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OPTIMALITY AND DUALITY FOR NONSMOOTH FRACTIONAL ROBUST OPTIMIZATION PROBLEMS WITH (V,ρ)-INVEXITY

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Abstract. We establish necessary and sufficient optimality conditions for a nonsmooth fractional robust optimization programming problems. Moreover, we prove the weak and strong duality theorems under (V,ρ)-invexity assumption.

1. Introduction

Let X be a Banach space, and let functions $f_i, g_i : X \to \mathbb{R}, i = 1, \cdots, p, j = 1, \cdots, m$ be given. Consider the following generalized nondifferentiable fractional optimization problem (GFP):

$$\begin{array}{lll} \text{(GFP)} & \text{Minimize} & \max\left\{\dfrac{f_i(x)}{g_i(x)} \mid i = 1, \cdots, p\right\} \\ & \text{subject to} & h_j(x, v_j) \leq 0,\ v_j \in V_j, \quad j = 1, \cdots, m, \end{array}$$

where v_j are uncertain parameters, and $v_j \in V_j$ for some sequentially compact topological space V_j, $j = 1, \cdots, m$ and $f_i : X \to \mathbb{R}$, $g_i : X \to \mathbb{R}, i = 1, \cdots, p$ and $h_j : X \times V_j \to \mathbb{R}, j = 1, \cdots, m$ are locally Lipschitz function. We assume that $f_i(x) \geqq 0$ and $g_i(x) > 0$, $i = 1, \cdots, p$.

Recently, Lee and Kim [5] considered a nonsmooth multiobjective robust optimization problem with more than two locally Lipschitz objective functions and locally Lipschitz constraint functions in the face of data uncertainty. In this paper, we establish necessary and sufficient optimality conditions for a nonsmooth fractional robust optimization programming problems. Moreover, we prove the weak and strong duality theorems under (V,ρ)-invexity assumption.

Now we give some notations for our results in this section;

Let a function $f\colon \mathbb{R}^n \to \mathbb{R}$ be given. We shall suppose that f is locally Lipschitz, that is, for each $x \in \mathbb{R}^n$, there exist an open neighborhood U and a

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constant $L > 0$ such that for all y and z in U,

$$|f(y) - f(z)| \leqq L\|y - z\|.$$

Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function. The subdifferential of g at $a \in \text{dom}g$ is defined by

$$\partial g(a) := \{v \in \mathbb{R}^n \mid g(x) \geqq g(a) + \langle v, x - a\rangle \quad \forall x \in \text{dom}g\},$$

where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathbb{R}^n$ and $\text{dom}g := \{x \in \mathbb{R}^n \ : \ g(x) < +\infty\}$.

Definition 1. A vector function $f : \mathbb{R}^n \to \mathbb{R}^p$ is said to be (V,ρ)-invex at $u \in \mathbb{R}^n$ with respect to the function η and $\theta_i : \mathbb{R}^n \to \mathbb{R}^n$ if there exists $\alpha_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+\backslash\{0\}$ and $\rho_i \in \mathbb{R}$, $i = 1,\dots,p$ such that for any $\xi_i \in \partial f_i(u)$, $i = 1,\dots,p$ and any $x \in \mathbb{R}^n$, and for all $i = 1,\dots,p$,

$$\alpha_i(x,u)[f_i(x) - f_i(u)] \geq \xi_i^T\eta(x,u) + \rho_i\|\theta_i(x,u)\|^2.$$

Lemma 1.1. [1] *Let f and g be Lipschitz near x and suppose that $g(x) \neq 0$. Then $\frac{f}{g}$ is Lipschitz near x, and one has*

$$\partial\left(\frac{f}{g}\right)(x) \subset \frac{g(x)\partial f(x) - f(x)\partial g(x)}{\{g(x)\}^2}.$$

If in addition $f(x) \geqq 0$, $g(x) > 0$ and if f and $-g$ are regular at x, then equality holds and $\frac{f}{g}$ is regular at x.

Theorem 1.2. [4] *Assume that f and g are vector-valued differentiable functions defined on $\mathbb{R}^n$ and $f(x) \geq 0$, $g(x) > 0$ for all $x \in \mathbb{R}^n$. If f and $-g$ are regular and (V,ρ)-invex at x_0, then $\frac{f}{g}$ is (V,ρ)-invex at x_0, where*

$$\bar{\alpha}_i(x,x_0) = \frac{g_i(x)}{g_i(x_0)}\alpha_i(x,x_0), \quad \bar{\theta}_i(x,x_0) = \left(\frac{1}{g_i(x_0)}\right)^{\frac{1}{2}}\theta_i(x,x_0).$$

Let V be a sequentially compact topological space and let $h : X \times V \to \mathbb{R}$ be a given function. Now, we will assume that the following conditions hold:

(C1) $h(x,v)$ is upper semicontinuous in (x,v).

(C2) h is locally Lipschitz in x, uniformly for v in V, that is, for each $x \in X$, there exist and open neighborhood U of x and a constant $L > 0$ such that for all y and z in U, and $v \in V$,

$$|h(y,v) - h(z,v)| \leqq L\|y - z\|.$$

(C3) $h_x^0(x,v;\cdot) = h_x'(x,v;\cdot)$, the derivatives being with respect to x.

(C4) the generalized gradient $\partial_x h(x,v)$ with respect to x is weak* upper semicontinuous in (x,v).

Remark 1. In a suitable setting, conditions (C2), (C3), and (C4) will follow if the function h is convex in x and continuous in v. These conditions on the function h also hold when the derivative $\nabla_x h(x,v)$ with respect to x exists and is continuous in (x,v).

We define a function $\psi : X \to \mathbb{R}$

$$\psi(x) := \max\{h(x,v) \mid v \in V\},$$

and we observe that our conditions (C1)-(C2) imply that ψ is defined and finite (with the maximum defining ψ attained) on X.

$$V(x) := \{v \in V \mid h(x,v) = \psi(x)\}.$$

It is easy to see that $V(x)$ is nonempty and closed for each x in X.

The following lemma, which is a nonsmooth version of Danskin's theorem [2] for max-functions, makes connection between the functions $\psi'(x;d)$ and $h^0(x,v;d)$.

Lemma 1.3. *Under the conditions (C1)-(C4), the usual one-sided directional derivative $\psi'(x;d)$ exists, and satisfies*

$$\begin{aligned}\psi'(x;d) = \psi^0(x;d) &= \max\{h_x^0(x,v;d) \mid v \in V(x)\}\\ &= \max\{\langle \xi, d\rangle \mid \xi \in \partial_x h(x,v),\ v \in V(x)\}.\end{aligned}$$

Lemma 1.4. [7] *In addition to the basic conditions (C1)-(C4), suppose that V is convex, and that $h(x,\cdot)$ is concave on V, for each $x \in U$. Then the following statements hold:*

(i) The set $V(x)$ is convex and sequentially compact.

(ii) The set

$$\partial_x h(x, V(x)) := \{\xi \mid \exists v \in V(x) \text{ such that } \xi \in \partial_x h(x,v)\}$$

is convex and weak compact.*

(iii) $\partial\psi(x) = \{\xi \mid \exists v \in V(x) \text{ such that } \xi \in \partial_x h(x,v)\}$.

2. Optimality theorems

Let $C := \{x \in X \mid h_j(x,v_j) \leq 0,\ v_j \in V_j,\ j = 1,\cdots,m\}$. Define $\psi_j(x) := \max_{v_j \in V_j} h_j(x,v_j)$ for each $j = 1,\cdots,m$. Then if h_j satisfy the conditions (C1) and (C2), $\psi_j : X \to \mathbb{R}, j = 1,\cdots,m$, are locally Lipschitz functions.

Let $x \in C$ and let us decompose $J := \{1,\cdots,m\}$ into two index sets $J = J_1(x) \cup J_2(x)$, where $J_1(x) = \{j \in J \mid \psi_j(x) = 0\}$ and $J_2(x) = J \setminus J_1(x)$. Then for each $j \in J_1(x)$,

$$V_j(x) := \{v_j \in V_j \mid h_j(x,v_j) = \psi_j(x)\}.$$

Definition 2. We define an Extended Nonsmooth Mangasarian-Fromovitz constraint qualification (ENMFCQ) at $x \in C$ as follows:

$$\exists d \in X \text{ such that } h_{jx}^0(x,v_j;d) < 0,\quad \forall v_j \in V_j(x),\quad \forall j \in J_1(x),$$

where $h_{jx}^0(x,v_j;d)$ denotes the generalized directional derivative of h_j with respect to x.

Now from Theorem 3.3 in [7], we can get the following necessary optimality theorem for a weakly robust efficient solution of (GFP); for simplicity, we give its proof.

Theorem 2.1. [7] *Assume that $f, -g$ are regular and $h_j, j = 1, \cdots, m$ satisfy the conditions (C1)–(C4). Suppose that for each $x \in X, h_j(x, \cdot)$ are concave on $V_j, j = 1, \cdots, m$. Let $x^* \in C$ be a weakly robust efficient solution of (GFP), then there exist $\lambda_i \geq 0$, $i \in I(x^*) := \{i \mid \max\left\{\frac{f_i(x^*)}{g_i(x^*)} \mid i = 1, \ldots, p\right\} = \frac{f_i(x^*)}{g_i(x^*)}\}$, $\sum_{i \in I(x^*)} \lambda_i = 1$ and $\mu_j \geq 0$, $j = 1, \ldots, m$, and $v_j^* \in V_j(x^*), j = 1, \cdots, m$ such that*

$$0 \in \sum_{i \in I(x^*} \lambda_i \partial \left(\frac{f_i}{g_i}\right)(x^*) + \sum_{j=1}^{m} \mu_j \partial_x h_j(x^*, v_j^*),$$
$$\mu_j h_j(x^*, v_j^*) = 0, \ j = 1, \cdots, m.$$

Moreover, if we further assume that the Extended Nonsmooth Mangasarian-Fromovitz constraint qualification (ENMFCQ) holds, then there exist $\lambda_i \geqq 0, i = 1, \cdots, p$, not all zero, $\mu_j \geqq 0$ and $v_j^* \in V_j(x^*), j = 1, \cdots, m$ such that

$$0 \in \sum_{i \in I(x^*)} \lambda_i \partial \left(\frac{f_i}{g_i}\right)(x^*) + \sum_{j=1}^{m} \mu_j \partial_x h_j(x^*, v_j^*),$$
$$\mu_j h_j(x^*, v_j^*) = 0, \ j = 1, \cdots, m.$$

Proof. Let $\phi_i(x) = \frac{f_i(x)}{g_i(x)}$, $i = 1, \ldots, p$. Let x^* be a solution of (GFP) and let $I(x^*) = \{i \mid \max\{\phi_i(x^*) \mid i = 1, \ldots, p\} = \phi_i(x^*)\}$. Then by Proposition 2.3.12 in [1], Corollary 5.1.8 in [9] and Theorem 3.3 [6], there exist $\mu_j \geqq 0, v_j^* \in V_j(x^*), j = 1, \cdots, m \ j = 1, \ldots, m$,

$$0 \in \text{co}\{\partial \phi_i(x^*) \mid i \in I(x^*)\} + \sum_{j=1}^{m} \mu_j \partial_x h_j(x^*, v_j^*) \tag{1}$$
$$\text{and} \ \ \mu_j h_j(x^*, v_j^*) = 0,$$

where co$A$ is the convexhull of the set A. By Lemma 1.2,

$$\partial \phi_i(x^*) = \frac{g_i(x^*)\partial f_i(x^*) - \partial g_i(x^*) f_i(x^*)}{(g_i(x^*))^2} = \partial \left(\frac{f_i}{g_i}\right)(x^*),$$

and hence from (1), there exist $\lambda_i \geq 0$, $i \in I(x^*)$, $\sum_{i \in I(x^*)} \lambda_i = 1$ and $\mu_j \geq 0, v_j^* \in V_j(x^*), j = 1, \cdots, m \ j = 1, \ldots, m$ such that

$$0 \in \sum_{i \in I(x^*)} \lambda_i \partial \left(\frac{f_i}{g_i}\right)(x^*) + \sum_{j=1}^{m} \mu_j \partial_x h_j(x^*, v_j^*)$$
$$\text{and} \ \ \sum_{j=1}^{m} \mu_j h_j(x^*, v_j^*) = 0.$$

□

Now we give a sufficient optimality theorem for weakly robust efficient solutions for (GFP):

Theorem 2.2. *Let x^* be a robust feaible solution of (GFP). Suppose that there exist $\lambda_i \geqq 0$, $i \in I(x^*)$, $\sum_{i\in I(x^*)} \lambda_i = 1$, $\mu_j \geqq 0$ and $v_j^* \in V_j(x^*), j = 1, \cdots, m$ such that*

$$0 \in \sum_{i\in I(x^*)} \lambda_i \partial \left(\frac{f_i}{g_i}\right)(x^*) + \sum_{j=1}^{m} \mu_j \partial_x h_j(x^*, v_j^*), \tag{2}$$
$$\mu_j h_j(x^*, v_j^*) = 0,\ j = 1, \cdots, m.$$

If each $f_i(\cdot), g_i(\cdot), i = 1, \cdots, p$ are (V, ρ)-invex at x^* and $h_j(\cdot, v_j^*), j = 1, \cdots, m$ are η-invex at x^* with respect to the same η and $\sum_{i=1}^{p} \lambda_i\rho_i\|\theta_i(x, x^*)\|^2 \geqq 0$, then x^* is a weakly robust efficient solution of (GFP).

Proof. Suppose that x^* is not a solution of (GFP). Then there exist a feasible solution x of (GFP) such that

$$\max_{1\leqq i\leqq p} \frac{f_i(x^*)}{g_i(x^*)} > \max_{1\leqq i\leqq p} \frac{f_i(x)}{g_i(x)}.$$

Then

$$\frac{f_i(x^*)}{g_i(x^*)} > \frac{f_i(x)}{g_i(x)}, \text{for all } i \in I(x^*),$$

and hence $\bar{\alpha}_i(x, x^*) > 0$,

$$\bar{\alpha}_i(x, x^*)\left[\frac{f_i(x)}{g_i(x)} - \frac{f_i(x^*)}{g_i(x^*)}\right] < 0.$$

Since $f(\cdot)$ and $-g(\cdot)$ are (V, ρ)-invex and regular at x_0, by Theorem 1.3, we have for any $w_i \in \partial\left(\frac{f_i}{g_i}\right)(x^*)$, $i \in I(x^*)$

$$w_i\eta(x, x^*) + \rho_i\|\bar{\theta}(x, x^*)\|^2 < 0.$$

Hence, there exist $\lambda_i \geqq 0$, $i \in I(x^*)$, $\sum_{i\in I(x^*)} \lambda_i = 1$ such that

$$\sum_{i\in I(x^*)} \lambda_i w_i\eta(x, x^*) + \sum_{i\in I(x^*)} \lambda_i\rho_i\|\bar{\theta}(x, x^*)\|^2 < 0.$$

Since $\sum_{i\in I(x^*)} \lambda_i\rho_i\|\bar{\theta}(x, x^*)\|^2 \geqq 0$,

$$\sum_{i\in I(x^*)} \lambda_i w_i\eta(x, x^*) < 0,$$

and so, it follows from (2) that there exist $\nu_j \in \partial_x h_j(x^*, v_j^*), v_j^* \in V_j(x^*)$, $j = 1, \ldots, m$ such that

$$\sum_{j=1}^{m} \mu_j\nu_j\eta(x, x^*) > 0.$$

Then, by the η-invexity of h, we have

$$\sum_{j=1}^{m} \mu_j h_j(x, v_j^*) > \sum_{j=1}^{m} \mu_j h_j(x^*, v_j^*).$$

Since $\sum_{j=1}^{m} \mu_j h_j(x^*, v_j^*) = 0$, we have $\sum_{j=1}^{m} \mu_j h_j(x, v_j^*) > 0$, which is a contradiction since $\mu_j \geqq 0$, $j = 1, \ldots, m$ and x is a feasible solution of (GFP). Consequently, x^* is a solution of (GFP). □

3. Duality Theorems

Now, we propose the following Mond-Weir type dual problem (DGFP):

$$\text{(DGFP)} \quad \text{Maximize} \quad \max\left\{\frac{f_i(u)}{g_i(u)} \mid i = 1, \ldots, p\right\}$$

$$\text{subject to} \quad 0 \in \sum_{i\in I(u)} \lambda_i \partial\left(\frac{f_i}{g_i}\right)(u) + \sum_{j=1}^{m} \mu_j \partial_x h_j(u, v_j) \tag{3}$$

$$\sum_{j=1}^{m} \mu_j h_j(u, v_j) = 0,$$

$$\lambda_i \geqq 0,\ i \in I(u),\ \sum_{i\in I(u)} \lambda_i = 1,$$

$$\mu_j \geqq 0,\ v_j \in V_j,\ j = 1, \ldots, m.$$

Now we show that the following weak duality theorem holds between (GFP) and (DGFP).

Theorem 3.1. (**Weak Duality**) *Assume that f and $-g$ are regular. Let x be a feasible for (GFP) and let (u, v, λ, μ) be feasible for (DGFP). Assume that $f(\cdot)$ and $-g(\cdot)$ are (V, ρ)-invex at u, and let $h_j(\cdot, v_j), j = 1, \cdots, m$ are η-invex at u with respect to the same η, and $\sum_{i\in I(u)} \lambda_i \rho_i \|\bar{\theta}_i(x, u)\|^2 > 0$. Then the following holds:*

$$\max\left\{\frac{f_i(x)}{g_i(x)} \mid i = 1, \ldots, p\right\} \geqq \max\left\{\frac{f_i(u)}{g_i(u)} \mid i = 1, \ldots, p\right\}.$$

Proof. Let x be any feasible for (GFP) and let (u, λ, μ) be any feasible for (DGFP). Then there exist $\mu_j \geqq 0$, $v_j \in V_j(x), j = 1, \cdots, m$ such that

$$\sum_{j=1}^{m} \mu_j h_j(x, v_j) \leqq 0 \leqq \sum_{j=1}^{m} \mu_j h_j(u, v_j).$$

By the η-invexity of $h_j(\cdot, v_j)$, $j = 1, \ldots, m$, there exists $\nu_j^* \in \partial_x h_j(u, v_j)$, $j = 1, \cdots, m$ such that

$$\sum_{j=1}^{m} \mu_j \nu_j^* \eta(x, u) \leqq 0.$$

Using (3), we have there exists $w_i^* \in \partial\left(\frac{f_i}{g_i}\right)(u),\ i \in I(u)$,

$$\sum_{i \in I(u)} \lambda_i\ w_i^* \eta(x,u) \geqq 0. \tag{4}$$

Now suppose that

$$\max\left\{\frac{f_i(x)}{g_i(x)} \mid i=1,\ldots,p\right\} < \max\left\{\frac{f_i(u)}{g_i(u)} \mid i=1,\ldots,p\right\}.$$

Then

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(u)}{g_i(u)},\ \text{ for all } i \in I(u).$$

By Theorem 1.3, we have there exists $w_i^* \in \partial\left(\frac{f_i}{g_i}\right)(u)$, $i \in I(u)$ such that

$$\begin{aligned} 0 > \bar{\alpha}_i(x,u)\left[\frac{f_i(x)}{g_i(x)} - \frac{f_i(u)}{g_i(u)}\right] \\ \geqq w_i^* \eta(x,u) + \rho_i \|\bar{\theta}_i(x,u)\|^2. \end{aligned}$$

By using $\lambda_i \geqq 0$, $i \in I(u)$, we have,

$$\sum_{i \in I(u)} \lambda_i w_i^* \eta(x,u) + \sum_{i \in I(u)} \lambda_i \rho_i \|\bar{\theta}_i(x,u)\|^2 < 0.$$

Since $\sum_{i \in I(u)} \lambda_i \rho_i \|\bar{\theta}_i(x,u)\|^2 \geqq 0$, we have

$$\sum_{i \in I(u)} \lambda_i w_i^* \eta(x,u) < 0,$$

which contradicts (4). Hence the result holds. □

Now we give a strong duality theorem which holds between (GFP) and (DGFP).

Theorem 3.2. *(***Strong Duality***) If $\bar{x}$ is a solution of (GFP) and suppose that the Extended Mangasarian-Fromovitz constraint qualification holds. Then there exist $\bar{\lambda} \in \mathbb{R}^p$ and $\bar{\mu} \in \mathbb{R}^m$ such that $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is feasible for (DGFP). Moreover if the weak duality holds, then $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is a solution of (DGFP).*

Proof. By Theorem 2.1, there exist $\bar{\lambda}_i \geq 0$, $i \in I(\bar{x}) := \{i \mid \max\{\frac{f_(\bar{x})}{g_i(\bar{x})} \mid i = 1,\ldots,p\} = \frac{f_(\bar{x})}{g_i(\bar{x})}\}$, $\sum_{i \in I(\bar{x})} \bar{\lambda}_i = 1$ and $\bar{\mu}_j \geq 0$, $j = 1,\ldots,m$ such that

$$0 \in \sum_{i \in I(\bar{x})} \bar{\lambda}_i \partial\left(\frac{f_i}{g_i}\right)(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j \partial_x h_j(\bar{x}, \bar{v}_j)$$

$$\text{and} \quad \sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{x}, \bar{v}_j) = 0.$$

Thus $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is a feasible for (DGFP). On the other hand, by weak duality (Theorem 3.1),

$$\max\left\{\frac{f_i(\bar{x})}{g_i(\bar{x})} \mid i = 1, \cdots, p\right\} \geq \max\left\{\frac{f_i(u)}{g_i(u)} \mid i = 1, \cdots, p\right\}$$

for any (DGFP) feasible solution $(u, \bar{v}, \lambda, \mu)$. Hence $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\mu})$ is a solution of (DGFP). □

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