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PRIMITIVE CIRCLE ACTIONS ON ALMOST COMPLEX MANIFOLDS WITH ISOLATED FIXED POINTS

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Abstract. Let the circle act on a compact almost complex manifold M with a non-empty discrete fixed point set. To each fixed point, there are associated non-zero integers called weights. A positive weight w is called primitive if it cannot be written as the sum of positive weights, other than w itself. In this paper, we show that if every weight is primitive, then the Todd genus $\mathrm{Todd}(M)$ of M is positive and there are $\mathrm{Todd}(M) \cdot 2^n$ fixed points, where $\dim M = 2n$. This generalizes the result for symplectic semi-free actions by Tolman and Weitsman [8], the result for semi-free actions on almost complex manifolds by the author [6], and the result for certain symplectic actions by Godinho [1].

1. Introduction

The main purpose of this paper is to define the notion of a primitive circle action on an almost complex manifold with isolated fixed points and prove that if a compact almost complex manifold admits a primitive circle action with isolated fixed points, then its Todd genus $\mathrm{Todd}(M)$ is positive, and there are $\mathrm{Todd}(M) \cdot 2^n$ fixed points, where $\dim M = 2n$. This generalizes the result for semi-free symplectic circle actions on symplectic manifolds with isolated fixed points by Tolman and Weitsman [8], the result for semi-free circle actions on almost complex manifolds with isolated fixed points by the author [6], and the result for certain symplectic circle actions on 6-dimensional symplectic manifolds with isolated fixed points by Godinho [1].

An **almost complex manifold** is a pair (M, J) where M is a manifold and $J : TM \longrightarrow TM$ is a smooth map, such that for any $m \in M$, J restricts to a linear map J_m on the tangent space T_mM to M at m, i.e., $J_m : T_mM \longrightarrow T_mM$, and $J_m^2 = -I_m$ on T_mM, where I_m is the identity map on T_mM.

Let the circle S^1 act on an almost complex manifold (M, J). Throughout this paper, we assume that the action preserves the almost complex structure J. That is, $dg \circ J = J \circ dg$ for any $g \in S^1$. Let p be an isolated fixed point. Then

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we can identify the tangent space T_pM to M at p with $\mathbb{C}^n$, where $\dim M = 2n$. The circle action on T_pM is then identified with

$$g \cdot (z_1, \cdots, z_n) = (g^{w_{p1}} z_1, \cdots, g^{w_{pn}} z_n),$$

where $g \in S^1 \subset \mathbb{C}$, z_i are complex coordinates, and w_{pi} are non-zero integers, $1 \leq i \leq n$. The non-zero integers w_{pi} are called **weights** at p. For results on circle actions on almost complex manifolds, see [2], [5], [6], [7], etc.

Let the circle act on a compact almost complex manifold M with isolated fixed points. Denote by $A = \{w_1, \cdots, w_l\}$ the multiset of the collection of the absolute values of all of the weights among all of the fixed points, counted with multiplicity, such that for each positive integer $w \in A$, the multiplicity of w in A is equal to

$$\max_{p \in M^{S^1}} |\{i \,:\, |w_{pi}| = w\}|.$$

In addition, for each positive integer j, denote by

$$A_j = \{w_{k_1} + w_{k_2} + \cdots + w_{k_j} \,|\, w_{k_i} \in A, k_1 < k_2 < \cdots < k_j\}.$$

Note that the A_i are multisets. A positive weight $w \in A$ is called **primitive**, if $w \notin A_j$ for $j \geq 2$. That is, w is never equal to the sum of the absolute values of weights among all the fixed points, counted with multiplicity, other than w itself. The circle action is called **primitive**, if every positive weight is primitive.

For instance, let the circle act on $S^2 \times S^2 \times S^2$ by rotating each 2-sphere 3 times, 5 times, and 7 times, respectively. The action has 8 fixed points, and the weights at each fixed point are $\{\pm 3, \pm 5, \pm 7\}$. We have that $A = A_1 = \{3, 5, 7\}$, $A_2 = \{8, 10, 12\}$, and $A_3 = \{15\}$. For any $w \in A$, we have that $w \notin A_2$ and $w \notin A_3$. Therefore, the action is primitive.

For an almost complex manifold M, the **Hirzebruch** χ_y**-genus** $\chi_y(M)$ of M is the genus belonging to the power series $\frac{x(1+ye^{-x(1+y)})}{1+e^{-x(1+y)}}$. If $\dim M = 2n$, then the Hirzebruch $\chi_y(M)$ of M can be written as $\chi_y(M) = \sum_{i=0}^n \chi^i(M) \cdot y^i$ for some integers $\chi^i(M)$, $0 \leq i \leq n$. The Hirzebruch χ_y-genus $\chi_y(M)$ with $y = 0$ is the Todd genus of M. That is, the **Todd genus** of M is the genus belonging to the power series $\frac{x}{1-e^{-x}}$. In this paper, we prove that if a compact almost complex manifold M admits a primitive circle action with isolated fixed points, then its Todd genus is positive, and there are precisely $\mathrm{Todd}(M) \cdot 2^{\frac{\dim M}{2}}$ fixed points.

Theorem 1.1. *Let the circle act on a compact almost complex manifold M with a non-empty discrete fixed point set. If the action is primitive, then Todd$(M) > 0$ and $\chi_y(M) = $ Todd$(M) \cdot (1-y)^n$, where $\dim M = 2n$. In particular, there are Todd$(M) \cdot 2^n$ fixed points, and the number of fixed points which have exactly i negative weights is equal to Todd$(M) \cdot \binom{n}{i}$ for each i such that $0 \leq i \leq n$.*

In other words, if an almost complex manifold M admits a primitive circle action with isolated fixed points, then the Hirzebruch χ_y-genus of M is $\chi_y(M) = \mathrm{Todd}(M) \cdot (1-y)^n = \sum_{i=0}^n (-1)^i \mathrm{Todd}(M) \cdot \binom{n}{i} \cdot y^i$. The proof of Theorem 1.1 is given in the last section. For an action of a group G on a manifold M, denote by M^G the fixed point set; that is, $M^G = \{p \in M \mid g \cdot p = p \text{ for every } g \in G\}$. An action of a group G on a manifold M is called **free**, if for every $m \in M$ we have $g \cdot m \neq m$ for any $g \in G$ such that $g \neq e$, where e is the identity element of G. An action of a group G on a manifold M is called **semi-free**, if the action is free on $M \setminus M^G$, that is, the action is free outside the fixed point set.

If a circle action on a compact almost complex manifold with isolated fixed points is semi-free, the author proves in [6] that there are $\text{Todd}(M) \cdot 2^n$ fixed points. If the action is semi-free, then every weight is equal to either $+1$ or -1. Then we have that $A = A_1 = \{1, \cdots, 1\}$ in which the multiplicity of 1 in A is n where $\dim M = 2n$, $A_2 = \{2, \cdots, 2\}, \cdots, A_n = \{n\}$. This means that any semi-free action is primitive. Therefore, Theorem 1.1 recovers the result for a semi-free circle action as a particular case.

Corollary 1.2. [6] *Let the circle act semi-freely on a compact almost complex manifold M with isolated fixed points. Then Todd$(M) > 0$ and $\chi_y(M) =$ Todd$(M) \cdot (1-y)^n$, where $\dim M = 2n$. In particular, there are Todd$(M) \cdot 2^n$ fixed points, and the number of fixed points which have exactly i negative weights is equal to Todd$(M) \cdot \binom{n}{i}$ for each i such that $0 \leq i \leq n$.*

In [8], Tolman and Weitsman prove that if the circle acts semi-freely and symplectically on a compact symplectic manifold with a non-empty discrete fixed point set, then the action must be Hamiltonian, and there are 2^n fixed points, where $\dim M = 2n$. This is reproved in [7]. Every symplectic manifold admits a compatible almost complex structure J, and any symplectic circle action on a symplectic manifold is a particular case of a circle action on an almost complex manifold (M, J) preserving the almost complex structure J. Let the circle act semi-freely and symplectically on a compact symplectic manifold with isolated fixed points. Since the action is semi-free, every weight is equal to either $+1$ or -1. It follows that the action is primitive. Therefore, by Theorem 1.1, it follows that $\text{Todd}(M) > 0$ and there are $\text{Todd}(M) \cdot 2^n$ fixed points. On the other hand, $\text{Todd}(M) = N_0$, where N_0 is the number of fixed points which have no negative weights. Since $N_0 > 0$, this implies that the action is Hamiltonian. Any Hamiltonian circle action has a unique fixed point which has no negative weights. This means that $\text{Todd}(M) = N_0 = 1$. Therefore, as a corollary of Theorem 1.1, we recover the result by Tolman and Weitsman, and by Li.

Corollary 1.3. [7], [8] *Let the circle act semi-freely and symplectically on a compact, connected symplectic manifold M with a non-empty discrete fixed point set. Then the action is Hamiltonian and $\chi_y(M) = (1-y)^n$, where $\dim M = 2n$. In particular, there are 2^n fixed points.*

Consider a symplectic circle action on a 6-dimensional compact connected symplectic manifold M with fixed points whose weights are $\{\pm a, \pm b, \pm c\}$ for some positive integers a, b, and c. Without loss of generality, let $a \leq b \leq c$. In [1], Godinho proves that if $a + b \neq c$, then the symplectic action is in fact

Hamiltonian, and there are 8 fixed points. Since $a \leq b \leq c$ and $a + b \neq c$, the action is primitive. As another corollary of Theorem 1.1, we recover the result by Godinho.

Corollary 1.4. [1] *Let the circle act symplectically on a 6-dimensional compact connected symplectic manifold M with fixed points, whose weights are $\{\pm a, \pm b, \pm c\}$ for some positive integers a, b, and c such that $a \leq b \leq c$. If $a + b \neq c$, then the action is Hamiltonian. Moreover, there are 8 fixed points.*

A detailed proof of Corollary 1.4 is given in the last section.

The conclusion of Theorem 1.1 need not hold if an action is not primitive. For instance, let the circle act on $\mathbb{CP}^2$ by

$$g \cdot [z_0 : z_1 : z_2] = [z_0 : g^2 z_1 : g^5 z_2],$$

where $g \in S^1 \subset \mathbb{C}$. The action has three fixed points $[1 : 0 : 0]$, $[0 : 1 : 0]$, and $[0 : 0 : 1]$, and the weights at the fixed points are $\{2, 5\}, \{-2, 3\}$, and $\{-3, -5\}$, respectively. We have that $A = A_1 = \{2, 3, 5\}$, $A_2 = \{5, 7, 8\}$, and $A_3 = \{10\}$. Since $5 \in A$ and $5 \in A_2$, the action is not primitive, and hence the number of fixed points need not be of the form $\mathrm{Todd}(M) \cdot 2^{\frac{\dim M}{2}} = \mathrm{Todd}(M) \cdot 2^2$.

2. Background

Throughout this section, we assume that M is a compact almost complex manifold, equipped with a circle action, having isolated fixed points. Fix an integer w. For each fixed point p, denote by $N_p(w)$ the number of times the weight w occurs at p. That is, $N_p(w) = |\{i : w_{pi} = w\}|$.

As in the Introduction, for an almost complex manifold M, the Hirzebruch χ_y-genus $\chi_y(M)$ of M is the genus belonging to the power series $\frac{x(1+ye^{-x(1+y)})}{1+e^{-x(1+y)}}$. Let $\chi_y(M) = \sum_{i=0}^{n} \chi^i(M) \cdot y^i$ for some integers $\chi^i(M)$, $0 \leq i \leq n$, where $\dim N = 2n$. For a circle action on a compact almost complex manifold M with isolated fixed points, in [7] Li proves that the equivariant index of Dolbeault type operator on M is rigid; it is independent of the choice of an element of S^1 and is equal to the Hirzebruch χ_y-genus of M. As a result, Li proves the following formula:

Theorem 2.1. [7] *Let the circle act on a $2n$-dimensional compact almost complex manifold M with isolated fixed points. Then*

$$\chi^i(M) = \sum_{p \in M^{S^1}} \frac{\sigma_i(t^{w_{p1}}, \cdots, t^{w_{pn}})}{\prod_{j=1}^{n}(1 - t^{w_{pj}})} = (-1)^i N_i = (-1)^{n-i} N_{n-i},$$

where t is an indeterminate, σ_i is the i-th elementary symmetric polynomial in n variables, and N_i is the number of fixed points which have i negative weights.

There is an intimate relationship between weights among all of the fixed points; for each time a weight w occurs, there exists a weight $-w$. This is proved in [2], and is reproved in [7] using Theorem 2.1.

Lemma 2.2. [2], [7] *Let the circle act on a compact almost complex manifold M with isolated fixed points. Fix an integer w. Then the number of times the weight w occurs among all of the fixed points, counted with multiplicity, is equal to the number of times the weight $-w$ occurs among all of the fixed points, counted with multiplicity. That is,*

$$\sum_{p \in M^{S^1}} N_p(w) = \sum_{p \in M^{S^1}} N_p(-w).$$

Lemma 2.2 enables us to associate a labeled, directed multigraph to M. First, assign a vertex to each fixed point. If a fixed point p has a positive weight w, by Lemma 2.2 there exists a fixed point q which has weight $-w$. Therefore, we can draw an edge from p to q, giving the label w to the edge. In fact, there exists a fixed point q which has weight $-w$ and $p \neq q$. Therefore, we can associate a labeled directed multigraph which does not have any self-loop. For this, see [3] and [6].

For a fixed point p, let n_p be the number of negative weights at p. The following lemma is the key lemma to prove Theorem 1.1, which is proved in [4].

Lemma 2.3. [4] *Let the circle act on a $2n$-dimensional compact almost complex manifold with isolated fixed points. Let w be a primitive weight. Then for each integer i such that $0 \leq i \leq n-1$, the number of times the weight w occurs at fixed point p with $n_p = i$, counted with multiplicity, is equal to the number of times the weight $-w$ occurs at fixed points p with $n_p = i+1$, counted with multiplicity. That is, for each integer i such that $0 \leq i \leq n-1$,*

$$\sum_{p \in M^{S^1}, n_p = i} N_p(w) = \sum_{p \in M^{S^1}, n_p = i+1} N_p(-w).$$

3. Proof of Theorem 1.1

Proof of Theorem 1.1. First, we show that there exists a fixed point p with $n_p = 0$. Let p be any fixed point. If $n_p = 0$, then the claim holds. Suppose that $n_p > 0$. Then p has a negative weight $-w$ for some positive integer w. Since the action is primitive, w is a primitive weight. Therefore, by applying Lemma 2.3 for $i = n_p - 1$ and the primitive weight w, there exists a fixed point p' such that $n_{p'} = n_p - 1$ and p' has weight $+w$. If $n_{p'} = 0$, then the claim holds. If $n_{p'} > 0$, then apply the same argument above; p' has a negative weight $-w'$ where w' is primitive, and hence by applying Lemma 2.3 for $i = n_{p'} - 1$ with the primitive weight w', there exists a fixed point p'' such that $n_{p''} = n_{p'} - 1$ and p'' has weight $+w'$. Continuing the argument, we have that there exists a fixed point p_0 such that $n_{p_0} = 0$.

Second, we prove that $N_k = N_0 \cdot \binom{n}{k}$ for each integer k such that $0 \leq k \leq n$, where $\dim M = 2n$ and N_k is the number of fixed points which have k negative weights. Consider N_0. For each positive weight w at a fixed point p_0 with $n_{p_0} = 0$, w is primitive by the assumption, and hence by Lemma 2.3 for $i = 0$ and w, there must exist a fixed point p_1 such that $n_{p_1} = 1$ and p_1 has weight

$-w$. The number of fixed points p_0 with $n_{p_0} = 0$ is N_0 and every fixed point p_0 with $n_{p_0} = 0$ has n positive weights, which are all primitive. Moreover, every fixed point p_1 with $n_{p_1} = 1$ has 1 negative weight. Therefore, by considering Lemma 2.3 for $i = 0$ and for all primitive weights, it follows that the total number $n \cdot N_0$ of positive weights at fixed points p_0 with $n_{p_0} = 0$ is equal to the total number N_1 of negative weights at fixed points p_1 with $n_{p_1} = 1$. That is, $N_1 = n \cdot N_0$.

Suppose that the claim holds for $k = m$ such that $k < n$. As above, since every positive weight w at a fixed point p_m with $n_{p_m} = m$ is primitive, by applying Lemma 2.3 for $i = m$ and w, there must exist a fixed point p_{m+1} such that $n_{p_{m+1}} = m + 1$ and $-w$ is a weight at p_{m+1}. By inductive hypothesis, the number N_m of fixed points p_m with $n_{p_m} = m$ is equal to $N_0 \cdot \binom{n}{m}$, and each fixed point p_m with $n_{p_m} = m$ has $(n - m)$ positive weights, which are all primitive. Moreover, every fixed point p_{m+1} with $n_{p_{m+1}} = m + 1$ has $m + 1$ negative weights. Therefore, by considering Lemma 2.3 for $i = m$ and for all primitive weights, it follows that the total number $(n - m) \cdot N_m$ of positive weights at fixed points p_m with $n_{p_m} = m$ is equal to the total number $(m + 1) \cdot N_{m+1}$ of negative weights at fixed points p_{m+1} with $n_{p_{m+1}} = m + 1$. It follows that $(m + 1) \cdot N_{m+1} = (n - m) \cdot N_m = (n - m) \cdot N_0 \cdot \binom{n}{m}$. Therefore, we have that $N_{m+1} = N_0 \cdot \binom{n}{m+1}$. Hence the claim holds.

Since the Todd genus $\text{Todd}(M)$ of M is equal to $\chi_y(M)|_{y=0}$ and $\chi_y(M) = \sum_{i=0}^{n} \chi^i(M) \cdot y^i$ for any y, it follows that $\text{Todd}(M) = \chi_0(M) = \chi^0(M) = N_0$. Because $N_0 > 0$ by the first claim, we have that $\text{Todd}(M) = \chi^0(M) = N_0 > 0$. This proves the first claim of the theorem. Moreover, by Theorem 2.1, we have that $\chi^i(M) = (-1)^i N_i = (-1)^i N_0 \cdot \binom{n}{i} = (-1)^i \text{Todd}(M) \cdot \binom{n}{i}$ for each integer i such that $0 \leq i \leq n$. Hence we have that $\chi_y(M) = \sum_{i=0}^{n} \chi^i(M) \cdot y^i = \sum_{i=0}^{n} (-1)^i \cdot \text{Todd}(M) \cdot \binom{n}{i} \cdot y^i = \text{Todd}(M)(1 - y)^n$, since $\sum_{i=0}^{n} (-1)^i \cdot \binom{n}{i} \cdot y^i = (1 - y)^n$.

We have shown that $N_i = N_0 \cdot \binom{n}{i} = \text{Todd}(M) \cdot \binom{n}{i}$ for $0 \leq i \leq n$. Therefore, the total number of fixed points is $\sum_{i=0}^{n} N_i = \sum_{i=0}^{n} N_0 \cdot \binom{n}{i} = \sum_{i=0}^{n} \text{Todd}(M) \cdot \binom{n}{i} = \text{Todd}(M) \cdot 2^n$. This proves the theorem. □

Proof of Corollary 1.4. We have that $A = A_1 = \{a, b, c\}$, $A_2 = \{a + b, a + c, b + c\}$, and $A_3 = \{a + b + c\}$. Since $a \leq b \leq c$ and $a + b \neq c$, for any $w \in A$ we have $w \notin A_2$ and $w \notin A_3$. This means that the action is primitive. Therefore, by Theorem 1.1, $\text{Todd}(M) > 0$ and there are $\text{Todd}(M) \cdot 2^3$ fixed points. Since $\text{Todd}(M)$ is equal to the number of fixed points which have no negative weights, it follows that the action is Hamiltonian. On the other hand, since the action is Hamiltonian, there is a unique fixed point which has no negative weights. This implies that $\text{Todd}(M) = 1$, and the number of fixed points is $\text{Todd}(M) \cdot 2^3 = 8$. □

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