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Teaching Practices for All Learners in the Mathematics Classroom

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In this paper, we articulate what is a lesson for all learners with different cognitive levels and what kind of teaching practices are required to implement this type of lesson. For all learners' own sense-making, open-ended tasks are the primary sources to bring their various mathematical ideas. These tasks can be meaningfully implemented by appropriate teaching practices: providing enough time (for thinking deeply and for preparing a reply), acting intentionally (alternative wrapping up activities and appointment of a struggling student), and cultivating collaborative classroom norms (respecting peer's thinking and learning from peers). This exploratory study has the potential to help practitioners and researchers understand the complexity of the work of teaching and clarify how to deal with such complexity.

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MESC Classification: C32, C72
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I. INTRODUCTION

Every teacher has some levels of expectation for students to gain "new sense-making" during lessons. Many lessons and studies might consider this "new sense-making" as an identical goal for all students in the classroom. However, in the authentic classroom situation, it would be not realistic to make this type of sense-making to happen. This is because each student has a different level of prior knowledge and reasoning ability. As a result, what students construct from the same lesson might be varied depending on their personal context and on teachers' teaching practices.

This does not mean that it is impossible to implement a successful lesson with all

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learners' own sense-making. Yes, it would be possible. However, to implement this idea, teachers might need to change the viewpoint to see how students think, understand, and learn mathematics. The authors would like to suggest the following perspective. At the beginning of the lesson, the student has different levels of prior knowledge and mathematical reasoning ability. Therefore, the individual students' accomplishment from the lesson should be also varied. That is, it might be difficult for all learners to have exactly the same level of understanding from the lessons. In this study, *a lesson for all learners* refers to a lesson which appreciates a wide spectrum of students' cognitive levels. This definition implies that teachers should implement their lessons not for only one level of students but diverse students who have different knowledge levels. In other words, some teachers might consider their diverse cognitive level of students as a single student with an exact level of cognition and treat them in the same manner. However, this is not appropriate. Teachers should appreciate their students' difference in terms of knowledge construction. Based on this premise, we suggest teaching practices to facilitate *a lesson for all learners*, which all learners can gain diverse sense-making.

II. OPEN-ENDED APPLICATION OF OPEN-ENDED TASKS

What types of tasks would be appropriate to implement the lesson for all learners? If we assume that students construct their own mathematical understanding from different cognitive levels, a closed task producing an identical solution would not be helpful to develop and extend students' understanding. In the same vein, Cognitive Guided Instruction approach (Carpenter, Fennema, Franke, Levi, & Empson, 2014), which emphasizes the diverse problem-solving strategies, also might be limited since students are expected to have only the same result from different strategies. Therefore, an optimal type of task for the lesson for all learners is a task which has multiple solutions with multiple strategies. For example, "Today's Number" (Kim & Yeo, 2019) was illustrated to show these multiple solutions from students' diverse strategies. To be specific, the task was introduced to first graders with the following prompt, " ______________= 20." Students identified various strategies to solve the task: addition (e.g., 10+10, 5+5+5+5), subtraction (e.g., 21-1, 22-2), multiplication (e.g., 2×10), and division (e.g., 40÷2). Note that the strategies students found are also solutions for the task. A student might solve the "Today's Number" task through addition and multiplication, 4+4+4+4+4 and 5×4. The student used different mathematical ideas to get the same result, 20. In this situation, both additive and multiplicative ideas are not only the strategies students employed but also the solutions students found. In this study, we focus on "The Number of Students in Our School" task to illustrate how the task for all learners could be implemented in the classroom situation. The goal of the task is to figure out the number of students from the first grade to the third grade. This task consists of two phases: fixed numbers and free numbers. In the fixed number phase, the instructor provided specific information for the number of students in each classroom (1st grade: 25, 24, 24, 2nd grade: 25, 24, 25, 3rd grade: 24, 24, 24). Students are expected to use a wide spectrum of mathematical ideas to find different strategies (e.g., $25 \times 3 + 24 \times 6$, $24 \times 9 + 3$, $25 + 24 + 24 + 25 + 24 + 25 + 24 + 24 + 24$). In the following free numbers phase, students make their own school with several classes of grade 1 to 3. One condition of the number of students for each classroom was to use only two numbers (e.g., 24 and 26) to set multiplicative situation by iterating the same numbers. This phase gave more open opportunities to consider different mathematical strategies as well as answers. Compared to the fixed number phase, every student might generate a different number of students in each classroom and this causes different answer. For example, Jungyoon (pseudonym) made a table to present the number of total students in her own created school (see Table 1). Then, she decomposed each number with 20 and leftover and calculate the sum of total students: $20 \times 15 + 6 \times 11 + 4 \times 4 = 300 + 66 + 16 = 382$.

Table 1. Jungyoon's generated information to find the number of students

	A	B	C	D	E
1st grade	26	26	26	24	26
2nd grade	26	26	26	26	24
3rd grade	26	26	26	24	24

III. TEACHING PRACTICES

Students might be engaged in diverse mathematical ideas through the use of open-ended tasks. However, these tasks do not guarantee students' successful knowledge construction during the lessons. It is important for teachers to implement the tasks by building up students' mathematical ideas with specific teaching practices. In this paper, we focus on the following teaching practices: providing enough time for students to think deeply, waiting time, respect for students' thinking, practices for a new type of wrapping up, appointing intentionally struggling students, and learning from each other.

1. Providing enough time for students to think deeply

The subject of understanding mathematics should be students, not teachers. Even though teachers' instruction emphasizes the mathematical understanding of students, if sense-making is originally stemmed from the teachers, then this approach is still a teacher-oriented. Some teachers argue their instruction built on teachers' understanding could be regarded as sense-making oriented lessons. In order to understand mathematics, students must have enough time for students to think about what they are learning. In this regard, one of the most important teaching practices is waiting for student enough time to developing their own strategies or understanding (Charalambous & Pitta-Pantazi, 2015; Yeh, Ellis, & Hurado, 2017). We know students' mathematical understanding is important for teaching and learning mathematics but teachers might only emphasize "making students understood." This makes some teachers speak so fast in order to let students understood during restricted lesson time (Black, Harrison, Lee, Marshall, & Wiliam, 2003).

When teachers try to provide enough time for their students, they tend to feel some concerns about covering a suggested curriculum in a designated semester or a school year (Chapin, O'Connor, & Anderson, 2013). Adversely, this concern might be identified more easily in the teacher-oriented classroom because students in the classroom usually do not learn mathematics in understanding it. When teachers care only about students' achievement of mathematical knowledge, they might be able to cover all suggested curriculum materials. However, this coverage is not directly related to their students' achievement. In other words, this type of teachers understands mathematical knowledge but fails to understand their students themselves. On the other hand, teachers who strive to understand mathematics itself and their students could provide more opportunities to learn on the basis of students' sense-making. With the development of understanding, students can comprehend new mathematical knowledge easily and their understanding can be developed further (Cho & Kim, 2011). That is, students' mathematical thinking has a reciprocal relationship with their mathematical understanding. Therefore, students are required to have enough time to think mathematically. For example, Engage NY which is online curricular materials launched by the New York State Education Department suggests total lesson time as 60 minutes including Fluency Practice, Application Problem, Concept Development, and Student Debrief. The amount of time for each phase is varied by sessions but the total amount of time is always 60 minutes. Compared to the time duration of the one-period lesson in Korea as 40 minutes, this extended lesson time of Engage NY is very impressive. It would be difficult for teachers to provide enough time for each activity during this short time duration with 3 to 4 distinct activities. To apply this extended lesson time to the Korean context, teachers can think about planning a series of lessons (e.g., consecutive 2 lessons) rather than a single lesson. With this additional time, teachers have more chances to lead the lessons built on students' mathematical ideas and then construct new mathematical knowledge from the main resource for the lessons (Kim, 2018).

2. Waiting Time

Rowe (1974: cited in Black et al., 2003) observed the time duration between teachers' questioning and students' answering, then found that teachers tended to spend a very short time to wait for the reply from students. This short preparing time might imply students only have a chance to reproduce what they have already known rather than to try to understand an intended mathematical idea in the question. That is, some teachers might think about what students have to do is memorizing and reproducing mathematical facts. Ironically, they also might criticize a lack of students' efforts for sense-making after the lessons. On the other hand, many studies showed when students have enough waiting time to prepare for an answer, the quality of the answer could be getting high (Chapin et al., 2013; Kamii, 1994; Kim, in press). It usually takes only 0.9 seconds from teachers' initial question to next involvement when students have no statement about the first question (Rowe, 1974: cited in Black et al., 2003). Rowe (1974) articulated the more teachers give waiting time, the longer and the more students have opportunities to answer. Students also showed more confidence to answer, challenged peer's statement to improve it, and provided an alternative idea. Black and colleagues (2003) recommended to apply this waiting time practice in a process of the formative assessment and found that the teachers, who employed the practice, noticed the similar students' response patterns as above. This practice also promotes a low achievement level of student to be engaged in the lessons.

3. Respect for Students' Thinking

Students tend to be engaged more in a lesson when they feel respect from teachers and their peers. They need to respect each other not only as an entity of the classroom community but also as their mathematical ideas. There are additional teaching practices for respecting students' mathematical ideas: listening to students' mathematical ideas without an evaluation of correctness and providing an opportunity for any student who wants to contribute. As the subject of mathematical thinking and understanding should be students, the decision of whether ideas are correct or wrong is up to students, too. Teachers' listening practice allows students to express their ideas freely. Modeling of the listening practice by teachers also conveys the implications of how caring about peers' opinions is important and how to listen to them including writing on the board, restating, and inquiring what they hear. These activities are exactly the same things students are required to do during the lessons. Meanwhile, teachers should give the equitable opportunity to share students' ideas in the discussion. That is, if any students want to share their ideas, the teacher should give an opportunity to share the ideas (Kim & Yeo, 2019). This approach might be contrasting with several studies, which emphasize the selection of a few strategies with teachers' specific intentions and share them in the whole group discussion (Chapin et al., 2013; Smith & Stein, 2018; Empson & Levi, 2011). The former approach would suggest all learners' ideas are considered as meaningful resources and used as means of communication in the individual or group activities. On the other hand, the latter approach might give a negative impression to some students who are not accepted in the whole group discussion. Students' such repeated experiences might cause to fail to construct the best idea from all learners in the classroom. Therefore, the latter might be not appropriate for all students to develop their own learning.

Regarding this argumentation, some teachers and researchers would advocate that to cover the suggested curriculum content and knowledge require enough time to complete. This might be because the instruction is focused on the same level of understanding according to suggested mathematics activities in the textbook. However, open-ended Tasks (e.g., "Today's Number" illustrated by Kim and Yeo (2019) and "The number of students in our school" in the beginning of this paper) might not be problematic to address this issue since students have a learning experience with a single task and make sense of mathematical knowledge at their own level in a series of lessons which are overlapped each other but have the possibility to emerge new mathematical ideas.

This concern is the same as students might not construct any knowledge from their activities. However, if teachers provide authentic and meaningful opportunities to students, this might not happen. That is, students have the ability to construct their knowledge and activate this ability to build up knowledge (Burns, 2001; Kamii, 1995; Kim, 2018).

To respect students' ideas, the next teaching practice is questioning such as "Is there any other thought?" Since all learners want to contribute their ideas, teachers are necessary to keep asking such a question. Under this learning environment, students could experience to share any ideas regardless of the value of the statement. When students' ideas are considered as respectable, they might have unique mathematical ideas and could develop their creativity. This is much more important than an acquisition of a specific piece of knowledge. Students also can recognize the relationships and make connections between emerged mathematical ideas. On the process of making connections, students would use meta-cognition and reflective abstractions and develop their mathematical thinking (Skemp, 1987).

4. Practices for a New Type of Wrapping up

The open-ended tasks which have multiple strategies and solutions such as 'Today's Number' (Kim & Yeo, 2019) and 'The Number of Students in Our School' in this paper give an opportunity for students to devise their own unique strategies and to develop various mathematical knowledge through the individual unique thinking.

All learners should construct their own mathematical knowledge. However, this does not mean they should digest all the presented mathematical ideas during the lessons. The learners can make sense of what they are able to understand on that day and continue to try to understand the rest of the presented mathematical ideas on the following day. This continuous knowledge building might be possible when teachers utilize a series of the open-ended tasks which allow to present a wide spectrum of mathematical ideas in the lessons and overlap those ideas between potential knowledge to be understood and difficult knowledge to be understood. In other words, learners are able to accumulate their own individual understandings in each lesson. This contrasts with learners' understanding from a traditional approach which pursues the same mathematical goals suggested by textbooks or curriculum regardless of each student's diverse background.

In the context of *the lesson for all learners*, it would be difficult to find the valuable meaning to wrap up at the end of the lesson by only summarizing what students have learned briefly. Instead, teachers might figure out how much students understand their contribution to developing individual knowledge and for the whole group discussion. The process of grasping students' understanding would be the formative assessment for lessons and teachers might choose the learning content for the following lessons based on the findings (Black et al., 2003). Further, this could be the role of teachers as a practitioner of a curriculum (Kim & Kim, 2013).

Without teachers' specific directions, learners could make records of what is meaningful, different from their understanding and what needs to be explained further at their notebooks and worksheets (see Figure 1). Additionally, at the end of the whole group discussion, teachers can also make the following statement, "You might ask the students directly if there is still something unclear or difficult to understand." Students have more chances to discuss what was happened during the lessons with their peers in the recess or after school (Burns, 2001; Kim, 2018, in press; Park, 2018, personal conversation). Teachers also might put students' notebooks and worksheet behind the classroom for students to share their friends' ideas by circulating their works.

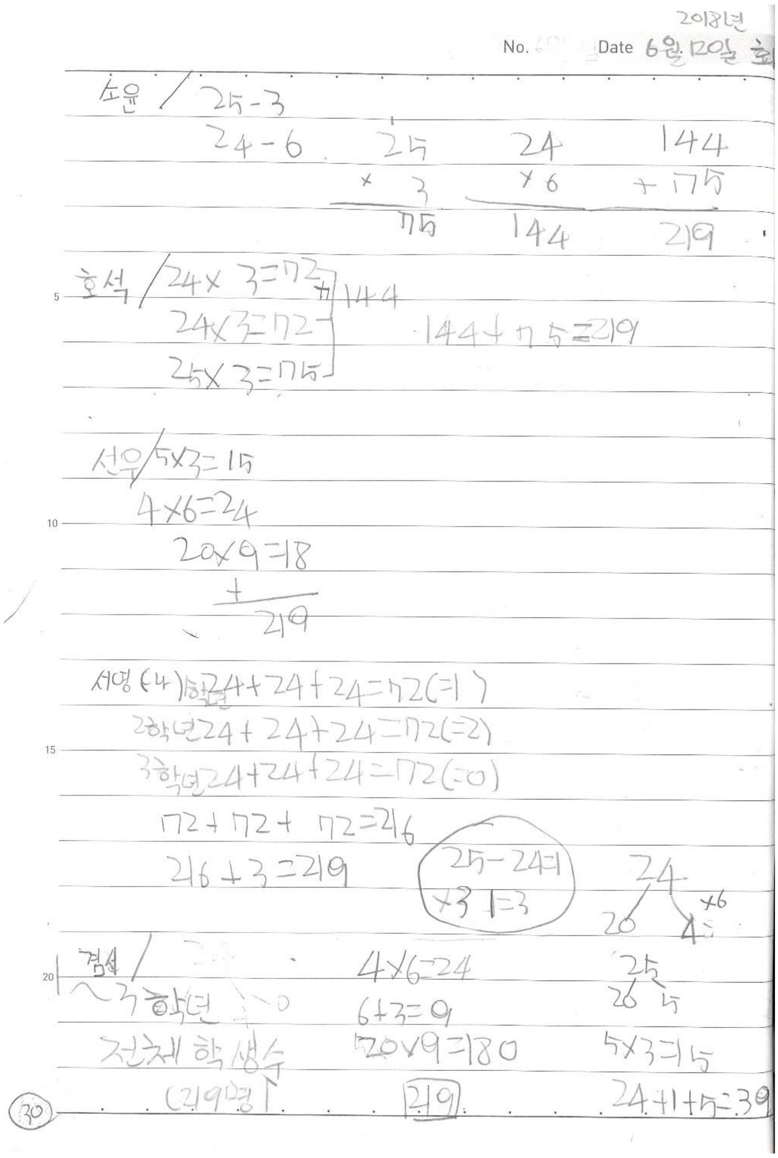

Figure 1. Jaeyoon’s notetaking of peer’s different strategies

5. Appointing Intentionally Struggling Students

There might be struggling students in mathematical content comprehension at any classrooms. As a factor of good mathematics instruction, teachers should make these struggling students engaged in the lesson to help them construct mathematical knowledge from their own levels.

Without the engagement, it might be difficult to have an opportunity to learn. For example, after the end of the last lesson focused on mathematical thinking for all students by the first author, a student reflected on how he had learned:

> Compared to using a mathematical textbook, I don't like the current style of lessons. My teacher used to ask no question. Also, I didn't bother my friends with silence. However, during the lessons, it was very annoying for me to answer such questions, "How do you think?", "Can you follow your friend's statement?"

His response might be understood by new interpretation, "I had to think a lot during the lessons. I strived to understand what my friends said. The teacher made me keep thinking. It was annoying though." However, this student's test score made significant improvement after a series of lessons. Not only to improve their achievement from the assessment, but teachers also have to make these struggling students engaged in the lessons. If teachers only weight for the ideas which have a value to discuss with peers, the struggling students have no room for contribution to building up mathematical knowledge with their own ideas. The limited opportunity to share their ideas in the whole group discussion might be related to isolating gradually the struggling students during the lessons and falling behind with little improvement.

One might be curious about how many struggling students make enhance their achievement with more opportunities to learn. Some studies show the significant improvement of achievement by the emphasis on this teaching practice (e.g., Chapin et al., 2013; Cho & Kim, 2011). For example, Cho and Kim (2011) investigated how the teaching practice to make struggling students engaged in the discussion influence their achievement. The following results (Table 2 and 3) show students' achievement for what they learned and what they did not were dependent on different achievement-level groups of students. In Table 1, the lower level showed the biggest gap between the three groups. Cho and Kim argued that the teaching approach had a significant impact on the lower level of achievement group of students. In Table 2, the mean of the treatment group at the lower level is greater than the mean of the control group in the middle level. This showed that even lower achievement group of students might be ready to understand the next mathematics content knowledge.

Table 2. Comparing mean about what students learn during treatment

Level	Group	*N*	mean	sd	Mean difference
Low	control	8	49.0	7.63	12.0
	treatment	8	61.0	12.23	
Middle	control	8	71.5	4.50	8.0
	treatment	8	79.5	7.23	
High	control	8	84.0	5.23	10.0
	treatment	8	94.0	3.70	

Table 3. Comparing mean about what students do not learn during treatment

Level	Group	*N*	mean	sd	Mean difference
Low	control	8	25.5	5.63	14.5
	treatment	8	40.0	6.04	
Middle	control	8	34.0	2.13	25.0
	treatment	8	59.0	5.12	
High	control	8	65	18.85	16.0
	treatment	8	81	7.01	

Teachers' practice to appoint the struggling students is crucial for engagement in the lessons and the use of open-ended Tasks is also important to contribute to building on knowledge based on their own ideas. One of the major features of open-ended Tasks is multiple entry points regardless of prior achievement levels, encouraging students to involve in the whole group discussion. Note that teachers should not give a compliment to only mathematically sophisticated statements among various ideas in the classroom. When students are eager to hear a teacher's complement, it would be possible to make a statement for building on knowledge not from their appropriate levels but from too easy or too difficult levels. Teachers might use the encouragement with "Is there any other thought?" and enough 'waiting time' for maximizing their engagement. Although these students might bring only a common idea, it is very precious ideas to themselves. Prior to sharing their ideas, the teachers need to check out what kind of idea students have by careful monitoring during an individual problem-solving phase. These struggling students might not be accustomed to present their ideas with a high voice. Therefore, teachers could restate what they said after the original student's statement or check other students' understanding of the statement by questioning.

6. Learning from Each Other

Compared to private tutoring, there are varied skills and knowledge level of students in mathematical classrooms to develop new knowledge together. In addition to the individual level of knowledge construction, thus, students are required to understand the ideas constructed by their peers. This is why we need a classroom as a learning community. For that, students should listen to peer's ideas when the peers try to contribute. Without listening, it would be difficult to involve in current discussion topics in the learning community. Some teachers believe their students tended to prefer sharing their own ideas but not to listen to their peer's (Chapin et al., 2013). Of course, this might be easy to happen at the beginning of the school year. However, if it happens consistently across the whole school year, the teachers might be regarded as have a failure in implementing their teaching practices appropriately.

It might be difficult for students to learn from each other without listening to peer's ideas. Which teaching practices are helpful to listen to peer's mathematical ideas? Teachers can be a good role model for this to inform this learning practice to students. Teachers also emphasize on understanding and extending students' mathematical ideas rather than merely presenting individual ideas. In this type of lesson, teachers can ask for students to be adjusted in this listening practice, "Who can explain the previous statement?", "(Appointing a specific person) Can you say what you heard again?", "What does the previous statement mean?", "Is there anyone to add on?" At the beginning of the semester, teachers can also cultivate classroom norms by praising the positive reactions from those questions.

IV. CONCLUSION

In this paper, we articulate what is a lesson for all learners with different cognitive levels and what kind of teaching practices are required to implement this type of lesson. It is evident each student has their own cognitive level but teachers often ignore this diversity. How can teachers make a differentiated lesson for all learners? We encourage to use open-ended tasks which have multiple strategies as well as multiple solutions. For example, to figure out "The Number of Students in Our School" task, children generated the number of students in each classroom at their own imagined school. This is not a trivial activity. When they decide the number of each class, it is required to consider why they choose the numbers and potentially to anticipate what mathematical ideas can be connected.

Additionally, open-ended tasks should be accompanied by specific teaching practices: providing enough time (for thinking deeply and for preparing a reply), acting intentionally (alternative wrapping up activities and appointment of a struggling student), and cultivating collaborative classroom norms (respecting peer's thinking and learning from peers). For example, teachers should wait enough time for giving a chance for students to think mathematically about the given tasks from their own cognitive level. When investigating their ideas, students also have enough time prior to answering about what they were asked. All these teaching practices have an overarching goal: how to appreciate children's varied mathematical thinking as teaching resources and to facilitate their thinking by making meaningful connections.

As the recognition of a problem is much easier than figuring out the best solution, some teachers might already realize that all students are able to make sense of what they taught in the lessons differently but hesitate how to deal with such diversity of sense-making. It is a more complicated question to ask how to utilize these various mathematical ideas to build up knowledge. Our approach, the lesson for all learners, suggests that teachers use open-ended tasks and specific teaching practices based on an alternative assumption for children’s learning and thinking. This exploratory study has the potential to help practitioners and researchers understand the complexity of the work of teaching and clarify how to deal with such complexity.

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