2 < \epsilon_0$ by the lemma 2.10.

3. Epsilon zero dense ace

In this section, we investigate about the concept of the ϵ_0-dense ace of a given ϵ_0-dense subset and research the shape of the point of the ϵ_0-dense ace. Throughout this section, $\epsilon_0 \geq 0$ denotes any, but fixed, non-negative real number.

DEFINITION 3.1. Let D be an ϵ_0-dense subset of an open subset E of R^m. For an element $a \in D$, the point a is called a point of the ϵ_0-dense ace of D in E if and only if $D - \{a\}$ is not ϵ_0-dense in E.

Note that $0-$dense subset of E has no points of the ϵ_0-dense ace.

LEMMA 3.2. *Let D be an ϵ_0-dense subset of an open subset E of R^m with $\epsilon_0 > 0$. For an element $a \in D$, if $a \notin D'_{(\epsilon_0)}$ then a is a point of the ϵ_0-dense ace of D. And the converse is not true in general.*

Proof. Suppose that $a \notin D'_{(\epsilon_0)}$. Then there is a positive real number ϵ_1 with $\epsilon_1 > \epsilon_0$ such that $B(a, \epsilon_1) \cap (D - \{a\}) = \emptyset$. By taking the minimum $\min(\epsilon_1, 2\epsilon_0)$, we may assume that $\epsilon_0 < \epsilon_1 \leq 2\epsilon_0$. Now pick up a vector $b \in E$ so close that $\|b - a\| \leq \frac{\epsilon_1 - \epsilon_0}{3}$. Indeed, this is possible since $a \in D \subseteq E$ and E is an open subset of R^m. Then we have, for any element $x \in B(b, \epsilon_0 + \frac{\epsilon_1 - \epsilon_0}{3})$,

$$\|x - a\| \leq \|x - b\| + \|b - a\| < \epsilon_0 + \frac{\epsilon_1 - \epsilon_0}{3} + \frac{\epsilon_1 - \epsilon_0}{3} < \epsilon_1$$

which implies that $x \in B(a, \epsilon_1)$. Hence $B(b, \epsilon_0 + \frac{\epsilon_1 - \epsilon_0}{3}) \subseteq B(a, \epsilon_1)$. Thus

$$B(b, \epsilon_0 + \frac{\epsilon_1 - \epsilon_0}{3}) \cap (D - \{a\}) \subseteq B(a, \epsilon_1) \cap (D - \{a\}) = \emptyset.$$

Since $\epsilon_0 + \frac{\epsilon_1 - \epsilon_0}{3} > \epsilon_0$, $b \notin D$ and $b \neq a$, this implies that

$$b \notin [D - \{a\}]'_{(\epsilon_0)} \cup (D - \{a\}).$$

Thus $D - \{a\}$ is not ϵ_0-dense in E. Hence a is a point of the ϵ_0-dense ace of D in E. On the other hand, put

$$D = [R^m - B((1.25, 0, \cdots, 0), 1.25)] \cup \{(1, 0, \cdots, 0)\}.$$

Then we have

$$\underset{a \in D}{\cup} \overline{B}(a, 1) = [R^m - B((1.25, 0, \cdots, 0), 0.25)] \cup B((1, 0, \cdots, 0), 1) = R^m.$$

Since D is closed, D is a $1-$dense subset of R^m by theorem 2.11. But we have

$$B((1.25, 0, \cdots, 0), 1.25) \cap (D - \{(1, 0, \cdots, 0)\} - \{(1.25, 0, \cdots, 0)\}) = \emptyset.$$

Thus we have

$$(1.25, 0, \cdots, 0) \notin [D - \{(1, 0, \cdots, 0)\}]'_{(1)} \cup (D - \{(1, 0, \cdots, 0)\})$$

which implies that $D - \{(1, 0, \cdots, 0)\}$ is not $1-$dense in R^m. Thus $(1, 0, \cdots, 0)$ is a point of the $1-$dense ace of D and $(1, 0, \cdots, 0) \in D'_{(1)}$.
□

Now we have the following theorem.

THEOREM 3.3. *Let D be an ϵ_0-dense subset of the non-empty and open subset E of R^m with $\epsilon_0 > 0$. For an element $a \in D$, a is a point of the ϵ_0-dense ace of D in E if and only if there is a real number $\epsilon_1 > \epsilon_0$ and a point $b \in E$ such that $B(b, \epsilon_1) \cap D = \{a\}$. In this case, the point $b \in E$ must satisfy the relation $\|a - b\| \leq \epsilon_0$.*

Proof. ($\Leftarrow$) Assume that $B(b, \epsilon_1) \cap D = \{a\}$ for some real number $\epsilon_1 > \epsilon_0$ and some element $b \in E$. Then $B(b, \epsilon_1) \cap (D - \{a\}) = \emptyset$. Hence we have $b \notin (D - \{a\})$ and $B(b, \epsilon_1) \cap (D - \{a\} - \{b\}) = \emptyset$. Since $\epsilon_1 > \epsilon_0$, this implies that $b \notin (D - \{a\})'_{(\epsilon_0)}$. Since $b \in E$, this implies that $D - \{a\}$ is not an ϵ_0-dense subset of E. Thus a is a point of the ϵ_0-dense ace of D in E. ($\Rightarrow$) Conversely, suppose that a is a point of ϵ_0-dense ace of D in E. Then $D-\{a\}$ is not ϵ_0-dense in E. Hence there is a point $b \in E$ such that

$$b \notin [D-\{a\}]'_{(\epsilon_0)} \cup (D-\{a\}).$$

Then we must have

$$b \notin [D-\{a\}]'_{(\epsilon_0)} \text{ and } \ b \notin (D-\{a\}) = D \cap \{a\}^C.$$

Since $b \in (D \cap \{a\}^C)^C = D^C \cup \{a\}$, we have the following two cases.

Case 1. The case where $b \notin [D-\{a\}]'_{(\epsilon_0)}$ and $\ b \in D^C$.

In this case, since $b \notin [D-\{a\}]'_{(\epsilon_0)}$, we have

$$\exists \epsilon_1 > \epsilon_0 \ s.t. B(b,\epsilon_1) \cap \{[D-\{a\}]-\{b\}\} = \emptyset.$$

But we must have $b \in D'_{(\epsilon_0)}$ since $b \in D'_{(\epsilon_0)} \cup D$ and $b \notin D$. Hence we have

$$\forall \epsilon > \epsilon_0, B(b,\epsilon) \cap \{D-\{b\}\} \neq \emptyset.$$

Since $\epsilon > \epsilon_0$ was arbitrary, we must have

$$B(b,\epsilon) \cap D = \{a\}$$

for all positive real number ϵ such that $\epsilon_0 < \epsilon \leq \epsilon_1$. Since $\epsilon_0 < \epsilon \leq \epsilon_1$ was arbitrary, we have $\overline{B}(b,\epsilon_0) \cap D = \{a\}$. In particular, we have

$$\exists \epsilon_1 > \epsilon_0 \ s.t. B(b,\epsilon_1) \cap D = \{a\}$$

for the point $b \in E$.

Case 2. The case where $b \notin [D-\{a\}]'_{(\epsilon_0)}$ and $\ b = a$.

In this case, since $b = a \notin [D-\{a\}]'_{(\epsilon_0)}$, we have

$$\exists \epsilon_1 > \epsilon_0 \ s.t. B(a,\epsilon_1) \cap \{[D-\{a\}]-\{a\}\} = \emptyset.$$

Therefore, we have $\exists \epsilon_1 > \epsilon_0 \ s.t. B(b,\epsilon_1) \cap D = \{a\}$ for the element $b = a$. This completes the proof of the sufficient condition in this theorem. Moreover, if the point $b \in E$ in this theorem satisfies $\|b-a\| > \epsilon_0$, then $b \notin D'_{(\epsilon_0)} \cup D$ since

$$\exists \epsilon_2 = \|b-a\| > \epsilon_0 \ s.t. B(b,\epsilon_2) \cap \{D-\{b\}\} = \emptyset \text{ and } \ b \notin D.$$

This is a contradiction to the fact that D is ϵ_0-dense in E. □

Let's denote the set of all the points of ϵ_0-dense ace of D in R^m by $dap_{\epsilon_0}(D)$ or $dap_{\epsilon_0}(D;R^m)$ and in E by $dap_{\epsilon_0}(D;E)$.

Corollary 3.4. *$dap_{\epsilon_0}(D;E)$ is countable and closed for any positive real number $\epsilon_0 > 0$.*

Proof. By the above theorem 3.3, $a \in dap_{\epsilon_0}(D;E)$ if and only if there is a positive real number $\epsilon_a > \epsilon_0$ and a point $b_a \in E$ such that $B(b_a, \epsilon_a) \cap D = \{a\}$. Hence any closed ball with radius ϵ_0 has at most finite number of the points of ϵ_0-dense ace of D in E. Therefore, $dap_{\epsilon_0}(D;E)$ is countable and closed for any positive real number $\epsilon_0 > 0$. □

THEOREM 3.5. *(Double Capacity) Let D be an ϵ_0-dense subset of R^m and $\epsilon_0 > 0$ be any, but fixed, positive real number. If $dap_{\epsilon_0}(D;R^m) \neq \emptyset$ then D is not $\frac{\epsilon_0}{2}-$dense in R^m. Equivalently, if D is ϵ_0-dense in R^m then $dap_{2\epsilon_0}(D;R^m) = \emptyset$.*

Proof. Choose an element $a \in dap_{\epsilon_0}(D;R^m) \neq \emptyset$. By the above theorem 3.3 with $E = R^m$, there is a positive real number $\epsilon_a > \epsilon_0$ and a point $b_a \in R^m$ such that $B(b_a, \epsilon_a) \cap D = \{a\}$. Now choose an element $c \in R^m$ such that

$$c = \begin{cases} \frac{1}{2}(2b_a + \epsilon_a \frac{b_a - a}{\|b_a - a\|}) & (\text{ if } \ b_a \neq a) \\ \frac{1}{2}\{2b_a + \epsilon_a(1, 0, \cdots, 0)\} & (\text{ if } \ b_a = a) \end{cases}$$

Note that c is the center point of the line segment joining the point b_a and the point $b_a + \epsilon_a \frac{b_a - a}{\|b_a - a\|}$ when $b_a \neq a$. Then we have $a \notin B(c, \frac{\epsilon_a}{2})$ and

$$\exists \epsilon_1 = \frac{\epsilon_a}{2} > \frac{\epsilon_0}{2}, \ s.t. B(c, \epsilon_1) \cap D = \emptyset.$$

Hence D is not $\frac{\epsilon_0}{2}-$dense in R^m by corollary 2.13. Finally, if D is ϵ_0-dense in R^m then D is $2\epsilon_0-$dense in R^m and $dap_{2\epsilon_0}(D;R^m) = \emptyset$. □

Note that the theorem above does not hold for an open subset E of R^m in general. For example, if we choose an open subset

$$E = B((0, \cdots, 0), 1) \cup B((6, 0, \cdots, 0), 1)$$

and a subset $D = \{(0, \cdots, 0), (6, 0, \cdots, 0)\}$ then D is $3-$dense subset of E and $dap_2(D;E) = D$. But D is also $1.5-$dense in E and $dap_6(D;E) = D \neq \emptyset$.

However, we have the following theorem.

THEOREM 3.6. *Let D be an ϵ_0-dense subset of an open subset E of R^m and $\epsilon_0 > 0$ be any, but fixed, positive real number. Suppose that $\bigcup_{b \in D} B(b, \epsilon_0) \subseteq E$. If $dap_{\epsilon_0}(D;E) \neq \emptyset$ then D is not $\frac{\epsilon_0}{2}-$dense in E. Equivalently, if D is ϵ_0-dense in E then $dap_{2\epsilon_0}(D;E) = \emptyset$.*

Proof. Choose an element $a \in dap_{\epsilon_0}(D;E) \neq \emptyset$. By the above theorem 3.3, there is a positive real number $\epsilon_a > \epsilon_0$ and a point $b_a \in E$

such that $B(b_a, \epsilon_a) \cap D = \{a\}$. Since the point $b_a \in E$ satisfies the condition $\|b_a - a\| \leq \epsilon_0$, we may assume without the loss of generality that $\epsilon_a < \frac{3}{2}\epsilon_0$. Now choose an element $c \in R^m$ such that

$$c = \begin{cases} b_a & (\text{ if } \ \|b_a - a\| > \frac{\epsilon_0}{2}) \\ \frac{1}{2}(a + b_a + \epsilon_a \frac{b_a - a}{\|b_a - a\|}) & (\text{ if } \ 0 < \|b_a - a\| \leq \frac{\epsilon_0}{2}) \\ a + \frac{\epsilon_0 + \epsilon_a}{4}(1, 0, \cdots, 0) & (\text{ if } \ b_a = a) \end{cases}$$

Now the following three cases occur.

Case 1. The case where $\|b_a - a\| > \frac{\epsilon_0}{2}$.

In this case, we have $c = b_a \in E$. Choose

$$\epsilon_1 = \frac{\epsilon_0}{2} + \frac{1}{2}(\|b_a - a\| - \frac{\epsilon_0}{2}) = \frac{\epsilon_0 + 2\|b_a - a\|}{4}.$$

Then we have

$$\begin{aligned} \forall x \in B(c, \epsilon_1) \Rightarrow \|x - b_a\| &= \|x - c\| < \epsilon_1 \\ &= \frac{\epsilon_0 + 2\|b_a - a\|}{4} \leq \frac{3}{4}\epsilon_0 < \epsilon_a. \end{aligned}$$

Hence we have $B(c, \epsilon_1) \subseteq B(b_a, \epsilon_a)$. And, since $\|b_a - a\| > \frac{\epsilon_0}{2}$, we have

$$\exists \epsilon_1 = \frac{\epsilon_0 + 2\|b_a - a\|}{4} > \frac{\epsilon_0}{2}, \exists c = b_a \in E \ s.t. B(c, \epsilon_1) \cap D = \emptyset.$$

Case 2. The case where $0 < \|b_a - a\| \leq \frac{\epsilon_0}{2}$.

In this case, let's pick up $c = \frac{1}{2}(a + b_a + \epsilon_a \frac{b_a - a}{\|b_a - a\|})$. Then, since $\epsilon_a < \frac{3}{2}\epsilon_0$, we have

$$\begin{aligned} \|c - a\| &= \|\frac{1}{2}(b_a - a + \epsilon_a \frac{b_a - a}{\|b_a - a\|})\| \\ &= \frac{\|b_a - a\|}{2}(1 + \frac{\epsilon_a}{\|b_a - a\|}) \\ &= \frac{1}{2}(\|b_a - a\| + \epsilon_a) \\ &< \frac{1}{2}(\frac{\epsilon_0}{2} + \frac{3\epsilon_0}{2}) = \epsilon_0. \end{aligned}$$

Hence $c \in B(a,\epsilon_0) \subseteq \underset{b\in D}{\cup} B(b,\epsilon_0) \subseteq E$. Now if we choose $\epsilon_1 = \frac{1}{2}(\|b_a - a\| + \epsilon_a)$ then $a \notin B(c,\epsilon_1)$ and $\epsilon_a > \epsilon_1 > \frac{\epsilon_a}{2} > \frac{\epsilon_0}{2}$. And we have

$$\begin{aligned} x \in B(c,\epsilon_1) \Rightarrow \|x - b_a\| &\leq \|x - c\| + \|c - b_a\| \\ &< \epsilon_1 + \|\frac{1}{2}(a - b_a + \epsilon_a \frac{b_a - a}{\|b_a - a\|})\| \\ &= \frac{\|b_a - a\| + \epsilon_a}{2} + \frac{1}{2}\|b_a - a\| \cdot | - 1 + \frac{\epsilon_a}{\|b_a - a\|}| \\ &= \frac{\|b_a - a\| + \epsilon_a}{2} + \frac{1}{2}(\epsilon_a - \|b_a - a\|) = \epsilon_a. \end{aligned}$$

Thus $B(c,\epsilon_1) \subseteq B(b_a,\epsilon_a)$. Therefore, we have

$$\exists \epsilon_1 = \frac{1}{2}(\|b_a - a\| + \epsilon_a) > \frac{\epsilon_0}{2}, \exists c \in E \ s.t. B(c,\epsilon_1) \cap D = \emptyset.$$

Case 3. The case where $b_a = a$.

In this case, let's pick up $c = a + \frac{\epsilon_0+\epsilon_a}{4}(1,0,\cdots,0)$. Then we have

$$\|c - a\| = \|\frac{\epsilon_0 + \epsilon_a}{4}(1,0,\cdots,0)\| < \frac{\epsilon_0 + \frac{3}{2}\epsilon_0}{4} = \frac{5}{8}\epsilon_0 < \epsilon_0.$$

Hence $c \in B(a,\epsilon_0) \subseteq \underset{b\in D}{\cup} B(b,\epsilon_0) \subseteq E$. Now if we choose $\epsilon_1 = \frac{\epsilon_0+\epsilon_a}{4} > \frac{\epsilon_0}{2}$ then we have

$$\begin{aligned} x \in B(c,\epsilon_1) \Rightarrow \|x - b_a\| &= \|x - a\| \\ &\leq \|x - c\| + \|c - a\| \\ &< \epsilon_1 + \|\frac{\epsilon_0 + \epsilon_a}{4}(1,0,\cdots,0)\| \\ &= \frac{\epsilon_0 + \epsilon_a}{4} + \frac{\epsilon_0 + \epsilon_a}{4} = \frac{\epsilon_0 + \epsilon_a}{2} < \epsilon_a. \end{aligned}$$

Thus $B(c,\epsilon_1) \subseteq B(b_a,\epsilon_a)$. Therefore, we have

$$\exists \epsilon_1 = \frac{\epsilon_0 + \epsilon_a}{4} > \frac{\epsilon_0}{2}, \exists c \in E \ s.t. B(c,\epsilon_1) \cap D = \emptyset.$$

Hence D is not $\frac{\epsilon_0}{2}-$dense in E by corollary 2.13. Finally, if D is ϵ_0-dense in E then D is $2\epsilon_0-$dense in E and $dap_{2\epsilon_0}(D;E) = \emptyset$. □

Definition 3.7. Let D be a subset of a non-empty and open subset E of R^m. We call the density number of D in E the minimum

$$DN(D;E) = \min\{\epsilon_0 \geq 0 | D \text{ is } \ \epsilon_0 - \text{dense in } \ E\}.$$

And we call the density number of $a \in D$ in E the minimum

$$DN(a;E) = \min\{\epsilon_0 \geq 0 | a \in dap_{\epsilon_0}(D;E)\}.$$

Note that, by theorem 2.14, D is ϵ_0-dense in E if and only if D is ϵ_1-dense in E for each positive real number $\epsilon_1 > \epsilon_0$. Hence the number $DN(D;E)$ is well-defined.

On the other hand, $DN(a;E)$ is also well-defined by the following lemma.

LEMMA 3.8. *Let D be a ϵ_0-dense subset of the non-empty and open subset E of R^m and $a \in dap_{\epsilon_0}(D;E)$. If $\beta = glb\{\epsilon \geq 0 | a \in dap_{\epsilon}(D;E)\}$ then*

$$DN(a;E) = \min\{\epsilon_0 \geq 0 | a \in dap_{\epsilon_0}(D;E)\} = \beta.$$

Proof. Suppose that $a \in dap_{\epsilon_0}(D;E)$. Since the set $\{\epsilon \geq 0 | a \in dap_{\epsilon}(D;E)\}$ contains the number ϵ_0, this set is non-empty and bounded below. Hence the infimum $glb\{\epsilon \geq 0 | a \in dap_{\epsilon}(D;E)\}$ exists. Now let $glb\{\epsilon \geq 0 | a \in dap_{\epsilon}(D;E)\} = \beta$. Then, for any positive real number γ such that $\beta < \gamma$, there is a positive real number ϵ_2 such that $\beta < \epsilon_2 < \gamma$ and $a \in dap_{\epsilon_2}(D;E)$. In particular, D is $\gamma-$dense in E since D is ϵ_2-dense and $\epsilon_2 < \gamma$. Since $\beta < \gamma$ was arbitrary, D is $\beta-$dense in E by theorem 2.14. Moreover, since $a \in dap_{\epsilon_2}(D;E)$, there is a real number $\epsilon_3 > \epsilon_2$ and a point $b \in E$ such that $B(b, \epsilon_3) \cap D = \{a\}$ by theorem 3.3. Since $\beta < \epsilon_3$, this implies that there is a real number $\epsilon_3 > \beta$ and a point $b \in E$ such that $B(b, \epsilon_3) \cap D = \{a\}$. Thus $a \in dap_{\beta}(D;E)$ by theorem 3.3. Therefore, the infimum β must be the minimum. □

On the other hand, for the points of ϵ_0-dense ace, we have the following lemma.

LEMMA 3.9. *Let $\{D_j | j \in J\}$ be a set of ϵ_0-dense subsets of the non-empty and open subset E of R^m. If $\underset{j \in J}{\cap} D_j = \emptyset$ then we have $dap_{\epsilon_0}(\underset{j \in J}{\cup} D_j; E) = \emptyset$.*

Proof. Suppose that $a \in dap_{\epsilon_0}(\underset{j \in J}{\cup} D_j; E)$ for some element $a \in \underset{j \in J}{\cup} D_j$. Then the subset $\underset{j \in J}{\cup} D_j - \{a\}$ is not an ϵ_0-dense subset of E in E by the definition of the point of ϵ_0-dense ace. But, since $a \notin \underset{j \in J}{\cap} D_j = \emptyset$, we have $a \notin D_{j_0}$ for some index $j_0 \in J$. Then we have $D_{j_0} \subseteq \underset{j \in J}{\cup} D_j - \{a\}$. Since D_{j_0} is an ϵ_0-dense subset of E, this implies that $\underset{j \in J}{\cup} D_j - \{a\}$ must be an ϵ_0-dense subset of E in E. This is a contradiction. Consequently, we have $dap_{\epsilon_0}(\underset{j \in J}{\cup} D_j; E) = \emptyset$. □

THEOREM 3.10. *Let $\{D_j | j \in J\}$ be a set of ϵ_0-dense subsets of the non-empty and open subset E of R^m. If a is a point of ϵ_0-dense ace of $\underset{j\in J}{\cup} D_j$ in E then $a \in \underset{j\in J}{\cap} dap_{\epsilon_0}(D_j; E)$. That is,*

$$dap_{\epsilon_0}(\underset{j\in J}{\cup} D_j; E) \subseteq \underset{j\in J}{\cap} dap_{\epsilon_0}(D_j; E).$$

The converse is not true in general.

Proof. We first show that $a \in \underset{j\in J}{\cap} D_j$. Assume that $a \notin \underset{j\in J}{\cap} D_j$. Then $a \notin D_{j_0}$ for some index $j_0 \in J$. Then we have $D_{j_0} \subseteq \underset{j\in J}{\cup} D_j - \{a\}$. Since D_{j_0} is an ϵ_0-dense subset of E, this implies that $\underset{j\in J}{\cup} D_j - \{a\}$ is an ϵ_0-dense subset of E. Hence a is not a point of ϵ_0-dense ace of $\underset{j\in J}{\cup} D_j$ in E. This contradiction implies that $a \in \underset{j\in J}{\cap} D_j$. Now, since $a \in dap_{\epsilon_0}(\underset{j\in J}{\cup} D_j; E)$, we have

$$\exists \epsilon_1 > \epsilon_0, \exists b \in E \ s.t. B(b, \epsilon_1) \cap (\underset{j\in J}{\cup} D_j) = \{a\}$$

by theorem 3.3. Since D_j is a subset of $\underset{j\in J}{\cup} D_j$ for each index $j \in J$, this implies that

$$\exists \epsilon_1 > \epsilon_0, \exists b \in E \ s.t. B(b, \epsilon_1) \cap D_j = \{a\}$$

for each index $j \in J$. Thus $a \in \underset{j\in J}{\cap} dap_{\epsilon_0}(D_j; E)$ by theorem 3.3. On the other hand, let D_1 and D_2 be two subsets of R such that

$$D_1 = (-\infty, -1) \cup \{0\} \cup (\frac{3}{2}, \infty) \quad \text{and} \quad D_2 = (-\infty, -\frac{3}{2}) \cup \{0\} \cup (1, \infty).$$

Then $0 \in dap_1(D_1; R) \cap dap_1(D_2; R)$. But 0 is not a point of $1-$dense ace of $D_1 \cup D_2$ in R since $D_1 \cup D_2 = (-\infty, -1) \cup \{0\} \cup (1, \infty)$. □

EXAMPLE 3.11. *Assume that the earth is a perfectly elliptical body. Let $F \subseteq R^3$ be the set of all the points on the surfaces consisting of the Korean land excluding all the islands. And let $E \supseteq F$ be an open subset of R^3 such that the distance between F and the boundary of E is less than or equal to 1 meter. Now let $D \subseteq E$ be the set of all the points on the surface F consisting of all the express highways in the Republic of Korea. Then D is $100-$dense subset of E with respect to the unit of kilometers since any closed ball with center at E and with radius r with $r > 100(km)$ contains at least one point of D and since $E \subseteq \underset{a\in D}{\cup} \overline{B}(a, \epsilon_1)$ for each positive real number $\epsilon_1 > 100$.*

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