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DEGREE OF CONVERGENCE FOR FOURIER SERIES OF FUNCTIONS IN THE CLASS L^p-BV

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Abstract. In this paper, the author introduces the class L^p-BV of functions which are of bounded variation in the sense of L^p-norm and investigates the degree of convergence for Fourier series of functions belonging to this class.

1. Introduction

In 1881 [4], Camille Jordan who was seeking a sufficient condition for a function to have an everywhere convergent Fourier series introduced the notion of functions of bounded variation. It is well known that functions of bounded variation have Fourier series which converge everywhere and converge uniformly on each closed interval of continuity [11]. The concept of functions of bounded variation was generalized by many authors and in various ways [2,5,6,7,9]. These generalizations had been studied mainly because of their applicability to the theory of Fourier series. For instance, the Fourier series of any function of harmonic bounded variation is everywhere pointwise convergent and in case of a continuous function the convergence is uniform [10]. In this note, we generalize the notion of functions of bounded variation in the mean [5] and study a sufficient condition for the convergence of Fourier series of a function in L^p-norm. In fact, we deal with functions that are of bounded variation in the sense of L^p-norm, namely L^p-BV and investigate the degree of convergence for Fourier series of a function belonging to the class L^p-BV.

2. Bounded variation in L^p-norm

Let $f \in L^p[-\pi, \pi]$ $(1 \leq p < \infty)$ be a 2π-periodic function and $P : -\pi = t_o < t_1 < ... < t_n = \pi$ be a partition of $[-\pi, \pi]$. Then f is said to be of bounded variation in L^p-norm if

$$V_p(f) = sup\{\sum_{k=1}^{n}(\int_{-\pi}^{\pi} |f(x+t_k) - f(x+t_{k-1})|^p dx)^{\frac{1}{p}}\} < \infty, \quad (1)$$

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where the supremum is taken over all partitions P of $[-\pi, \pi]$. We denote the class of all functions that are of bounded variation in L^p-norm by L^p-BV. First of all, we have

Theorem 2.1. *The class L^p-BV is a linear space.*

Proof. For $f, g \in L^p$-BV and $a, b \in R$,

$$\sum_{k=1}^{n}\left(\int_{-\pi}^{\pi} |(af+bg)(x+t_k)-(af+bg)(x+t_{k-1})|^p dx\right)^{\frac{1}{p}}$$

$$=\sum_{k=1}^{n}\left(\int_{-\pi}^{\pi} |a(f(x+t_k)-f(x+t_{k-1}))+b(g(x+t_k)-g(x+t_{k-1}))|^p dx\right)^{\frac{1}{p}},$$

by virtue of Minkowski inequality, the last equality leads to

$$\leq \sum_{k=1}^{n}\left\{|a|\left(\int_{-\pi}^{\pi} |f(x+t_k)-f(x+t_{k-1})|^p dx\right)^{\frac{1}{p}}+|b|\left(\int_{-\pi}^{\pi} |g(x+t_k)-g(x+t_{k-1})|^p dx\right)^{\frac{1}{p}}\right\}$$

$$=|a|\sum_{k=1}^{n}\left(\int_{-\pi}^{\pi} |f(x+t_k)-f(x+t_{k-1})|^p dx\right)^{\frac{1}{p}}+|b|\sum_{k=1}^{n}\left(\int_{-\pi}^{\pi} |g(x+t_k)-g(x+t_{k-1})|^p dx\right)^{\frac{1}{p}}$$

$$\leq |a|V_p(f)+|b|V_p(g)$$

and this implies that $af+bg$ is in the class L^p-BV. This completes the proof. □

Also it is easy to see that the class L^p-BV is a normed linear space equipped with norm

$$||f||_{L^p-BV} = ||f||_p + V_p(f), \tag{2}$$

where $||f||_p = (\int_{-\pi}^{\pi} |f(x)|^p dx)^{\frac{1}{p}}$. In fact, we have

Theorem 2.2. *The class L^p-BV with the norm $||\cdot||_{L^p-BV}$ is a Banach space.*

Proof. Let $\{f_n\}_{n\in N}$ be a Cauchy sequence in the class L^p-BV. Then for any $\varepsilon > 0$, there exists a positive integer n_o such that

$$||f_n - f_m||_{L^p-BV} < \varepsilon \tag{3}$$

whenever $n, m \geq n_o$.
It follows from (1) and (2) that we have

$$||f_n - f_m||_p \leq ||f_n - f_m||_{L^p-BV} < \varepsilon.$$

The last inequality implies that $\{f_n\}_{n\in N}$ is a Cauchy sequence in the L^p space which is complete. Thus $\lim_{n\to\infty} f_n$ exists, call it f. Taking into account Fatou's lemma and (1), we obtain

$$||f_n - f||_{L^p-BV} \leq \lim\inf_{m\to\infty}(||f_n - f_m||_p + V_p(f_n - f_m)) \leq \varepsilon \quad (4)$$

whenever $n \geq n_o$.
From (4) it follows that

$$||f||_{L^p-BV} \leq ||f_{n_o}||_{L^p-BV} + ||f - f_{n_o}||_{L^p-BV} \leq ||f_{n_o}||_{L^p-BV} + \varepsilon < \infty.$$

Therefore f belongs to the class L^p-BV. This completes the proof. □

The modulus of continuity of f in L^p-norm is defined by

$$\omega_p(f,\delta) = sup_{|t|\leq\delta}(\int_{-\pi}^{\pi} |f(x+t) - f(x)|^p dx)^{\frac{1}{p}}. \quad (5)$$

Setting

$$\psi_x(t) = f(x+t) + f(x-t) - 2f(x), \quad (6)$$

we define

$$\Omega_p(\psi, I) = sup_{t,s\in I}(\int_{-\pi}^{\pi} |\psi_x(t) - \psi_x(s)|^p dx)^{\frac{1}{p}} \quad (7)$$

for a subinterval $I \subset [-\pi,\pi]$.
From now on, subintervals $I_{k,n}$ denote $[\frac{k\pi}{n+1}, \frac{(k+1)\pi}{n+1}], k = 0,1,...,n$. It is easy to see that

$$\Omega_p(\psi, I_{k,n}) \leq 4\omega_p(f, \frac{\pi}{n+1}). \quad (8)$$

In the sequel, we shall see that this quantity for subintervals $I_{k,n}$ of $[0,\pi]$ provides the degree of convergence for Fourier series of f in L^p-norm.

Lemma 2.3. *Let f belong to the class L^p-BV. Then we have*

$$(\int_{-\pi}^{\pi} |s_n(f,x) - f(x)|^p dx)^{\frac{1}{p}} = O\big(\sum_{k=0}^{n} \frac{1}{k+1}\Omega_p(\psi, I_{k,n})\big),$$

where $s_n(f,x)$ denotes the nth partial sum of Fourier series of f.

Proof. It follows from Titchmarsh [8] that

$$s_n(f,x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \psi_x(t)\frac{sin(n+\frac{1}{2})t}{sin(\frac{t}{2})}dt.$$

Hence we have

$$(\int_{-\pi}^{\pi} |s_n(f,x) - f(x)|^p dx)^{\frac{1}{p}} = \frac{1}{2\pi}(\int_{-\pi}^{\pi} |\int_0^{\pi} \psi_x(t)\frac{sin(n+\frac{1}{2})t}{sin(\frac{t}{2})}dt|^p dx)^{\frac{1}{p}},$$

by virtue of Minkowski inequality, the last equality leads to

$$\leq \frac{1}{2\pi}(\int_{-\pi}^{\pi}|\int_{I_{o,n}}\psi_x(t)\frac{sin(n+\frac{1}{2})t}{sin(\frac{t}{2})}dt|^p dx)^{\frac{1}{p}}+\frac{1}{2\pi}(\int_{-\pi}^{\pi}|\sum_{k=1}^{n}\int_{I_{k,n}}\psi_x(t)\frac{sin(n+\frac{1}{2})t}{sin(\frac{t}{2})}dt|^p dx)^{\frac{1}{p}}$$

$$\leq \frac{1}{2\pi}(\int_{-\pi}^{\pi}|\int_{I_{o,n}}\psi_x(t)\frac{sin(n+\frac{1}{2})t}{sin(\frac{t}{2})}dt|^p dx)^{\frac{1}{p}}+$$

$$+\frac{1}{2\pi}(\int_{-\pi}^{\pi}|\sum_{k=1}^{n}\int_{I_{k,n}}(\psi_x(t)-\psi_x(\frac{k\pi}{n+1}))\frac{sin(n+\frac{1}{2})t}{sin(\frac{t}{2})}dt|^p dx)^{\frac{1}{p}}+$$

$$+\frac{1}{2\pi}(\int_{-\pi}^{\pi}|\sum_{k=1}^{n}\psi_x(\frac{k\pi}{n+1})\int_{I_{k,n}}\frac{sin(n+\frac{1}{2})t}{sin(\frac{t}{2})}dt|^p dx)^{\frac{1}{p}}$$

$$\leq \frac{1}{2\pi}(\int_{-\pi}^{\pi}(\int_{I_{o,n}}|\psi_x(t)\frac{sin(n+\frac{1}{2})t}{sin(\frac{t}{2})}|dt)^p dx)^{\frac{1}{p}}+$$

$$+\sum_{k=1}^{n}\frac{1}{2\pi}(\int_{-\pi}^{\pi}(\int_{I_{k,n}}|\psi_x(t)-\psi_x(\frac{k\pi}{n+1})||\frac{sin(n+\frac{1}{2})t}{sin(\frac{t}{2})}|dt)^p dx)^{\frac{1}{p}}+$$

$$+\frac{1}{2\pi}(\int_{-\pi}^{\pi}|\sum_{k=1}^{n}\psi_x(\frac{k\pi}{n+1})\int_{I_{k,n}}\frac{sin(n+\frac{1}{2})t}{sin(\frac{t}{2})}dt|^p dx)^{\frac{1}{p}}. \tag{9}$$

Firstly, by virtue of Minkowski's integral inequality, we get

$$\frac{1}{2\pi}(\int_{-\pi}^{\pi}(\int_{I_{o,n}}|\psi_x(t)\frac{sin(n+\frac{1}{2})t}{sin(\frac{t}{2})}|dt)^p dx)^{\frac{1}{p}}$$

$$\leq \frac{1}{2\pi}\int_{I_{o,n}}(\int_{-\pi}^{\pi}|\psi_x(t)\frac{sin(n+\frac{1}{2})t}{sin(\frac{t}{2})}|^p dx)^{\frac{1}{p}}dt$$

$$=\frac{1}{2\pi}\int_{I_{o,n}}|\frac{sin(n+\frac{1}{2})t}{sin(\frac{t}{2})}|(\int_{-\pi}^{\pi}|\psi_x(t)|^p dx)^{\frac{1}{p}}dt,$$

by virtue of the inequality [11]

$$|\frac{sin(n+\frac{1}{2})t}{sin(\frac{t}{2})}|\leq 2n+1$$

and the fact that $\psi_x(0)=0$, the last equality leads to

$$=\frac{1}{2\pi}\int_{I_{o,n}}|\frac{sin(n+\frac{1}{2})t}{sin(\frac{t}{2})}|(\int_{-\pi}^{\pi}|\psi_x(t)-\psi_x(0)|^p dx)^{\frac{1}{p}}dt$$

$$\leq \frac{1}{2\pi}\Omega_p(\psi,I_{o,n})\int_{I_{o,n}}|\frac{sin(n+\frac{1}{2})t}{sin(\frac{t}{2})}|dt$$

$$\leq \Omega_p(\psi,I_{o,n}). \tag{10}$$

Secondly, taking into account Minkowski's integral inequality, we have

$$\sum_{k=1}^{n}\frac{1}{2\pi}(\int_{-\pi}^{\pi}(\int_{I_{k,n}}|\psi_x(t)-\psi_x(\frac{k\pi}{n+1})||\frac{sin(n+\frac{1}{2})t}{sin(\frac{t}{2})}|dt)^p dx)^{\frac{1}{p}}$$

$$\leq\sum_{k=1}^{n}\frac{1}{2\pi}\int_{I_{k,n}}(\int_{-\pi}^{\pi}(|\psi_x(t)-\psi_x(\frac{k\pi}{n+1})||\frac{sin(n+\frac{1}{2})t}{sin(\frac{t}{2})}|)^p dx)^{\frac{1}{p}}dt$$

$$=\sum_{k=1}^{n}\frac{1}{2\pi}\int_{I_{k,n}}|\frac{sin(n+\frac{1}{2})t}{sin(\frac{t}{2})}|(\int_{-\pi}^{\pi}|\psi_x(t)-\psi_x(\frac{k\pi}{n+1})|^p dx)^{\frac{1}{p}}dt,$$

by taking into account the inequality

$$sin\frac{t}{2}\geq\frac{t}{\pi},(0\leq t\leq\pi),$$

the last equality leads to

$$\leq\sum_{k=1}^{n}\frac{1}{2\pi}\Omega_p(\psi,I_{k,n})\int_{I_{k,n}}|\frac{sin(n+\frac{1}{2})t}{sin(\frac{t}{2})}|dt$$

$$\leq\sum_{k=1}^{n}\frac{1}{2k}\Omega_p(\psi,I_{k,n}). \tag{11}$$

Thirdly, let us denote

$$P_{k,n}=\int_{\frac{k\pi}{n+1}}^{\pi}\frac{sin(n+\frac{1}{2})t}{sin(\frac{t}{2})}dt.$$

Then it follows from summation by parts that

$$\frac{1}{2\pi}(\int_{-\pi}^{\pi}|\sum_{k=1}^{n}\psi_x(\frac{k\pi}{n+1})\int_{I_{k,n}}\frac{sin(n+\frac{1}{2})t}{sin(\frac{t}{2})}dt|^p dx)^{\frac{1}{p}}$$

$$=\frac{1}{2\pi}(\int_{-\pi}^{\pi}|\sum_{k=1}^{n}\psi_x(\frac{k\pi}{n+1})(P_{k,n}-P_{k+1,n})|^p dx)^{\frac{1}{p}}$$

$$=\frac{1}{2\pi}(\int_{-\pi}^{\pi}|\sum_{k=1}^{n}(\psi_x(\frac{k\pi}{n+1})-\psi_x(\frac{(k-1)\pi}{n+1}))P_{k,n}|^p dx)^{\frac{1}{p}},$$

by using the inequality [1],

$$|\int_{x}^{\pi}\frac{sin(n+\frac{1}{2})t}{sin(\frac{t}{2})}dt|\leq\frac{3\pi}{(n+\frac{1}{2})x},(0<x\leq\pi)$$

the last equality yields from Minkowski inequality

$$\leq\frac{1}{2\pi}\sum_{k=1}^{n}\frac{6}{k}(\int_{-\pi}^{\pi}|\psi_x(\frac{k\pi}{n+1})-\psi_x(\frac{(k-1)\pi}{n+1})|^p dx)^{\frac{1}{p}}$$

$$\leq\frac{3}{\pi}\sum_{k=1}^{n}\frac{1}{k}\Omega_p(\psi,I_{k-1,n}). \tag{12}$$

Combing (9),(10),(11) and (12), we have the desired result. □

As consequences, the following theorems can be derived from lemma 2.3:

Theorem 2.4. *Let f belong to the class L^p-BV. Then*

$$\Big(\int_{-\pi}^{\pi} |s_n(f,x) - f(x)|^p dx\Big)^{\frac{1}{p}} = O\Big(\sum_{k=0}^{n} \frac{1}{k+1}\omega_p(f, \frac{\pi}{n+1})\Big).$$

Proof. From lemma 2.3 and (8), the result follows. □

Theorem 2.5. *Let f belong to the class L^p-BV. Then*

$$\Big(\int_{-\pi}^{\pi} |s_n(f,x) - f(x)|^p dx\Big)^{\frac{1}{p}} = o\big(1\big).$$

Proof. It is well known [3] that for any function $f \in L^p[-\pi,\pi]$

$$\lim_{n\to\infty} \omega_p(f, \frac{\pi}{n+1}) = 0.$$

By considering the inequality (8), we have , for a fixed l

$$\lim_{n\to\infty} \sum_{k=1}^{l} \frac{1}{k+1}\Omega_p(\psi, I_{k,n}) = 0.$$

On the other hand, from (1),(6) and (7), it follows that

$$\sum_{k=l+1}^{n} \frac{1}{k+1}\Omega_p(\psi, I_{k,n}) \le \frac{4}{l+2}V_p(f).$$

The right hand side of the last inequality can be made as small as we wish as long as l is sufficiently large. Hence the claim follows. □

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